Maths 761 Lecture 20

Topic for today
Bifurcations of fixed points in maps

Reading for this lecture
Glendinning Chapter 9 to end of §9.1

Suggested exercise
Worksheet 7

Today's handouts
Lecture 20 notes
Worksheet 7: Local bifurcations in maps

Local bifurcations in maps

We consider maps of the form

$$x_{n+1} = f(x_n)$$

where $x \in \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth invertible map. Local bifurcations occur at parameter values for which a fixed point or periodic orbit is non-hyperbolic. (Remember that fixed points of a map are non-hyperbolic if one or more eigenvalues of the Jacobian lie on the unit circle.)

Many of the bifurcation results valid for differential equations carry over to maps with only small modifications.

The Centre Manifold Theorem for Maps
Consider the map $x_{n+1} = f(x_n)$ where $x \in \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth invertible map with a fixed point at $x = 0$. Let E^u, E^s, and E^c be the generalised eigenspaces associated with the eigenvalues of $Df(0)$ with magnitude greater than one, less than one, and equal to one, respectively.

Then there exist smooth unstable, stable, and centre manifolds, $W^u(0)$, $W^s(0)$, and $W^c(0)$, tangent at $x = 0$ to E^u, E^s, and $E^c(0)$, respectively. All three manifolds are invariant but W^c is not necessarily unique.

Just as for differential equations, the Centre Manifold Theorem allows us to reduce the dimension of a local bifurcation problem by restricting attention to the centre manifold (or the extended centre manifold). Centre manifolds for maps are computed using power series expansions, just as for differential equations.

We look at the simplest local bifurcations, with one- and two-dimensional centre manifolds. There are three cases:

- $Df(0)$ has one eigenvalue equal to +1;
- $Df(0)$ has one eigenvalue equal to -1;
- $Df(0)$ has a complex conjugate pair of eigenvalues on the unit circle (but not at +1 or -1).

Case 1: A single simple eigenvalue of +1
Without loss of generality, assume that a map

$$w_{n+1} = f(w_n; \mu)$$

has a fixed point at $x = 0$ when $\mu = 0$, and that $Df(0; 0)$ has a single eigenvalue of +1 with no other eigenvalues on the unit circle. Then there is a one-dimensional centre manifold on which the dynamics is of the form

$$x_{n+1} = g(x_n; \mu)$$

for $x \in \mathbf{R}$ and for some function g with

$$g(0; 0) = 0 \quad \text{and} \quad \frac{\partial g}{\partial x}(0; 0) = 1.$$

Expanding g in a Taylor series about $(x, \mu) = (0, 0)$ gives

$$g(x; \mu) = x + g_\mu \mu + \frac{1}{2}\left(g_{xx} x^2 + 2g_{x\mu}\mu x + g_{\mu\mu}\mu^2\right) + \ldots$$

where all the derivatives are evaluated at $(0; 0)$. Depending on the values of the terms g_μ, g_{xx}, etc., the bifurcation at $\mu = 0$ can be a saddle-node, pitchfork, or transcritical bifurcation, just as for differential equations.

- If $g_\mu(0; 0) \neq 0$ and $g_{xx}(0; 0) \neq 0$ there is a saddle-node bifurcation. For example, the map $x_{n+1} = x_n + \mu - x^2$ has a saddle-node bifurcation at $\mu = 0$; where a pair of fixed points of different stabilities is created as μ is increased through zero.
- If $g_\mu = 0$, $g_{xx} \neq 0$, and $g_{\mu x}^2 - g_{xx}g_{\mu\mu} \neq 0$ there is a transcritical bifurcation.For example, the map $x_{n+1} = x_n + \mu x - x^2$ has a transcritical bifurcation at $\mu = 0$; two fixed points pass through one another as μ is increased through zero, and they exchange stabilities as they do so.
- If $g_\mu = g_{xx} = 0$, $g_{\mu x} \neq 0$, and $g_{xxx} \neq 0$ there is a pitchfork bifurcation. For example, the map $x_{n+1} = x_n + \mu x - x^3$ has a pitchfork bifurcation at $\mu = 0$; a fixed point changes stability and a pair of fixed points with the same stability as each other is created as μ passes through zero.

More detail about the existence and stability of fixed points near non-hyperbolic equilibria is contained in the theorems.

The saddle-node bifurcation theorem for maps
Suppose that for $\mu = 0$ the origin is a non-hyperbolic fixed point for the map $x_{n+1} = g(x_n; \mu)$ and that $g_x(0; 0) = 1$, where $x \in \mathbf{R}$.

Then if f $g_\mu(0; 0) \neq 0$ and $g_{xx} \neq 0$ there is a continuous curve of fixed points in a neighbourhood of $(0, 0)$ in the $x\mu$ plane; the curve is tangent to the line $\mu = 0$ at $(0, 0)$.

If $g_\mu g_{xx} < 0$ (resp. $g_\mu g_{xx} > 0$) there are no fixed points near $(0, 0)$ for $\mu < 0$ (resp. $\mu > 0$) while for each positive (resp. negative) value of μ sufficiently close to zero there are two hyperbolic fixed points with x values near zero. The upper fixed point is stable and the lower one unstable if $g_{xx} < 0$. The stabilities are reversed if $g_{xx} > 0$.

The transcritical bifurcation theorem for maps
Suppose that for $\mu = 0$ the origin is a non-hyperbolic fixed point for the map $x_{n+1} = g(x_n; \mu)$ and that $g_x(0; 0) = 1$, where $x \in \mathbf{R}$.

If $g_\mu(0; 0) = 0$, $g_{xx} \neq 0$, and $g_{\mu x}^2 - g_{xx} g_{\mu\mu} > 0$ there are two continuous curves of fixed points in some neighbourhood of $(0, 0)$ in the $x\mu$ plane. The curves intersect transversally at $(0, 0)$.

For each sufficiently small, nonzero μ there are two hyperbolic stationary solutions near $x = 0$, with the upper one stable (resp. unstable) and the lower one unstable (resp. stable) if $g_{xx} < 0$ (resp. $g_{xx} > 0$).

The pitchfork bifurcation theorem for maps
Suppose that for $\mu = 0$ the origin is a non-hyperbolic fixed point for the map $x_{n+1} = g(x_n; \mu)$ and that $g_x(0; 0) = 1$, where $x \in \mathbf{R}$.

If $g_\mu(0; 0) = g_{xx} = 0$, $g_{\mu x} \neq 0$ and $g_{xxx} \neq 0$, there are two continuous curves of stationary solutions in some neighbourhood of $(0, 0)$ in the $x\mu$ plane. One of the curves passes through $(0, 0)$ transverse to the axis $\mu = 0$ while the other is tangent to $\mu = 0$ at $(0, 0)$.

If $g_{\mu x} g_{xxx} < 0$ there exist three stationary points near $x = 0$ if $\mu > 0$ (the outer pair being stable and the inner one unstable if $g_{xxx} < 0$), and one stationary point near $x = 0$ (stable if $g_{xxx} < 0$) if $\mu < 0$.

If $g_{\mu x} g_{xxx} > 0$ there exist three stationary points near $x = 0$ if $\mu < 0$ (the outer pair being stable and the inner one unstable if $g_{xxx} < 0$), and one stationary point near $x = 0$ (stable if $g_{xxx} < 0$) if $\mu > 0$. The stabilities are reversed if $g_{xxx} > 0$.