Maths 761 Lecture 1 Summary

Topics for today

Introduction to dynamical systems
Existence and uniqueness of solutions to ODEs
Phase space and flows
Special types of solutions

Reading for this lecture

READ THE STUDY GUIDE
Review material from Maths 260
Glendinning: Chapter 1
Strogatz: Chapters 1 and 2, Chapter 6 to end of §6.2.
Hirsch and Smale: See reference in notes below.

Reading for next lecture

Glendinning: §2.1-§2.4
Strogatz: §5

Introduction

In this course you will learn to use analytic and numerical techniques to discover the qualitative properties of solutions to nonlinear ordinary differential equations (as opposed to looking for exact solutions to the equations). For a particular system we might ask:

- What types of solution can occur?
- Which solutions are observable (i.e. 'attracting')?
- What happens if there is more than one attracting solution?
- How does the long term behaviour of solutions depend on initial conditions?
- How does the long term behaviour depend on parameters in the system?
- What happens if we perturb the system?

In this course you will learn methods to answer each of these questions.

For the purposes of this course we will mainly be interested in differential equations of the form

$$\frac{d\mathbf{x}}{dt} \equiv \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}; \mu) \tag{1}$$

where $t \in \mathbb{R}$ is the independent variable, $\mathbf{x} \in \mathbb{R}^n$ is a vector of dependent variables, $\mu \in \mathbb{R}^m$ is a vector of parameters and $\mathbf{f}$ is a vector-valued nonlinear function. This is a first order system of ordinary differential equations; recall that almost all ordinary differential equations can be written in this form. A solution to equation (1) is a vector $\mathbf{x}(t)$ which satisfies

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}(t, \mathbf{x}(t); \mu). \tag{2}$$

We will also look at discrete systems which can be written as maps:

$$\dot{\mathbf{x}}_{t+1} = \mathbf{g}(t, \mathbf{x}_t; \mu)$$

where now $t \in \mathbb{N}$ is a discrete independent variable.

Existence and Uniqueness of Solutions to ODEs

Existence and Uniqueness Theorem: Consider the first order system of equations

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n,\ t \in \mathbb{R},\ \mathbf{f} \in C^r,\ r \geq 1$ (that is, $\mathbf{f}$ is r times continuously differentiable) and $\mathbf{f}$ is defined on some set $U \subset \mathbb{R} \times \mathbb{R}^n$. If $(t_0, \mathbf{x}_0) \in U$ there is a unique solution to the system of equations (3) passing through the point $\mathbf{x} = \mathbf{x}_0$ when $t = t_0$ and defined for $|t - t_0|$ sufficiently small. Furthermore, the solution is a C^r function of t_0, $\mathbf{x}_0$ and t.

Proof: For details, see *"Differential Equations, Dynamical Systems, and Linear Algebra"* by M. Hirsch and S. Smale §8.3, §8.4.

The idea of the proof is to use Picard iteration to construct a sequence of functions that satisfy the initial conditions, and then show that the sequence converges to a solution of the differential equation. Technical steps in the proof include establishing that the Picard iterates live in a complete metric space, and that the sequence constructed is a Cauchy sequence. These two facts yield the result.

Phase space and flows

In a system of equations such as (1), each dependent variable can be identified with a coordinate axis; the space spanned by these axes is called the *phase space*. For example, for the system (1) the phase space is $\mathbb{R}^n$. Geometrically, a solution defines a one-dimensional curve in the phase space, parametrised by time. The equation

$$\dot{\mathbf{x}}|_{(t,x)=(t_0,x_0)} = \mathbf{f}(t_0, \mathbf{x_0})$$

defines the tangent vector to the solution curve through the point (t_0, x_0). If the differential equation is autonomous (that is, $\mathbf{f}$ is independent of t), the tangent vector is uniquely defined at each point in the phase space, and different solution curves or paths will not intersect transversely in the phase space.

A non-autonomous system in $\mathbb{R}^n$ can be rewritten as an autonomous system in $\mathbb{R}^{n+1}$ by defining a new independent variable, say $\theta = t$, and adding one more equation to the system, i.e., $\dot{\theta} = 1$.

The *flow* of a system of differential equations is, loosely speaking, the set of all solutions to the differential equation. Formally, we define the flow for a system of autonomous differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

with $\mathbf{x} \in \mathbb{R}^n$ to be the function $\phi(\mathbf{x}, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying the conditions

1. $\phi(\mathbf{x}_0, 0) = \mathbf{x}_0$;

2. for all t,
$$\frac{d}{dt}(\phi(\mathbf{x}, t)) = \mathbf{f}(\phi(\mathbf{x}, t)).$$

In other words, the flow function $\phi(\mathbf{x}_0, t)$ gives the position at time t of the solution that is at $\mathbf{x} = \mathbf{x}_0$ when $t = 0$. We talk about flows when we want to emphasize the dependence of solutions on initial conditions rather than the dependence of solutions on time.

Some special types of solution

Consider a system of autonomous ODEs $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with corresponding flow $\phi(\mathbf{x}, t)$.

1. A point $\mathbf{x_0}$ is a *stationary point* (or *fixed point* or *equilibrium solution*) of the flow if and only if $\phi(\mathbf{x_0}, t) = \mathbf{x_0}$ for all t. Equivalently, we say $\mathbf{x_0}$ is a stationary point if $\mathbf{f}(x_0) = 0$.

2. A point $\mathbf{x_0}$ is a *periodic point* of the flow, with (least) period T, if and only if $\phi(\mathbf{x_0}, t+T) = \phi(\mathbf{x_0}, t)$ for all t and $\phi(\mathbf{x_0}, t+s) \neq \phi(\mathbf{x_0}, t)$ for any s with $0 < s < T$. If $\mathbf{x_0}$ is a periodic point then the orbit $\{\phi(\mathbf{x_0}, t),\ t \in \mathbb{R}\}$ is a *periodic orbit* passing through $\mathbf{x_0}$.

3. An orbit $\phi(\mathbf{x_0}, t)$ is a *homoclinic orbit* of the flow if $\phi(\mathbf{x_0}, t) \to \mathbf{y}$ as $t \to \infty$ and as $t \to -\infty$, where $\mathbf{y}$ is a stationary point of the flow and $\mathbf{y} \neq \mathbf{x}_0$.

4. An orbit $\phi(\mathbf{x_0}, t)$ is a *heteroclinic orbit* of the flow if $\phi(\mathbf{x_0}, t) \to \mathbf{y}_0$ as $t \to \infty$ and $\phi(\mathbf{x_0}, t) \to \mathbf{y}_1$ as $t \to -\infty$, where $\mathbf{y}_0$ and $\mathbf{y}_1$ are distinct stationary points of the flow and $\mathbf{x_0} \neq \mathbf{y}_0$, $\mathbf{x_0} \neq \mathbf{y}_1$.

5. A set M is *invariant under the flow* if for all $\mathbf{x} \in M$, $\phi(\mathbf{x}, t) \in M$ for all t.
6. A set M is *forward invariant under the flow* if for all $\mathbf{x} \in M$, $\phi(\mathbf{x}, t) \in M$ for all $t > 0$.
7. A set M is *backward invariant under the flow* if for all $\mathbf{x} \in M$, $\phi(\mathbf{x}, t) \in M$ for all $t < 0$.

Recurrent behaviour and limit sets

In determining the long term behaviour of solutions to ODEs, we will be interested in two types of invariant set: nonwandering sets, which capture recurrent behaviour; and attracting sets, which capture observable behaviour.

1. A point $\mathbf{x}$ is *recurrent* in a flow if for any neighbourhood U of $\mathbf{x}$ there exists arbitrarily large t such that
$$\phi(\mathbf{x}, t) \in U,$$
Stationary and periodic points are recurrent, but there exist non-periodic recurrent points.

2. A point $\mathbf{x}$ is *nonwandering* in a flow if for any neighbourhood U of $\mathbf{x}$ there exists arbitrarily large t such that
$$\phi(U, t) \cap U \neq \emptyset,$$
i.e. a nonwandering point lies on or near a solution that returns arbitrarily close to the point. The set of all nonwandering points of a flow is the *nonwandering set* for that flow.

   All recurrent points are also nonwandering, but, for example, non-equilibrium points on most homoclinic orbits are nonwandering but not recurrent.

3. A point $\mathbf{y}$ is an *ω-limit point* of another point $\mathbf{x}$ if there exists a sequence of times t_n, with $t_n < t_{n+1}$ and $t_n \to \infty$ as $n \to \infty$, such that $\phi(\mathbf{x}, t_n) \to \mathbf{y}$ as $n \to \infty$. The *ω-limit set of point* $\mathbf{x}$, $\omega(\mathbf{x})$, is the set of all ω-limit points of $\mathbf{x}$. In other words, $\omega(\mathbf{x})$ is the set of points to which $\mathbf{x}$ tends in forward time. The *ω-limit set of a flow* is the union of all the ω-limit sets of individual points.

4. A point $\mathbf{y}$ is an *α-limit point* of another point $\mathbf{x}$ if there is a sequence of times t_n, with $t_n > t_{n+1}$ and $t_n \to -\infty$ as $n \to \infty$, such that $\phi(\mathbf{x}, t_n) \to \mathbf{y}$ as $n \to \infty$. The *α-limit set of point* $\mathbf{x}$, $\alpha(\mathbf{x})$, is the set of all α-limit points of $\mathbf{x}$. In other words, $\alpha(\mathbf{x})$ is the set of points to which $\mathbf{x}$ tends in backward time. The *α-limit set of a flow* is the union of all the α-limit sets of individual points.

The sets $\omega(\mathbf{x})$ and $\alpha(\mathbf{x})$ are invariant under the flow, i.e., if $\mathbf{y} \in \omega(\mathbf{x})$ then $\phi(\mathbf{y}, t) \in \omega(\mathbf{x})$ for all t, and similarly for $\alpha(\mathbf{x})$.