MATHS750
Some 3-manifolds from sewing

1. **$\mathbb{S}^3$ from two balls.**

   Let $\mathbb{B}_1^3$ and B_2^3 be two disjoint copies of the 3-ball $\{x \in \mathbb{R}^3 \ / \ |x| \leq 1\}$. Let $h : \partial B_1^3 \to \partial \mathbb{B}_2^3$ be any homeomorphism. Define $\sim$ on $\mathbb{B}_1^3 \cup \mathbb{B}_2^3$ to be generated by $x \sim h(x)$ for each $x \in \mathbb{B}_1^3$. Topologise $\mathbb{B}_1^3 \cup \mathbb{B}_2^3$ as a topological sum. Then $\mathbb{B}_1^3 \cup \mathbb{B}_2^3/\sim$ is homeomorphic to $\mathbb{S}^3$. A standard notation is to write $\mathbb{B}_1^3 \cup_h \mathbb{B}_2^3$ for $\mathbb{B}_1^3 \cup \mathbb{B}_2^3/\sim$

   The same idea works for spheres of other dimensions.

2. **$\mathbb{S}^3$ from two solid tori.**

   We can embed $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^3$ as

   $$\left\{(x, y, z) \ / \ \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\}.$$

   Let V be a "solid torus," ie the region of $\mathbb{R}^3$ bounded by the embedded torus described above. On the surface of V we may choose two vital curves called the *latitude* or *meridian* and the *longitude*. Denote these curves by m and l respectively.

   Now let V_1 and V_2 be two disjoint copies of V. Suppose that $h : \partial V_1 \to \partial V_2$ is a homeomorphism. Concentrate on the images $h(m)$ and $h(l)$, and especially $h(m)$. The curve $h(m)$ will travel around the torus a few times along the longitude and a few times along the meridian, say p and q respectively. Then p and q are coprime integers. Just as in 1 define $\sim$ on $V_1 \cup V_2$ to be generated by $x \sim h(x)$. Set $L(p, q) = V_1 \cup_h V_2$. The space $L(p, q)$ is called the *lens space of type* (p, q).

   $$L(1, q) \approx \mathbb{S}^3; L(0, 1) \approx \mathbb{S}^2 \times \mathbb{S}^1; L(2, 1) \approx \mathbb{RP}^3.$$

   Some facts

   - $L(p, q) \approx L(p, -q) \approx L(-p, q) \approx L(-p, -q) \approx L(p, q + kp)$ for any integer k.
   - $L(p, q) \approx L(p, q')$ if $\pm qq' \equiv 1(\text{mod} p)$.

3. Let H be a *handlebody* of genus g: H itself is best thought of as a quotient space. Let H_1 and H_2 be two disjoint copies of H and let $h : \partial H_1 \to \partial H_2$ be a homeomorphism. Form the quotient space $H_1 \cup_h H_2$.

   The triple (H_1, H_2, h) is called the *Heegaard diagram* or *Heegaard splitting* of *genus* g of $H_1 \cup_h H_2$.

   An important theorem of 3-manifold theory says that every closed orientable 3-manifold has a Heegaard diagram.

4. Denote by $\mathbb{O}(3)$ the group of orthogonal transformations of the vector space $\mathbb{R}^3$, ie those of determinant ± 1. Denote by $\mathbb{SO}(3)$ the special orthogonal transformations, ie those of determinant 1. Both these spaces have natural topologies. Then $\mathbb{SO}(3)$ is $\mathbb{S}^3/(\text{antipodal identification})$. This space is called *real projective 3-space*, $\mathbb{RP}^3$.

5. Let $\Gamma \subset \mathbb{SO}(3)$ be the group of oriented symmetries of the regular dodecahedron: Γ is a group of order 60. Let $\mathbb{SO}(3)/\Gamma$ denote the usual group quotient; this inherits the quotient topology. This space is also a 3-manifold, *Poincaré's homology sphere*. It may also be thought of as the space of all regular dodecahedra in $\mathbb{S}^2$.