Department of Mathematics

Maths 761 Assignment 2

August 10, 2010 Due: 4pm, Thursday August 29th

This assignment is worth 10% of the final grade for this course. Hand in this assignment to your lecturer on the correct day. You are encouraged to use XPP to check your answers, but you must make sure to carefully justify analytically any statements you make, and show all your working. Starred (*) questions are trickier than the rest, and should not be attempted until you have completed all the other questions.

1. (5 marks) Consider the map
$$x_{n+1} = x_n^2(x_n - 1).$$
   (a) Find all fixed points and their stability.
   (b) Sketch a cobweb plot for this map, and show a few iterations of the map near each of the fixed points. Check that the behaviour matches your computed stability in (a).

2. (10 marks) Consider the following system of differential equations:
$$\begin{aligned}\dot{x} &= -2x + y^2,\\ \dot{y} &= y - x^2.\end{aligned}$$
   (a) Sketch a global phase portrait for this system. Be sure to justify carefully the existence of, or lack of, any periodic orbits.
   (b) Use XPP to confirm your answers, and hand in a phase portrait plotted with XPP with your answers. On your XPP plot add shading to indicate the basin of attraction of the non-zero equilibrium.

3. (10 marks) Consider the system
$$\begin{aligned}\dot{x} &= -x(x^2 + y^2 - 2x - 3) + y,\\ \dot{y} &= -y(x^2 + y^2 - 2x - 3) - x.\end{aligned}$$
   (a) Show that there is only one equilibrium, and determine its stability.
   (b) Change coordinates in the following way: first write $\xi = x - 1$, and then convert to polar coordinates for ξ and y.
   (c) Use the Poincaré–Benedixson theorem to prove that there exists a periodic orbit. Be very careful about defining the region to which you apply the theorem.

4. (15 marks) The periodically forced Duffing equations are:

$$\dot{x} = y$$
$$\dot{y} = -\delta y + x - x^3 + \gamma \cos(t + \psi)$$

where $\delta > 0$, $\gamma \geq 0$ and $0 < \psi \leq 2\pi$ are parameters.

(a) For the system with $\gamma = 0$ (i.e. no forcing), prove that there are no periodic orbits and sketch a global phase portrait of the system. (You may assume that $\delta^2 < 8$).

(b) Now consider the system with $\gamma \neq 0$, $\psi = 0$. Consider a subset of points from a trajectory $(x(t), y(t))$ given by $\{x(t_j), y(t_j)\}$, for $t_j = 0 \mod 2\pi$. Use XPP to plot (approximately) 4000 points from such a set, for $\delta = 0.2$, $\gamma = 0.3$. Do your results depend qualitatively on your initial conditions?

(c) Notice that if we want to look at the set of points with $t_j = c \mod 2\pi$ (for some $c \neq 0$) in the system with $\psi = 0$, then this is equivalent to setting $\psi = c$ and looking at the points where $t = 0 \mod 2\pi$. With this in mind, use XPP to plot sets of points in the $x - y$ plane for $t = c \mod 2\pi$, for a range of $c \in [0, 2\pi)$. Print out a selection of pictures which show the stretching and folding behaviour of these equations and hand these in with your solutions. Give a brief explanation of the dynamics which your pictures are showing.

5. * (10 marks) The Hénon map is:

$$x_{n+1} = y_n + 1 - ax_n^2$$
$$y_{n+1} = bx_n$$

Throughout the following, you should consider b to be fixed at some positive value, and consider a as a parameter.

(a) Find all fixed points, and show that they exist only if $a > a_0$, where a_0 is a function of b to be determined.

(b) Find the period-two points, and show that they only exist for $a > a_1$, where a_1 is a function of b to be determined. *Hint: you will find the algebra easier if you don't expand powers of* $(1 - b)$.

(c) Use XPP to investigate numerically what happens for various values of a. Include some printouts from XPP showing at least four qualitatively different types of behaviour, and explain the changes in behaviour that you observe. *Hint: a good value for* b *is* $b = 0.3$. *Think carefully about how best to present your results in order to show clearly the different types of behaviour.*