Maths 761 Lecture 4 Summary

Topics for today

Invariant manifolds in non-linear systems

Reading for this lecture

Glendinning: §4.2, §4.6

Recommended problems

Glendinning: §4: 2
Strogatz: 6.1.14

Reading for next lecture

Glendinning: §5.5-§5.8
Strogatz: §6.8, §7.0-§7.3

Stable and Unstable Manifolds in Nonlinear Systems

Recall from Lecture 2 that a stationary point, x_0, for a nonlinear system $\dot{x} = f(x)$ is *hyperbolic* if and only if $Df(x_0)$ has no zero or purely imaginary eigenvalues.

Hyperbolic stationary points persist under small perturbations of the defining differential equations. In other words, and loosely speaking, if the equation $\dot{x} = f(x)$ has a hyperbolic stationary point at $x = x_0$ then for small enough ϵ the equation $\dot{x} = f(x) + \epsilon v(x)$ will have a hyperbolic stationary solution that is not very far from $x = x_0$.

We wish to extend to nonlinear systems the definitions given earlier for the stable and unstable manifolds of a stationary solution. Let U be some neighbourhood of a stationary point x. Define

$$\begin{aligned} W^s_{loc}(x) &= \text{ the local stable manifold of } x \\ &= \{y \in U \mid \phi(y,t) \to x \text{ as } \ t \to \infty, \ \phi(y,t) \in U \ \ \forall t \geq 0\} \end{aligned}$$

and

$$\begin{aligned} W^u_{loc}(x) &= \text{ the local unstable manifold of } x \\ &= \{y \in U \mid \phi(y,t) \to x \text{ as } \ t \to -\infty, \ \phi(y,t) \in U \ \forall t \leq 0\} \end{aligned}$$

The stable manifold theorem gives some information about the local stable and unstable manifolds of a hyperbolic stationary point.

Stable Manifold Theorem: Suppose that $x = 0$ is a hyperbolic stationary point for

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

where $f \in C^r$, and E^s and E^u are the stable and unstable manifolds of the origin in the linear system

$$\dot{\mathbf{x}} = Df(0)\mathbf{x}.$$

Then there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ for the non-linear system, of the same dimension as E^s and E^u respectively. These manifolds are tangential to E^s and E^u, respectively, at the origin, and are C^r.

The local stable and unstable manifolds can be extended to global stable and unstable manifolds. Define:

$$\begin{aligned} W^s(x) &= \text{the global stable manifold of } x \\ &= \bigcup_{t \leq 0} \phi(W^s_{loc}(x), t), \end{aligned}$$

and

$$\begin{aligned} W^u(x) &= \text{the global unstable manifold of } x \\ &= \bigcup_{t \geq 0} \phi(W^u_{loc}(x), t). \end{aligned}$$

Approximate expressions for $W^u_{loc}(x)$ and $W^s_{loc}(x)$ can sometimes be obtained from power series expansions, as demonstrated in the examples on the handout.