6 Set Theoretic Topology

6.1 The Zermelo-Fraenkel Axioms

Like most other mathematical structures studied in Pure Mathematics, Set Theory begins with a collection of axioms. There are various collections of axioms which somehow display the essentials of Set Theory. We shall state the Zermelo-Fraenkel Axioms. Once one states one's axiom system one usually then gives examples of objects which satisfy these axioms. While there are ways to do this we will not do so. All of the axioms will be given for completeness but we will not necessarily explain all of the terminology required to appreciate them fully. Actually we also neglect to define some important terms, for example $\in$.

The way that sets are built up reduces the distinction between a collection of sets and a collection of elements of sets. In our study of topology we have begun with a set X, whose elements were denoted by $x, y, \ldots$. Then we have taken a collection $\mathcal{T}$ of subsets of X satisfying certain properties to be a topology. So we have a hierarchy starting with elements $x, y, \ldots$, going on to sets of elements $X, Y, \ldots$ then to sets of sets of elements $\mathcal{X}, \mathcal{Y}, \ldots$, but we have not gone further. In Set Theory the hierarchy may go on for ever. Rather than looking for yet new ways of drawing symbols, Set Theorists have used the same kind of symbol for elements, sets, sets of sets, sets of sets of sets, ...

1. **Axiom of Extensionality** If two sets have identical elements then they are equal.
2. **Null Set Axiom** There is a set which has no members. This set is denoted by $\varnothing$.
3. **Power Set Axiom** If x is a set, there is a set that consists of all and only the subsets of x. This set is denoted $\mathcal{P}(x)$.
4. **Axiom of Union** If x is a set, there is a set whose members are precisely the members of the members of x. This set is denoted $\cup x$.
5. **Axiom of Infinity** There is a set x such that $\varnothing \in x$ and such that $\{a\} \in x$ whenever $a \in x$.
6. **Axiom of Replacement** Let $\varphi(u, v)$ be formula involving the two variables u and v such that for each set a there is a unique set b such that $\varphi(a, b)$. Let x be a set. Then there is a set y consisting of precisely those b such that $\varphi(a, b)$ for some $a \in x$.
7. **Axiom of Subset Selection** Let x be a set and $\varphi(v)$ a formula. Then there is a set consisting of those $a \in x$ for which $\varphi(a)$.
8. **Axiom of Foundation** If x is a nonempty set there is an $a \in x$ such that $a \cap x = \varnothing$.
9. **Axiom of Choice** Let $\mathcal{S}$ be a set of pairwise disjoint nonempty sets. Then there is a set S that consists precisely of one element from each member of $\mathcal{S}$.

Axioms 1-8 are usually called the *Zermelo-Fraenkel Axioms*. Collectively Axioms 1-9 are denoted by ZFC.

6.2 Ordinal and Cardinal Numbers

Definition 6.1 *Let n be a positive integer and x a set. An n-*ary relation *on x is a subset of the n-fold product of x, ie of $x \times x \times \ldots \times x$, with n factors. (This, of course, begs the question what do we mean by $x \times x \times \ldots \times x$? We are only going to use 2-ary relations, which are usually called* binary relations.*)*

If R is a binary relation on x then we usually rewrite aRb to mean $(a, b) \in R$. Our generic symbol for a binary relation will be $\leq$. If $a \leq b$ and $a \neq b$ then we write $a < b$.

Definition 6.2 *A* partial ordering *of a set x is a binary relation on x which is reflexive, anti-symmetric and transitive. So if $\leq$ is the relation then $a \leq a$, $a \leq b$ and $a \neq b \Longrightarrow b \not\leq a$, and $a \leq b$ and $b \leq c \Longrightarrow a \leq c$. A pair $(x, \leq)$ consisting of a set x and a partial ordering $\leq$ of x is called a* poset *(partially ordered set)*

A total ordering *of a set x is a partial ordering $\leq$ such that for each $a, b \in x$ either $a \leq b$ or $b \leq a$. The pair $(x, \leq)$ is then called a* toset *(totally ordered set).*

If $(x, \leq)$ is a poset and $y \subset x$ then $a \in y$ is a minimal element *provided that if $b \in y$ is such that $b \leq a$ then $b = a$.*

A well-ordering *of a set x is a total ordering $\leq$ such that every nonempty subset has a minimal element. The pair $(x, \leq)$ is then called a* woset *(well-ordered set).*

For a woset $(x, \leq)$ and $a \in x$ denote by x_a the segment *consisting of all $y \in x$ satisfying $y < a$.*

An ordinal *is a woset $(x, \leq)$ for which $x_a = a$ for each $a \in x$.*

At first sight this might seem to be a strange definition. However one can gradually build up exactly such a set using the ZFC axioms, starting with the Null Set Axiom and then using especially the Power Set and Subset Selection Axioms.

- Take $0 = \varnothing$, a woset with no need to define the well-ordering.
- Take $1 = \{0\}$; again there is no need to define $\leq$. Note that the only possibility for $a \in 1$ is $a = 0$, and $1_0 = \varnothing = 0$ as required.
- Take $2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$, and declare $0 < 1$. Then $2_0 = \varnothing = 0$ and $2_1 = \{0\} = 1$ as required.
- Take $3 = \{0, 1, 2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}$, and declare $0 < 1 < 2$. Then $3_0 = \varnothing = 0$, $3_1 = \{0\} = 1$ and $3_2 = \{0, 1\} = 2$ as required.

Continuing in this way leads to $n = \{0, 1, \ldots, n-1\}$ with the obvious definition of $\leq$. The first infinite ordinal is $\omega = \{0, 1, 2, \ldots, n, n+1, \ldots\}$, the second is $\omega + 1 = \{0, 1, 2, \ldots, n, n+1, \ldots, \omega\}$, etc.

Ordinals are usually denoted by lower case Greek letters. Each ordinal α has an immediate successor $\alpha + 1 = \alpha \cup \{\alpha\}$.

Definition 6.3 *An ordinal of the form $\alpha + 1$ for some ordinal α is called a* successor ordinal *while all other ordinals are called* limit ordinals.

Here are the names for some ordinals. $0, 1, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega 2, \ldots, \omega 2 + 2002, \ldots, \omega 3, \ldots, \ldots, \omega^2, \ldots, p(\omega), \ldots, \omega^\omega, \ldots, \ldots$. The expression $p(x)$ is a polynomial in x with non-negative integer coefficients, and all possibilities are there. This implies that there are concepts of addition, multiplication and exponentiation for ordinal numbers. Already addition of 1 (and hence of any finite ordinal) has been discussed. I will not discuss precisely how to define the other operations but will note that the definitions of addition and multiplication give operations which are not commutative. This is why we write $\omega 2$ rather than 2ω.

There are many results about ordinals. For example a set is an ordinal if and only if it is transitive and totally ordered by $\in$, hence $\alpha \leq \beta$ if and only if $\alpha \subset \beta$; the union of any set of ordinals is an ordinal. (Of course, $\in$ is *transitive* means that if $x \in y \in z$ then $x \in z$.) There is also a prime number theory for ordinals.

Definition 6.4 *Let λ be a limit ordinal and let $\langle \alpha_\xi \ / \ \xi < \lambda \rangle$ be a λ-sequence of ordinals. Write $\alpha = \lim_{\xi<\lambda} \alpha_\xi$ if and only if*

$$\text{for each } \beta < \alpha \text{ there is } \xi < \lambda \text{ such that for all } \zeta \text{ with } \xi < \zeta < \lambda, \quad \beta < \alpha_\zeta \leq \alpha.$$

If such α exists then it is unique and is the limit *of the sequence.*

We are mostly interested in ω-sequences.

Lemma 6.5 *Let λ be a limit ordinal and $\langle \alpha_\xi \ / \ \xi < \lambda \rangle$ an increasing λ-sequence of ordinals. Then $\langle \alpha_\xi \rangle$ has limit $\cup_{\xi<\lambda} \alpha_\xi$.* ∎

This definition of limit is closely tied to the interval topology on any ordinal α, which has a base the collection $\{(\beta, \gamma) \ / \ 0 \leq \beta < \gamma < \alpha\} \cup \{[0, \gamma) \ / \ 0 < \gamma < \alpha\}$. This is the natural topology to be imposed on an ordinal and whenever we require a topology on an ordinal it will be assumed that this is the topology used. In this topology successor ordinals and 0 are isolated while arbitrary neighbourhoods of any other limit ordinal contain the tails of all sequences converging to that ordinal.

Recall that two finite sets have the same number of elements if and only if there is a bijection between them.

Definition 6.6 *For any set X, the* cardinality *of X, denoted by $|X|$, is the least ordinal α for which there is a bijection between X and α. Not all ordinals are the cardinality of a set: for example $\omega + 1$, $\omega 2$. A* cardinal number *is an ordinal α such that for no $\beta < \alpha$ is there a bijection between α and β. A set X is* countable *provided that there is a bijection between X and α for some $\alpha \leq \omega$: if $\alpha < \omega$ then X is* finite *while if $\alpha = \omega$ then X is* countably infinite. *If X is not countable then X is called* uncountable.

Lemma 6.7 *Let $\lambda > 0$ be a countable limit ordinal. Then there is a strictly increasing sequence $\langle \alpha_n \rangle$ such that $\lim_{n\to\omega} \alpha_n = \lambda$.*

Proof. As λ is countably infinite, there is a bijection $\varphi : \omega \to \lambda$. Let $\alpha_0 = \varphi(0)$. Given α_n, say $m_n \in \omega$ is that integer for which $\alpha_n = \varphi(m_n)$. Note that, because λ is a limit ordinal, the set

$$\{m \in \omega \ / \ m > m_n \text{ and } \varphi(m) > \alpha_n\}$$

is nonempty so has a least member: call that least member m_{n+1}. Set $\alpha_{n+1} = \varphi(m_{n+1})$. Then $\langle \alpha_n \rangle$ is increasing. Further $\lim_{n\to\omega} \alpha_n = \lambda$. ∎

We are especially interested in the smallest uncountable ordinal: it is denoted by ω_1. Note that ω_1 is precisely the set of all countable ordinals.

Actually one can index naturally all infinite cardinal numbers by the ordinal numbers as ω_α and we have just declared the case $\alpha = 1$. The case $\alpha = 0$ is usually abbreviated (as we have) to ω. [Aside, never to be used again: evidently for every α we have $\omega_\alpha \geq \alpha$, and it seems as though ω_α increases at a much more rapid rate than does α. However define the sequence $\langle \alpha_n \rangle$ by $\alpha_0 = 0$ and given α_n let $\alpha_{n+1} = \omega_{\alpha_n}$. As an increasing sequence, $\langle \alpha_n \rangle$ converges; let $\lim_{n<\omega} \alpha_n = \alpha$. Then $\alpha = \lim_{n<\omega} \alpha_{n+1} = \lim_{n<\omega} \omega_{\alpha_n}$ from which we can deduce that $\omega_\alpha = \alpha$!]

It is common practice to use the Hebrew letter $\aleph$ to denote cardinal numbers. So $\aleph_0 = |\omega|$ and $\aleph_\alpha = |\omega_\alpha|$.

Lemma 6.8 *For any $\alpha < \omega_1$ there is a continuous injection $f : [0, \alpha] \to [0, \infty)$ such that $\beta \leq \gamma$ if and only if $f(\beta) \leq f(\gamma)$, for each $\beta, \gamma < \alpha$ (ie f is* order-preserving*).*

Proof. Suppose not. Let α be the least ordinal for which the conclusion fails. We consider two cases.

1. Suppose that α is a successor ordinal, say $\alpha = \beta + 1$. Then there is a continuous order-preserving injection $g : [0, \beta] \to [0, \infty)$. Let $h : [0, \infty) \to [0, 1)$ be a continuous order-preserving bijection. Define $f : [0, \alpha] \to [0, \infty)$ by letting $f(\gamma) = hg(\gamma)$ if $\gamma < \alpha$ and $f(\alpha) = 1$.

2. Suppose that α is a limit ordinal. Clearly $\alpha > 0$. Thus by Lemma 6.7 there is a strictly increasing sequence $\langle \alpha_n \rangle$ such that $\lim_{n<\omega} \alpha_n = \alpha$. As $\alpha_n < \alpha$ for each n it follows that there is a continuous order-preserving injection $f_n : [0, \alpha_n] \to [0, \infty)$. Inductively construct continuous order-preserving bijections $h_n : [0, \infty) \to [0, 1)$ so that $h_{n+1} f_{n+1}(\alpha_n) = h_n f_n(\alpha_n)$. Define $f : [0, \alpha] \to [0, \infty)$ by letting $f(\gamma) = h_{n+1} f_{n+1}(\gamma)$ if $\gamma \in [\alpha_n, \alpha_{n+1}]$ and $f(\alpha) = \lim_{n<\omega} h_n f_n(\alpha_n)$. ∎

Theorem 6.9 (Pressing Down Lemma) *Let $f : \omega_1 \to \omega_1$ satisfy $f(\alpha) < \alpha$ for each $\alpha > 0$ (such a function is called* regressive*). Then there is $\beta < \omega_1$ such that for each $\gamma < \omega_1$ there is δ with $\gamma < \delta < \omega_1$ and $f(\delta) = \beta$.*

Proof. Suppose not. Define a sequence $\langle \alpha_n \rangle$ by $\alpha_0 = 0$. Given α_n there is $\alpha_{n+1} > \alpha_n$ such that for each $\delta > \alpha_{n+1}$ we have $f(\delta) > \alpha_n$. Let $\alpha = \lim_{n<\omega} \alpha_n$.

As $\alpha > \alpha_{n+1}$ for each n we have $f(\alpha) > \alpha_n$ for each n so that $f(\alpha) \geq \alpha$, which contradicts the assumption that f is regressive. ∎

Lemma 6.10 *For any set x we have $|x| < |\mathcal{P}(x)|$.*

Proof. Suppose $f : x \to \mathcal{P}(x)$ is any function. It suffices to show that f cannot be surjective, ie find some subset y of x which is not $f(z)$ for any $z \in x$. Let $y = \{w \in x \ / \ w \notin f(w)\}$.

Suppose that $z \in x$ is such that $f(z) = y$. Either $z \in f(z)$ or $z \notin f(z)$. In the former case z does not satisfy the condition for belonging to y so $z \notin y$ and hence $z \notin f(z)$ as $y = f(z)$, so the former case cannot arise. In the latter case z does satisfy the condition for belonging to y so also $z \in f(z)$ as $y = f(z)$, so the latter case cannot arise either. ∎

6.3 The Long Line

Before describing the long line we consider a way of describing the real line. The real line may be constructed by gluing together two copies of the closed ray $[0, \infty)$. The gluing takes place at the 0 end of each by identifying these two 0's. More precisely $\mathbb{R}$ is the quotient space obtained from $[0, \infty) \times \{+, -\}$ by declaring $(0, +) \sim (0, -)$ but no other pairs of distinct points are $\sim$-equivalent. So now we have to construct $[0, \infty)$, and it is $\omega \times [0, 1)$ with the lexicographic order topology. Intuitively what we have done is take $|\omega|$ copies of $[0, 1)$ and laid them out one after the other so that the pair $(n, t) \in \omega \times [0, 1)$ corresponds to the non-negative real number $n + t$. The topology ensures that $\lim_{t \to 1}(n + t) = n + 1$.

Definition 6.11 *The* closed long ray*, denoted $\mathbb{L}_+$, is $\omega_1 \times [0, 1)$ with the lexicographic order topology. The* long line *is obtained from $\mathbb{L}_+ \cup \mathbb{L}_-$ by identifying the points $(0, 0)$ from each, where $\mathbb{L}_-$ is just a second, disjoint, copy of $\mathbb{L}_+$. The* open long ray*, $\mathbb{L}_o = \mathbb{L}_+ - \{(0, 0)\}$.*

What we have done now is take $|\omega_1|$ copies of $[0, 1)$ and laid them out one after the other. It is natural to think of the pair $(\alpha, t) \in \omega_1 \times [0, 1)$ as corresponding to a 'number' $\alpha + t$ (so

$\alpha + 0$ is just abbreviated to α). The topology ensures that $\lim_{t\to 1}(\alpha + t) = \alpha + 1$, where, of course, $\alpha + 1$ is the ordinal immediately succeeding α as already discussed. Often $\mathbb{L}_+$ is called the long line but that is really a misnomer. For any $x \in \mathbb{L}_+$ (resp. $\mathbb{L}_-$) denote the corresponding point of $\mathbb{L}_-$ (resp. $\mathbb{L}_+$) by $-x$: of course $-(-x) = x$. Note that there is a natural way to write every member of $\mathbb{L}$ in the form $\pm(\lambda + t)$, where λ is a limit ordinal and t a nonnegative real number. Just as the lexicographic order on $[0, \infty)$ induces the usual order on $\mathbb{R}$, so there is a standard order on $\mathbb{L}$, viz given $a, b \in \mathbb{L}$, declare $a \leq b$ provided either $a, b \in \mathbb{L}_+$ and $a \leq b$ in the lexicographic order on $\omega_1 \times [0, 1)$, or $a \in \mathbb{L}_-$ and $b \in \mathbb{L}_+$, or $a, b \in \mathbb{L}_-$ and $-b \leq -a$.

Proposition 6.12 *Let $a, b \in \mathbb{L}$ with $a < b$. Then there is an increasing homeomorphism $h : [a, b] \to [0, 1]$.*

Proof. We only consider the case where $a = 0$, and assume that $b \in \omega_1$, so write $b = \beta$. By Lemma 6.8 $[0, \beta] \cap \omega_1$ order-preserving embeds in $[0, \infty)$ hence $[0, 1]$: let $h : [0, \beta] \cap \omega_1 \to [0, 1]$ be an order-preserving embedding. For each $x \in [0, \beta]$ there is $\alpha \in \beta$ and $t \in [0, 1]$ such that $x = \alpha + t$. Set $h(x) = h(\alpha) + t(h(\alpha + 1) - h(\alpha))$. ∎

Corollary 6.13 *The long line $\mathbb{L}$ is path-connected, hence connected.*

Corollary 6.14 *The long line $\mathbb{L}$ is Hausdorff.*

Corollary 6.15 *The long line $\mathbb{L}$ and long ray $\mathbb{L}_o$ are 1-manifolds.*

Corollary 6.16 *The long line $\mathbb{L}$ is Tychonoff.*

Lemma 6.17 *Every increasing sequence $\langle x_n \rangle$ in $\mathbb{L}$ converges.*

Proof. Let $\langle x_n \rangle$ be an increasing sequence in $\mathbb{L}$. We will assume that $x_n \geq 0$ for each n. We may write uniquely $x_n = \lambda_n + t_n$, where λ_n is a limit ordinal and t_n is a nonnegative real number. There are two cases.

- for some n_0 and some limit ordinal λ we have $\lambda_n = \lambda$ for each $n > n_0$. Then either $\lim_{n\to\omega} t_n = t$ for some $t \in [0, \infty)$ or $\lim_{n\to\omega} t_n = \infty$. In the first subcase $\lim_{n\to\omega} x_n = \lambda + t$ while in the second $\lim_{n\to\omega} x_n$ is $\lambda + \omega$, the least limit ordinal greater than λ.

- for each $n \in \omega$ there is $m \in \omega$ with $m > n$ such that $\lambda_m > \lambda_n$. By Lemma 6.5, $\lim_{n\to\omega} \lambda_n$ converges, necessarily to a limit ordinal, say λ. Then $\lim_{n\to\omega} x_n = \lambda$. ∎

Corollary 6.18 *The long line $\mathbb{L}$ is sequentially compact.*

Proof. Let $\langle x_n \rangle$ be a sequence. Then $\langle x_n \rangle$ has a subsequence which is either increasing or decreasing. By Lemma 6.17 this subsequence converges. ∎

Proposition 6.19 *The long line $\mathbb{L}$ is normal.*

Proof. First we show that $\mathbb{L}_+$ is normal. Suppose that $A, B \subset \mathbb{L}_+$ are disjoint closed sets. Then at least one of A and B is bounded. Indeed, otherwise we can construct an increasing sequence $\langle x_n \rangle$ in $\mathbb{L}_+$ so that $x_n \in A$ for each even index $n \in \omega$ and $x_n \in B$ for each odd index $n \in \omega$. By Lemma 6.17 the sequence $\langle x_n \rangle$ converges, say to $x \in \mathbb{L}$. But then $x \in A \cap B$ as A and B are closed and $\langle x_n \rangle$ has subsequences lying entirely in A and in B.

Assume that A is bounded, say $\alpha \in \mathbb{L}$ is such that $A \subset [0, \alpha]$. Consider the disjoint closed subsets A and $B \cap [0, \alpha+1]$ of $[0, \alpha+1]$. By Proposition 6.12, $[0, \alpha+1]$ is normal so has disjoint open subsets U and V' such that $A \subset U \subset [0, \alpha+1)$ and $B \cap [0, \alpha+1] \subset V'$. Let $V = V' \cup (\alpha+1, \omega_1)$. Then U and V are disjoint open subsets of $\mathbb{L}_+$ containing A and B respectively. Thus $\mathbb{L}_+$ is normal.

To see that $\mathbb{L}$ is normal, follow the same sort of idea. Given disjoint closed subsets $A, B \subset \mathbb{L}$, find $\alpha \in \omega_1$ so that one of A and B is bounded above by α and one of these sets is bounded below by $-\alpha$. Initially separate $A \cap [-(\alpha+1), \alpha+1]$ and $B \cap [-(\alpha+1), \alpha+1]$ then extend the separating open sets over $(-\omega_1, -(\alpha+1))$ and $(\alpha+1, \omega_1)$. ∎

Theorem 6.20 *The long line $\mathbb{L}$ is not homeomorphic to the real line $\mathbb{R}$.*

Proof. Suppose that there is a homeomorphism $h : \mathbb{R} \to \mathbb{L}$. Either h is increasing or it is decreasing. Indeed either $h(0) < h(1)$ or $h(0) > h(1)$: we show that in the former case h is increasing. Suppose $a, b \in \mathbb{R}$ with $a < b$. Choose an interval $[c, d] \subset \mathbb{L}$ containing the points $h(0), h(1), h(a)$ and $h(b)$. By Proposition 6.12 there is an increasing homeomorphism $g : [c, d] \to [0, 1]$. Then $gh : h^{-1}([c, d]) \to [0, 1]$ is a homeomorphism so must be increasing as $gh(0) < gh(1)$. In particular $gh(a) < gh(b)$ and hence $h(a) < h(b)$.

In the case that h is decreasing we may follow it by a homeomorphism which changes sign to get an increasing homeomorphism $\mathbb{R} \to \mathbb{L}$. Thus we may assume that h is increasing. For each $n \in \omega$ let $x_n = h(n)$. Then $\langle x_n \rangle$ is an increasing sequence so by the observation above $\langle x_n \rangle$ converges, say to x. As h^{-1} is continuous, we must have that the sequence $\langle h^{-1}(x_n) \rangle$ converges to $h^{-1}(x)$, but this sequence is $\langle n \rangle$ which clearly does not converge in $\mathbb{R}$. ∎

Proposition 6.21 *The long line $\mathbb{L}$ is not Lindelöf, hence not compact.*

Proof. The collection $\{(-\alpha, \alpha) \;/\; \alpha \in \omega_1\}$ is an open cover of $\mathbb{L}$ having no countable subcover. ∎

Corollary 6.22 *The long line $\mathbb{L}$ is not paracompact (hence not metrisable), second countable or separable.* ∎

6.4 The Continuum Hypothesis, Martin's Axiom

In this section we consider two axioms which are independent of the ZFC axioms in the sense that it has been shown that if there are models of Set Theory which satisfy ZFC then there are models of ZFC in which the extra axiom is true and there are models in which the extra axiom is false.

As noted in Lemma 6.10 the cardinality of the power set of any set is always bigger than the cardinality of the set. One might naturally ask whether $|\mathcal{P}(\omega)|$ equals $\aleph_1$ or is strictly greater. This is addressed by the Continuum Hypothesis. If one explores more precisely what is meant by cardinal exponentiation then it is not too hard to show that $|\mathcal{P}(\lambda)| = 2^\lambda$ for any cardinal λ. It is also not too hard to show, using binary expansions, that $|[0, 1]| = 2^{\aleph_0}$ so, as there is a bijection between $[0, 1]$ and $\mathbb{R}$, it follows that $|\mathcal{P}(\omega)| = |\mathbb{R}|$. Because of its topology, $\mathbb{R}$ is frequently referred to as the continuum. So we have:

Continuum Hypothesis (CH) $|\mathbb{R}| = \aleph_1$.

CH was first posed as a question by Cantor over a century ago. It was the first of the 23 major mathematical problems posed for the 20th century by David Hilbert in 1900: see, for example, http://mathworld.wolfram.com/HilbertsProblems.html for a list of all of those problems. Contrary to expectations, it was not possible to prove (or disprove!) the statement $|\mathbb{R}| = \aleph_1$. In the 1930's Gödel showed that CH is consistent with ZFC and in the 1960s Cohen showed that ¬CH (the negation of CH) is also consistent with ZFC.

One of the big uses of CH is in constructions involving transfinite induction. Transfinite induction is like the familiar induction except that now the induction parameter may be any ordinal up to some particular ordinal, rather than up to ω (which is just the finite ordinals, ie 0 and the positive integers). The procedure is almost exactly the same. Induction begins at step 0. Given that the proof or construction has reached step α some procedure is used to show that it gets to step $\alpha + 1$. Now, however, we need also show that for any limit ordinal λ, if the induction has been shown up to step α for any $\alpha < \lambda$ then it also gets to step λ. As an example of this sort of thing we revisit the long line.

Example 6.23 *An alternative construction of the closed long ray* $\mathbb{L}_+$.

Let $D = [0, 1)$ and for each ordinal $\alpha < \omega_1$ let $I_\alpha \cong [0, 1)$, all topologically distinct. Set $M_\alpha = D \cup (\bigcup_{\beta<\alpha} I_\beta)$ and $M = \bigcup_{\alpha<\omega_1} M_\alpha$. Then $\mathbb{L}_+ = M$. When we constructed $\mathbb{L}_+$ before we specified the underlying set more elegantly.

Inductively topologise M_α by $\mathcal{T}_\alpha$ so that:

1. $(M_\alpha, \mathcal{T}_\alpha) \cong (D, \text{usual})$;
2. for each $\beta < \alpha$, $(M_\beta, \mathcal{T}_\beta)$ is an open subspace of $(M_\alpha, \mathcal{T}_\alpha)$.

$\mathcal{T}_0$ is the usual topology on $M_0 = [0, 1)$.

For α a limit ordinal, $\mathcal{T}_\alpha$ has $\bigcup_{\beta<\alpha} \mathcal{T}_\beta$ as basis.

For $\alpha = \beta + 1$, $\mathcal{T}_\alpha$ is the topology sticking $0 \in I_\beta (= M_\alpha - M_\beta)$ onto the 1 end of $M_\beta \cong [0, 1)$.

The topology $\mathcal{T}$ on M has basis $\bigcup_{\alpha<\omega_1} \mathcal{T}_\alpha$. ■

Often an inductive construction on $\mathbb{R}$ is such that at any stage of the induction the process may interfere with previous stages but in such a way that interference with only countably many steps does not disrupt the induction while interference with uncountably many may. It is in these circumstances that CH may be particularly valuable because the underlying set on which the induction is carried out may be $\mathbb{R}$ or a set with the same cardinality. Under CH, any set having the cardinality of $\mathbb{R}$ may be well-ordered by ω_1; more precisely there is a bijection $f : \mathbb{R} \to \omega_1$. In a weak way the construction in Example 6.23 displays this feature: Proposition 6.12 shows that our construction preserves the topological nature of the space being constructed as each M_α is homeomorphic to $[0, 1)$ but the final outcome of the construction is not (so if the construction continued beyond ω_1 then the space M_α would not be homeomorphic to $[0, 1)$).

There are several versions of the next axiom but we begin with the most topological. First a definition. A space has the *countable chain condition* (abbreviated ccc) provided that every pairwise disjoint family of open sets is countable.

Martin's Axiom (MA) *In every compact, ccc, Hausdorff space the intersection of fewer than* $2^{\aleph_0}$ *dense open sets is dense.*

Recall that a space X is a *Baire space* provided that the intersection of any countable collection of dense open subsets of X is dense in X. One version of the Baire category theorem tells us that every locally compact Hausdorff space is a Baire space. From this it follows that CH⇒MA.

MA is independent of ZFC in the same sense as CH is. Thus we might expect three possible kinds of ZFC Set Theories in which combinations of CH and MA appear: ZFC+CH, ZFC+MA+¬CH and ZFC+¬MA. All three possibilities do occur. Usually in applications other

equivalent versions of MA are used.
Martin's Axiom (MA) Let $(X, \leq)$ be a poset which has the ccc and let $\mathcal{D}$ be a collection of fewer than $2^{\aleph_0}$ dense subsets of X. Then there is a subset $Y \subset X$ such that:

- for each $x, y \in Y$ there is $z \in Y$ such that $z \leq x$ and $z \leq y$;
- for each $x \in X$ and $y \in Y$, if $x > y$ then $x \in Y$;
- for each $D \in \mathcal{D}$ we have $D \cap Y \neq \varnothing$.

Let $(X, \leq)$ be a poset. Elements $x, y \in X$ are *compatible* if there is some $z \in X$ such that $z \leq x$ and $z \leq y$. $(X, \leq)$ has the *countable chain condition*, abbreviated ccc, provided that there is no uncountable pairwise incompatible subset. A subset $D \subset X$ is *dense* if for each $x \in X$ there is $d \in D$ such that $d \leq x$.