

DEPARTMENT OF MATHEMATICS
MATHS 761 Laboratory 5 notes

In this laboratory session we are going to use XPP to investigate using Poincaré sections in ODEs.

- To add a Poincare section in XPP, click on n(U)meric, (P)oincare map.
- You then have three options - (S)ection, (M)ax/min or (P)eriod. We will only use (S)ection and (P)eriod today.
- If you select (S)ection, you can set up a Poincaré section such as, e.g. $x = 1$, by typing X as the variable and 1 for the section. The Direction box tells you which way the trajectory crosses the section. Type N for the last variable.
- If you select (P)eriod, you should set the variable to T , and then the section to, say, T_1 , then each time $T = 0 \pmod{T_1}$, a point is recorded. This is useful for periodically forced equations.

When viewing the output, you will need to change the line style to plot only points (see notes for Lab 4 if you have forgotten how to do this).

1. Write a .ode file for the equations

$$\begin{aligned}\dot{x} &= \mu x - \omega y - ax(x^2 + y^2) \\ \dot{y} &= \omega x + \mu y - ay(x^2 + y^2)\end{aligned}$$

- (a) First show that if $a, \mu > 0$ the system has a periodic orbit.
- (b) Start XPP with the parameter values $\mu = 0.1, \omega = 2, a = 1$. First integrate the equations as normal and find the periodic orbit.
- (c)
 - i. Set up a Poincaré section along $\{y = 0\}$ as described above (using (S)ection). When you next integrate the equations, the only points that should be plotted are those along the line $y = 0$.
 - ii. Plot the x -coordinates when the trajectory crosses this section as a function of time.
- (d) It's also useful to plot the $(n + 1)$ th point versus the n th point; this gives us an idea of what the Poincaré map looks like. This is easily done in XPP.
 - i. First set up the window so that it is a 2D plot with X on both axes: Click (V)iew axes (2d) and put X on both axes. Make sure your range includes the fixed point. You will see a diagonal line representing X vs X as expected. (It will not be a solid line, but, instead, a series of points.)
 - ii. Click n(U)meric and then r(U)elle plot. This can set negative shifts of the data on either axes. We will shift the X coordinate by 1 so that we will then plot $X(n)$ vs $X(n - 1)$. Enter '1' for x -axis shift, and '0' for y -axis and z -axis shift.
 - iii. Hit Esc to go back to the main menu and (R)estore to plot the shifted data.
- (e) Locate the fixed point on the x_{n-1} vs x_n plot and check it corresponds to the location of the periodic orbit.

2. Write a .ode file for the periodically forced Duffing equations:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\delta y + \beta x - \alpha x^3 + \gamma \cos(t + \psi)\end{aligned}$$

- (a) Use initial parameters of $\alpha = 1$, $\beta = 1$, $\delta = 0.2$, $\gamma = 0.3$, $\psi = 0$. First integrate the equations as normal for a variety of initial conditions. What sort of behaviour do you see? Why do the lines cross in the phase plane?
- (b) Set up a Poincaré section to record points whenever $t = 0 \pmod{2\pi}$ (as described above, using (P)eriod). Note that you will have to write out the value of 2π numerically! Plot the points in the x - y plane (remember to change the line style to points). You should get a nice pretty picture if you have plotted enough points.
- (c) Show that if we want to look at points when $t = c \pmod{2\pi}$ (for some $c \neq 0$) in the system with $\psi = 0$, then this is equivalent to setting $\psi = c$ and looking at the points where $t = 0 \pmod{2\pi}$.
- (d) With this in mind, experiment changing the value of ψ to a range of values between 0 and 2π . Print out a nice selection of pictures to show the stretching and folding behaviour of these equations.