Maths 761 Lecture 2 Summary

Topics for today

Stability of solutions
Liapounov stability
Stationary solutions in linear systems
Hyperbolicity

Reading for this lecture

Glendinning: §2.1-§2.4
Strogatz: §2 and §5

Recommended problems

Glendinning: §2, 1, 3, 7, 9, §3: 2
Strogatz: 2.4.1-7, 5.2.3-10

Reading for next lecture

Glendinning: §3.3-§3.4, §4.1, §4.2, §4.5
Strogatz: §6.1-§6.3

Stability of solutions

A solution to a differential equation is said to be stable if, loosely speaking, nearby solutions do not go too far away as time increases. More precisely, we have the following types of stability:

1. A point $\mathbf{x}$ is *Liapounov (Lyapunov) stable* if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$ then $|\phi(\mathbf{x}, t) - \phi(\mathbf{y}, t)| < \epsilon$ for all $t \geq 0$.

2. A point $\mathbf{x}$ is *quasi-asymptotically stable* if and only if there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$ then $|\phi(\mathbf{x}, t) - \phi(\mathbf{y}, t)| \to 0$ as $t \to \infty$.

3. A point $\mathbf{x}$ is *asymptotically stable* if and only if it is both Liapounov stable and quasi-asymptotically stable.

4. A periodic orbit Γ is *Liapounov orbitally stable* if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $d(\mathbf{x}_0, \Gamma) < \delta$ then $d(\phi(\mathbf{x}_0, t), \Gamma) < \epsilon$ for all $t \geq 0$, where $d(\mathbf{y}, \Gamma)$ is the shortest distance between the point $\mathbf{y}$ and the curve Γ.

5. A periodic orbit, Γ is *quasi-asymptotically orbitally stable* if and only if there exists $\delta > 0$ such that if $d(\mathbf{x}_0, \Gamma) < \delta$ then $d(\phi(\mathbf{x}_0, t), \Gamma) \to 0$ as $t \to \infty$, where $d(\mathbf{y}, \Gamma)$ is the shortest distance between a point $\mathbf{y}$ and the curve Γ.

6. A periodic orbit is *asymptotically orbitally stable* if and only if it is Liapounov orbitally stable and quasi-asymptotically orbitally stable.

Stationary Solutions in Linear Systems

We can gain information about the nonlinear system by studying the linearized system near stationary solutions. Here we look at the different possible stability types for stationary solutions in linear systems.

Definition: A stationary solution of the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is *hyperbolic* if no eigenvalues of $\mathbf{A}$ have zero real part.

Theorem: If $\mathbf{A}$ is an $n \times n$ matrix with real entries, and all the eigenvalues of $\mathbf{A}$ have negative real part, then all solutions $\mathbf{x}(t)$ of the system of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ satisfy

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}.$$

See Glendinning §2.3 for a proof of this theorem.

When the conditions of the theorem hold, the stationary solution at the origin is called a *sink*. We get an analogous statement if all the eigenvalues of $\mathbf{A}$ have positive real part; in that case, all solutions $\mathbf{x}(t)$ of the system of equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ satisfy

$$\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$$

and the stationary solution at the origin is called a *source*.

In addition to sinks and sources as above, there is a third class of hyperbolic stationary solution in a linear system, i.e., *saddles*, which occur if the matrix $\mathbf{A}$ has at least one eigenvalue with positive real part and at least one eigenvalue with negative real part (and no eigenvalues with zero real part). These classes can be further subdivided: a sink or a source is a *node* if all the eigenvalues of $\mathbf{A}$ are real, and is a *focus* or *spiral* if at least one pair of eigenvalues is complex.

There are two ways in which a stationary solution can be nonhyperbolic: either by having a pair of purely imaginary eigenvalues, in which case the stationary solution is often called a *centre*; or by having a zero eigenvalue. Nonhyperbolic stationary solutions will be discussed further in the section on bifurcation theory.