Maths 761 Lecture 6 Summary

Topic for today

Dynamics in $\mathbb{R}^2$
Dynamics near nonhyperbolic equilibria

Reading for this lecture

Strogatz: §6.1-§6.4
Glendinning: §5.1-§5.4. Note that there is a major error in Glendinning, p. 108, where equation (5.12) should read

$$s = \frac{\mathrm{Tr}L \pm \sqrt{(\mathrm{Tr}L)^2 - 4\det L}}{2}.$$

Recommended problems

Glendinning: §5: 4, 5
Strogatz: 6.1.1-6.1.6, 6.4.1-6.4.3, 6.7.1

Reading for next lecture

Glendinning: §6.1, §6.5

Sketching planar phase portraits

When asked to sketch the (global) phase portrait for a planar system, you should do most or all of the following.

1. Find all stationary solutions and determine their types and stabilities.
2. Check for invariant axes or other obvious invariant lines/curves.
3. Draw locally valid phase portraits for all stationary solutions. You may wish to calculate the first few nonlinear terms in the power series expansions of W^s_{loc} and W^u_{loc}, but this is usually unnecessary.
4. Consider the possibilities for the relative positions of global stable and unstable manifolds, paying particular attention to the global behaviour of the stable manifold of any saddle-type stationary solution in the system.
5. Check for periodic orbits.
6. Use nullclines to find places where solutions have slope 0 or ∞;
7. Sketch a neat phase portrait that is consistent with everything you know about the system. Include solutions curves that illustrate all the different kinds of behaviour that can occur.
8. Check your results by investigating the equations numerically (e.g., with XPP). If some questions were not answered by the steps above (e.g., if you were unable to determine analytically whether or not there are periodic orbits) try to find the answer numerically. Note that a phase portrait drawn by or copied from XPP is

not an appropriate substitute for the steps outlined above. If you claim a certain type of behaviour is present in your system, you must back up your claim with analytic proof of the truth of the claim whenever possible.

Dynamics Near Nonhyperbolic Stationary Solutions
Near a nonhyperbolic stationary solution, the dynamics is not fully determined by looking at the linearised flow; the following techniques may be useful for calculating the behaviour of solutions near the non-hyperbolic equilibrium. Note that these techniques can be used in two or higher dimensions.

1. Look for invariant axes or lines passing through the stationary solution. If there is an appropriate invariant line, determine the direction of flow on that line.
2. Use nullclines to determine the direction of flow near the equilibrium.
3. Use results from Centre Manifold theory as discussed in later lectures.

The following systems are good examples for practising sketching planar phase portraits:

Example: Sketch the phase portrait for the system

$$\begin{aligned}\dot{x} &= x(1-x-y),\\ \dot{y} &= y(2-x-y).\end{aligned}$$

Example: Sketch the phase portrait for the system

$$\begin{aligned}\dot{x} &= x(3-x+2y),\\ \dot{y} &= -y(2-2x+y).\end{aligned}$$

Example: (Exam 2005)
Consider the following system of differential equations:

$$\begin{aligned}\dot{x} &= x(-3+x+y),\\ \dot{y} &= y(-5+x-y).\end{aligned}$$

1. Find all equilibrium solutions and determine their types. For each equilibrium you find, sketch a phase portrait showing the behaviour of solutions near that equilibrium.
2. Prove that there are no periodic solutions.
3. Carefully draw the global phase portrait. Use nullclines to help draw the picture accurately.