Maths 761 Lecture 9 Summary

Topic for today

Stable manifolds in maps

Reading for today's lecture

Glendinning: §11.3-5

Reading for next lecture

Glendinning: §6.1-6.4
Strogatz: §8.7

Stable and Unstable manifolds in maps

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function and consider the map $x_{n+1} = f(x_n)$. A fixed point x of the map is *hyperbolic* if and only if $|\lambda_i| \neq 1$ for all eigenvalues λ_i of the matrix $Df(x)$.

The local and global stable and unstable subspaces and manifolds for fixed points of nonlinear maps are defined in a similar way to the corresponding subspaces and manifolds in flows. Let U be some neighbourhood of a fixed point x of the map associated with the function f. We define:

1. $W^s_{loc}(x)$, the local stable manifold of x, is

$$W^s_{loc}(x) = \{y \in U \mid \lim_{n\to\infty} f^n(y) = x \text{ and } \quad f^i(y) \in U \ \forall i \geq 0\}$$

2. $W^u_{loc}(x)$, the local unstable manifold of x, is

$$W^u_{loc}(x) = \{y \in U \mid \lim_{n\to\infty} f^{-n}(y) = x \text{ and } \quad f^{-i}(y) \in U \ \forall i \geq 0\}$$

3. $W^s(x)$, the global stable manifold of x, is

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_{loc}(x))$$

4. $W^u(x)$, the global unstable manifold of x, is

$$W^u(x) = \bigcup_{n\geq 0} f^{n}(W^u_{loc}(x)).$$

There are analogues for maps to the Stable Manifold and Hartman-Grobman theorems:

Stable Manifold Theorem for Maps
Suppose that $x = 0$ is a hyperbolic fixed point for the map $x_{n+1} = f(x_n)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible C^r function, and E^s and E^u are the stable and unstable manifolds of the origin in the linear system $x_{n+1} = Df(0)x_n$. Then there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ for the nonlinear system, of the same dimension as E^s and E^u respectively. These manifolds are tangential to E^s and E^u, respectively, at the origin, and are C^r.

Hartman-Grobman Theorem for Maps
If $x = 0$ is a hyperbolic stationary point of $x_{n+1} = f(x_n)$, and f is as above, there is a continuous invertible map defined on some neighbourhood of $x = 0$ which takes orbits of the nonlinear map to those of the linear map $y_{n+1} = Df(0)y_n$.