4 COMPACTNESS AXIOMS

Definition 4.1 *Let X be a set and $A \subset X$. A* cover *of A is a family of subsets of X whose union contains A. A* subcover *of a given cover is a subfamily which is also a cover. A* refinement *of a cover $\mathcal{C}$ is another cover $\mathcal{D}$ so that for each $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $D \subset C$. A family of subsets of X is said to have the* finite intersection property (fip) *iff every finite subfamily has a non-empty intersection.*

Now suppose that X is a topological space. An open cover *of A is a cover consisting of open subsets of X.* Local *and* point finiteness *of covers are obvious extensions of Definition 2.23*

A space X is compact *iff every open cover of X has a finite subcover. A subset of X is* compact *iff it is compact as a subspace iff every cover of the subset by open subsets of X has a finite subcover.*

In one sense compactness is a generalisation of finiteness; every finite space is compact. If the topology on a space has only finitely members or has a finite basis then the space is also compact. Using the completeness axiom on the reals, it is readily shown that any closed bounded interval of $\mathbb{R}$ is compact; indeed this generalises to the Heine-Borel theorem which says that any subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Theorem 4.2 *Let X be a topological space. The following are equivalent:*

(i) X is compact;

(ii) if $\mathcal{C}$ is any family of closed subsets of X having the fip then $\cap_{C\in\mathcal{C}} C \neq \varnothing$;

(iii) if $\mathcal{F}$ is any family of subsets of X having the fip then $\cap_{F\in\mathcal{F}} \bar{F} \neq \varnothing$.

Proof. (i)⇒(ii): let $\mathcal{C}$ be a family of closed subsets of X such that $\cap_{C\in\mathcal{C}} C = \varnothing$. Consider $\mathcal{D} = \{X - C \;/\; C \in \mathcal{C}\}$. Then $\mathcal{D}$ is an open cover of X; let $\{X - C_1, \dots, X - C_n\}$ be a finite subcover. Then $\cap_{i=1}^n C_i = X - \cup_{i=1}^n (X - C_i) = X - X = \varnothing$, so $\mathcal{C}$ cannot have the fip.

(ii)⇒(iii): let $\mathcal{F}$ be any family of subsets of X having the fip. Then $\mathcal{C} = \{\bar{F} \;/\; F \in \mathcal{F}\}$ is a family of closed sets having the fip, so by (ii), $\cap_{F\in\mathcal{F}} \bar{F} = \cap_{C\in\mathcal{C}} C \neq \varnothing$.

(iii)⇒(i): let $\mathcal{C}$ be an open cover of X. Then $\mathcal{F} = \{X - C \;/\; C \in \mathcal{C}\}$ is a family of subsets of X with $\cap_{F\in\mathcal{F}} \bar{F} = \varnothing$. Thus $\mathcal{F}$ cannot have the fip, i.e. there are $C_1, \dots, C_n \in \mathcal{C}$ so that $\cap_{i=1}^n (X - C_i) = \varnothing$. Then $\{C_1, \dots, C_n\}$ is a finite subcover of $\mathcal{C}$. ■

Proposition 4.3 *Every closed subset of a compact space is compact.*

Proof. Let $\mathcal{C}$ be an open cover of the closed subset C of the compact space X. Then $\mathcal{C} \cup \{X - C\}$ is an open cover of X. Let $\mathcal{D}$ be a finite subfamily of $\mathcal{C} \cup \{X - C\}$ covering X. Then $\mathcal{D} - \{X - C\}$ is a finite subcover of $\mathcal{C}$. ■

Proposition 4.4 *The continuous image of a compact set is compact.*

Proof. Easy. ■

Proposition 4.5 *Every compact subset of a Hausdorff space is closed.*

Proof. Let C be a compact subset of the Hausdorff space X, and let $x \in X - C$. For each $y \in C$, we have $x \neq y$, so there are disjoint open sets U_y and V_y so that $x \in U_y$ and $y \in V_y$. Then $\{V_y \;/\; y \in C\}$ is an open cover of C: let $\{V_{y_1}, \dots, V_{y_n}\}$ be a finite subcover. Then $\cap_{i=1}^n U_{y_i}$ is an open set containing x and contained in $X - C$. Thus $X - C$ is open so C is closed. ■

Compacta need not be closed in arbitrary spaces. For example, any subset of an indiscrete space is compact.

Theorem 4.6 *Suppose that X is a compact space and Y is a Hausdorff space. Then every continuous bijection $f : X \to Y$ is a homeomorphism.*

Proof. We use criterion (v) of Theorem 1.19 for continuity of the inverse function $f^{-1} : Y \to X$. Let C be any closed subset of X. By Proposition 4.3, C is compact, so by Proposition 4.4, $f(C)$ is also compact, hence closed by Proposition 4.5. As $(f^{-1})^{-1}(C) = f(C)$, it follows that f^{-1} is continuous. ∎

One significant aspect of this result is that if $\mathcal{T}$ is a compact Hausdorff topology on a set X, then any topology $\mathcal{S} \subset \mathcal{T}$, with $\mathcal{S} \neq \mathcal{T}$is non-Hausdorff, and any topology $\mathcal{U} \supset \mathcal{T}$, with $\mathcal{U} \neq \mathcal{T}$ is non-compact; however, $\mathcal{S}$ is compact and $\mathcal{U}$ is Hausdorff. Loosely speaking, open sets abound in Hausdorff spaces but are rather scarce in compact spaces. We should therefore expect compact Hausdorff spaces to be rather special.

Theorem 4.7 *Every compact Hausdorff space is normal.*

Proof. Let A and B be disjoint closed subsets of the compact Hausdorff space X. Then A and B are compact. Let $x \in A$. For each $y \in B$, there are open sets S_y and T_y so that $x \in S_y$, $y \in T_y$, and $S_y \cap T_y = \varnothing$. As in Proposition 4.5, use compactness of B to obtain open sets U_x and V_x with $x \in U_x$, $B \subset V_x$, and $U_x \cap V_x = \varnothing$. Now use compactness of A to obtain open sets U and V so that $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$. ∎

Theorem 4.8 *Let X be a non-empty compact Hausdorff space in which every point is an accumulation point of X. Then X is uncountable.*

Proof. Suppose that $f : \mathbb{N} \to X$ is a function. It suffices to show that f is not surjective. Using induction on $n \in \mathbb{N} \cup \{0\}$, we construct non-empty open sets V_n so that if $n \geq 1$ then $V_n \subset V_{n-1}$ and $f(n) \notin \overline{V_n}$.

Set $V_0 = X$. Given V_{n-1}, choose a point $y \in V_{n-1}$ other than $f(n)$: if $f(n) \notin V_{n-1}$ this is easy; if $f(n) \in V_{n-1}$, then since $f(n)$ is an accumulation point of X, $V_{n-1} \cap (X - \{f(n)\}) \neq \varnothing$, by Proposition 1.17. Since X is Hausdorff, $f(n)$ and y have disjoint neighbourhoods, say U and V are disjoint open sets with $f(n) \in U$ and $y \in V$. Let $V_n = V \cap V_{n-1}$. Then V_n satisfies the inductive requirements.

Consider the family $\{\overline{V_n} \;/\; n = 1, 2, \ldots\}$. This has the fip, so by Theorem 4.2(iii), $\cap_{n=1}^{\infty} \overline{V_n} \neq \varnothing$, say $x \in \cap_{n=1}^{\infty} \overline{V_n}$. Then for each n, $x \neq f(n)$, since $x \in \overline{V_n}$ but $f(n) \notin \overline{V_n}$. Thus f cannot be surjective. ∎

Corollary 4.9 $\mathbb{R}$ *is uncountable.*

Proof. $[0, 1]$ (or any non-trivial closed interval) satisfies the conditions of Theorem 4.8 so is uncountable. Thus $\mathbb{R}$ is uncountable. ∎

There are many different proofs of the following theorem, all based on some form of the Axiom of Choice (to which Tychonoff's theorem is logically equivalent). The version we give appeals to Zorn's lemma. Criterion (iii) of Theorem 4.2 is used in this proof.

Theorem 4.10 (Tychonoff's theorem) *Let $\{X_\alpha \;/\; \alpha \in A\}$ be a family of compact spaces. Then ΠX_α is compact.*

Proof. Let $\mathcal{F}$ be any family of subsets of $X = \Pi X_\alpha$ having the fip. In order to show that $\cap_{F \in \mathcal{F}} \bar{F} \neq \varnothing$, i.e. to exhibit an element of $\cap_{F \in \mathcal{F}} \bar{F}$, the temptation may be to look at $\{\pi_\alpha(F) \;/\; F \in \mathcal{F}\}$ and appeal to compactness of X_α to find $x_\alpha \in \cap_{F \in \mathcal{F}} \overline{\pi_\alpha(F)}$. The problem with this approach is that we have too much freedom in our choice of x_α and we might choose the different points x_α (as α varies) in such a way that when we put them together to get a point x of X, this point x does not lie in $\cap_{F\in\mathcal{F}}\bar{F}$ even though $x_\alpha \in \cap_{F\in\mathcal{F}}\overline{\pi_\alpha(F)}$ for each $\alpha \in A$. To overcome this problem, we firstly enlarge $\mathcal{F}$ using Zorn's lemma so that the choice of x_a is forced on us.

Let $\Gamma = \{\mathcal{A} \ / \ \mathcal{A}$ is a family of subsets of X having the fip, and $\mathcal{F} \subset \mathcal{A}\}$. Order Γ by $\subset$. Then $(\Gamma, \subset)$ is a poset. Furthermore $(\Gamma, \subset)$ satisfies the conditions of Zorn's lemma, for suppose $\Delta \subset \Gamma$ is such that $(\Delta, \subset)$ is a non-empty toset and consider

$$\mathcal{G} = \cup\Delta = \{A \subset X \ / \ \text{there is } \mathcal{A} \in \Delta \text{ such that } A \in \mathcal{A}\}.$$

Then $\mathcal{G} \in \Gamma$, for clearly $\mathcal{F} \subset \mathcal{G}$, and if $A_1, \ldots, A_n \in \mathcal{G}$, then there exist $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \Delta$ with $A_i \in \mathcal{A}_i$. Since $(\Delta, \subset)$ is a toset, one of the families $\mathcal{A}_1, \ldots, \mathcal{A}_n$ contains all of the others, say it is $\mathcal{A}_n$. Then for any i, $A_i \in \mathcal{A}_n$, so, since $\mathcal{A}_n \in \Gamma$, $\cap_{i=1}^n A_i \neq \varnothing$. Thus $\mathcal{G} \in \Gamma$. Moreover, $\mathcal{G}$ is an upper bound for Δ. Thus $(\Gamma, \subset)$ satisfies the conditions of Zorn's lemma.

Now let $\mathcal{G}$ be a maximal element for Γ. Thus $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{G}$ has the fip. It is this family $\mathcal{G}$ which we use to choose the x_α.

Claim I. $\mathcal{G}$ is closed under finite intersections, for suppose that $G_1, G_2 \in \mathcal{G}$. Then $\mathcal{G} \cup \{G_1 \cap G_2\} \in \Gamma$, so by maximality of $\mathcal{G}$, $G_1 \cap G_2 \in \mathcal{G}$.

Now for each $\alpha \in A$, $\{\pi_\alpha(G) \ / \ G \in \mathcal{G}\}$ has the fip, so by compactness of X_α, $\cap_{G\in\mathcal{G}}\overline{\pi_\alpha(G)} \neq \varnothing$, say $x_\alpha \in \cap_{G\in\mathcal{G}}\overline{\pi_\alpha(G)}$. Amalgamating the points x_α, we get a point $x \in X$ so that for each $\alpha \in A$, $\pi_\alpha(x) = x_\alpha$. It remains to show that $x \in \cap_{G\in\mathcal{G}}\bar{G}$, since $\mathcal{F} \subset \mathcal{G}$, so that $\cap_{G\in\mathcal{G}}\bar{G} \subset \cap_{F\in\mathcal{F}}\bar{F}$.

Claim II. For each $\alpha \in A$, and each open $U_\alpha \subset X_\alpha$, so that $x_\alpha \in U_\alpha$, we have $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{G}$. In fact, $U_\alpha \cap \pi_\alpha(G) \neq \varnothing$ for each $G \in \mathcal{G}$, so $\pi_\alpha^{-1}(U_\alpha) \cap G \neq \varnothing$. Thus by maximality of $\mathcal{G}$ and Claim I, $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{G}$.

Claim III. For each open $U \subset X$ such that $x \in U$ and and each $G \in \mathcal{G}$, $U \cap G \neq \varnothing$. In fact, given U open in X with $x \in U$, we may assume that U is basic, say $U = \Pi U_\alpha$ where $U_\alpha = X_\alpha$ unless $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$. Then by claim II, $\pi_{\alpha_i}^{-1}(U_{\alpha_i}) \in G$, so by claim I, $U \in \mathcal{G}$. Thus for each $G \in \mathcal{G}$, $U \cap G \neq \varnothing$.

It follows immediately from claim III that $x \in \bar{G}$ for each $G \in \mathcal{G}$. ∎

Definition 4.11 *A space X is* locally compact *iff every point of X has a compact neighbourhood.*

Contrast this definition with that of local connectedness: in that definition it was required that the family of connected neighbourhoods forms a neighbourhood basis whereas here we require only that the family of compact neighbourhoods be non-empty. Thus we have two ways of localising a topological property: a strong method where the family of neighbourhoods satisfying the property forms a neighbourhood basis, and a weak method where it is merely required that there be at least one neighbourhood satisfying the property. There seems to be a certain amount of arbitrariness in deciding which method to use for a given topological property. In the weak method, a topological space which enjoys a property also enjoys it locally. Thus compact spaces are locally compact.

Proposition 4.12 *Let X be a locally compact regular space. Then at each point of X, the family of closed compact neighbourhoods forms a neighbourhood basis.*

Proof. Let $x \in X$ and let N be a neighbourhood of x. Let K be a compact neighbourhood of x. Then $K \cap N$ is a neighbourhood of x. Thus by Theorem 2.11, there is a closed neighbourhood C of x with $C \subset K \cap N$. As a closed subset of a compact set, C is also compact. ∎

Proposition 4.13 *Every locally compact Hausdorff space is Tychonoff.*

Proof. Let X be a locally compact Hausdorff space. Let $x \in X$ and C be a closed subset of X with $x \notin C$. Let K be a compact neighbourhood of x and let $A = (K - \text{int}K) \cup (K \cap C)$. Then A is a closed subset of the compact Hausdorff space K and $x \in K - A$. Thus by Theorems 4.7 and 2.20, there is a map $f_1 : K \to [0, 1]$ with $f_1(x) = 0$ and $f_1(A) = 1$. Define $f : X \to [0, 1]$ by $f|K = f_1$ and $f(X - \text{int}K) = 1$. By Proposition 4.5, K is closed, so by Theorem 1.22, f is continuous. Since $f(x) = 0$ and $f(C) = 1$, we deduce that X is Tychonoff. ∎

Note that locally compact Hausdorff spaces need not be normal, the tangent discs topology of Example 2.14 being a counterexample.

Definition 4.14 *A space X is* σ-compact *iff it is expressible as a countable union of compact subsets. A space X is* Lindelöf *iff every open cover of X has a countable subcover.*

Thus $\mathbb{R}^n$ with the usual topology is σ-compact. Every subspace of $\mathbb{R}^n$ is Lindelöf. Clearly every σ-compact space is Lindelöf, as is every second countable space. Each converse is false, eg the cocountable topology of section 1 Problem 3 is Lindelöf but neither σ-compact nor second countable.

Proposition 4.15 *Let X be a locally compact Lindelöf space. Then X is σ-compact.*

Proof. For each $x \in X$, let K_x be a compact neighbourhood of x. Then $\{\ \text{int}K_x\ /\ x \in X\}$ is an open cover of X. Let $\{\ \text{int}K_{x_n}\ /\ n = 1, 2, \ldots\}$ be a countable subcover. Then $X = \cup K_{x_n}$ is a union of countably many compact subsets. ∎

Proposition 4.16 *Every regular Lindelöf space is normal.*

Proof. The same proof as that of Theorem 3.10 works, except that we replace the (not necessarily countable) cover $\{U_x\ /\ x \in A\}$ by a countable subcover $\{U_n\ /\ n = 1, 2, \ldots\}$. ∎

Definition 4.17 *A space X is* paracompact *iff every open cover has a locally finite open refinement; and is* metacompact *iff every open cover has a point-finite open refinement.*

In contrast to compactness, we cannot say that the fewer the number of open sets the greater the likelihood of paracompactness. In fact discrete and indiscrete spaces are always paracompact.

Example 4.18 *The* long line *is a connected T_4 space which is not paracompact.*

Let $(A, <)$ be an uncountable well-ordered set in which each element has only countably many predecessors, cf Corollary 6 of the Appendix.

For each $a \in A$, let $I_a = \{a\} \times (0, 1)$, and let $\mathbb{L} = A \cup (\cup_{a \in A} I_a)$. Extend $<$ to $\mathbb{L}$ by specifying $x < y$ in each of the following cases:

- $x \in A$, $y \in I_a$, $x \leq a$;
- $x \in I_a$, $y \in A$, $a < y$;
- $x \in I_a$, $y \in I_b$, $a < b$;
- $x, y \in I_a$, say $x = (a, x')$ and $y = (a, y')$, and $x' < y'$.

Topologise $\mathbb{L}$ using the order topology, i.e. sets of the form $(x, y) = \{z \in \mathbb{L}\ /\ x < z < y\}$ together with those of the form $[0, y) = \{z \in \mathbb{L}\ /\ z < y\}$ form a basis for the topology. With this topology, $\mathbb{L}$ is called the long line (really it should be called the long ray). Basically, $\mathbb{L}$ is constructed from A much as $[0, \infty)$ is constructed from $\{0\} \cup \mathbb{N}$ by inserting an interval between a and its successor.

Sorgenfrey's square, Example 2.15, is an example of a non-paracompact product of two paracompact spaces.

Lemma 4.19 *Let X be a paracompact space and let A and B be a pair of disjoint subsets with B closed. Suppose that for each $x \in B$, there are open sets U_x and V_x so that $A \subset U_x$, $x \in V_x$ and $U_x \cap V_x = \varnothing$. Then there are open sets U and V with $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.*

Proof. The family $\{X - B\} \cup \{V_x \ / \ x \in B\}$ is an open cover of X. Let $\{W_\mu \ / \ \mu \in M\}$ be a locally finite open refinement. Let $M_1 = \{\mu \in M \ / \ \text{there is } x \in B \text{ with } W_\mu \subset V_x\}$. Then for each $\mu \in M_1$, $A \cap \overline{W_\mu} = \varnothing$. Further, $\cup\{\overline{W_\mu} \ / \ \mu \in M_1\} = \overline{\cup\{W_\mu \ / \ \mu \in M_1\}}$, cf Problem 13 below, so $U = X - \cup\{\overline{W_\mu} \ / \ \mu \in M_1\}$ is an open set containing A. The set $V = \cup\{W_\mu \ / \ \mu \in M_1\}$ is an open set containing B. Clearly $U \cap V = \varnothing$.

Theorem 4.20 *Every paracompact Hausdorff space is regular and every paracompact regular space is normal.*

Proof. Immediate consequence of Lemma 4.19. ∎

Theorem 4.21 *Let X be a paracompact Hausdorff space. Then every open cover of X has a subordinate partition of unity.*

Proof. Let $\mathcal{U} = \{U_\alpha \ / \ \alpha \in A\}$ be an open cover of X and let $\mathcal{V} = \{V_\beta \ / \ \beta \in B\}$ be a locally finite open refinement of $\mathcal{U}$. Choose $\varphi : B \to A$ to satisfy: for each $\beta \in B$, $V_\beta \subset U_{\varphi(\beta)}$. For each $\alpha \in A$, let $W_\alpha = \cup\{V_\beta \ / \ \varphi(\beta) = \alpha\}$. Then $\mathcal{W} = \{W_\alpha \ / \ \alpha \in A\}$ is also a locally finite open refinement of $\mathcal{U}$ ($\mathcal{W}$ is called a *precise refinement*). By Theorem 4.20, X is normal, so by Theorem 2.26, $\mathcal{W}$ has a subordinate partition of unity. This partition is also subordinate to $\mathcal{U}$.

∎

Theorem 4.22 *Every metrisable space is paracompact.*

Proof. Let (X, d) be a metric space and let $\{U_\alpha \ / \ \alpha \in A\}$ be an open cover of X. Well-order A by $<$.

For each $n \in \mathbb{N}$, and each $\alpha \in A$, define an open set $V_{\alpha,n}$ by induction on n by:

$$\begin{aligned} V_{\alpha,n} = \bigcup \Big\{ B\left(x; \frac{1}{2^n}\right) \ / \quad & x \notin U_\beta \text{ for each } \beta < \alpha; \quad B\left(x; \frac{3}{2^n}\right) \subset U_\alpha; \\ & \text{and } x \notin V_{\beta,j} \text{ for each } j < n, \text{ and each } \beta \in A \Big\}. \end{aligned}$$

(I) $\{V_{\alpha,n} \ / \ \alpha \in A, n \in \mathbb{N}\}$ covers X, for given $x \in X$, there is a least $\alpha \in A$ so that $x \in U_\alpha$. Choose n so large that $B\left(x; \frac{3}{2^n}\right) \subset U_\alpha$. Then either $x \in V_{\alpha,n}$ or there are $j < n$ and $\beta \in A$ so that $x \in V_{\beta,j}$.

(II) $\{V_{\alpha,n} \ / \ \alpha \in A, n \in \mathbb{N}\}$ refines $\{U_\alpha \ / \ \alpha \in A\}$: in fact $V_{\alpha,n} \subset U_\alpha$.

(III) $\{V_{\alpha,n} \ / \ \alpha \in A, n \in \mathbb{N}\}$ is locally finite, for suppose that $x \in X$. Let α be the least element of A for which there is $n \in \mathbb{N}$ so that $x \in V_{\alpha,n}$, and choose $m \in \mathbb{N}$ such that $B\left(x; \frac{1}{2^m}\right) \subset V_{\alpha,n}$. It is claimed that the neighbourhood $B\left(x; \frac{1}{2^{m+n}}\right)$ of x meets only finitely many of the sets $V_{\beta,l}$. We consider two cases:

(a) if $l \geq m + n$, then $B\left(x; \frac{1}{2^{m+n}}\right) \cap V_{\beta,l} = \varnothing$ for each β.

Indeed, if this were not the case, then there are $\beta \in A$ and $y \in X$ so that the following hold:

* $y \notin U_\gamma$ for each $\gamma < \beta$;
* $B\left(y; \dfrac{3}{2^l}\right) \subset U_\beta$;
* $y \notin V_{\gamma,k}$ for each $k < l$ and $\gamma \in A$; and
* $B\left(x; \dfrac{1}{2^{m+n}}\right) \cap B\left(y; \dfrac{1}{2^l}\right) \neq \varnothing$, say $z \in B\left(x; \dfrac{1}{2^{m+n}}\right) \cap B\left(y; \dfrac{1}{2^l}\right) \neq \varnothing$.

Thus $d(x,z) < \dfrac{1}{2^{m+n}}$ and $d(y,z) < \dfrac{1}{2^l}$. As $l \geq m+n$, by the third condition of the definition of $V_{\beta,l}$, $y \notin V_{\alpha,n}$: thus $y \notin B\left(x; \dfrac{1}{2^m}\right)$, so we have

$$\frac{1}{2^m} \leq d(x,y) \leq d(x,z) + d(y,z) < \frac{1}{2^{m+n}} + \frac{1}{2^l} \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}},$$

which is impossible, so (a) holds.

(b) if $l < m+n$, then $B\left(x; \dfrac{1}{2^{m+n}}\right) \cap V_{\beta,l} \neq \varnothing$ holds for at most one β.

To verify this, it is sufficient to show that if $y \in V_{\beta,l}$ and $z \in V_{\gamma,l}$ with $\beta < \gamma$ then $d(y,z) > \dfrac{1}{2^l}$ for then y and z cannot both belong to $B\left(x; \dfrac{1}{2^{m+n}}\right)$.

By the definition of $V_{\beta,l}$ and $V_{\gamma,l}$, there are u, v so that $y \in B\left(u; \dfrac{1}{2^l}\right) \subset V_{\beta,l}$ and $z \in B\left(v; \dfrac{1}{2^l}\right) \subset V_{\gamma,l}$; further, $B\left(u; \dfrac{3}{2^l}\right) \subset U_\beta$. By the definition of $V_{\beta,l}$ and $\beta < \gamma$, we have $v \notin U_\beta$, so $d(u,v) \geq \dfrac{3}{2^l}$. Thus

$$d(y,z) \geq d(u,v) - d(u,y) - d(v,z) > \frac{3}{2^l} - \frac{1}{2^l} - \frac{1}{2^l} = \frac{1}{2^l}.$$

This verifies (b).

Combining (a) and (b), we see that $\{V_{\alpha,n} \ / \ \alpha \in A, n \in \mathbb{N}\}$ is locally finite.

Thus X is paracompact. ■

Exercises

1. Let $f : X \to Y$ be a continuous surjection, X being compact and Y Hausdorff. Suppose that $g : Y \to Z$ is a function and that $gf : X \to Z$ is continuous. Prove that g is continuous.

2. Let A and B be disjoint compact subsets of the Hausdorff space X. Prove that there are open sets U and V so that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.

3. Let A and B be disjoint subsets of the Tychonoff space X, with A compact and B closed. Prove that there is a continuous function $f : X \to [0,1]$ so that $f(A) = 0$ and $f(B) = 1$.

4. Prove that all cubes are T_4.

5. Consider the product space $X \times Y$, where Y is compact. Suppose that $x_0 \in X$ and N is an open subset of $X \times Y$ containing $\{x_0\} \times Y$. Prove that there is an open set $U \subset X$ such that $\{x_0\} \times Y \subset U \times Y \subset N$.

6. Let $f : X \to Y$ be a function where Y is compact and Hausdorff. Prove that f is continuous iff the *graph* of f, $\Gamma(f) = \{(x, f(x)) \ / \ x \in X\}$, is closed in $X \times Y$. [Hint: if $\Gamma(f)$ is closed and N is a neighbourhood of $f(x_0)$, apply Problem 5 to the set $(X \times N) \cup (X \times Y - \Gamma(f))$.]

7. Let X and Y be compact Hausdorff spaces and $f, g : X \to Y$ be continuous functions. Show that there is an $x \in X$ with $f(x) = g(x)$ if and only if for each open cover $\mathcal{C}$ of Y there is an $x \in X$ and a $U \in \mathcal{C}$ with $f(x), g(x) \in U$.

8. Let X be paracompact and Y compact. Prove that $X \times Y$ is paracompact.

9. Let A be a dense subset of a space X and $f : A \to Y$ continuous, where Y is compact and Hausdorff. Prove that there is a continuous $g : X \to Y$ with $g|A = f$ iff for each pair B, C of disjoint closed subsets of Y, $\overline{f^{-1}(B)} \cap \overline{f^{-1}(C)} = \varnothing$.

10. Suppose that infinitely many of the spaces $\{X_\alpha \ / \ \alpha \in A\}$ are non-compact. Show that any compact subset K of ΠX_α is nowhere dense, i.e. $\mathrm{int} K = \varnothing$.

11. What does Problem 10 tell us about the local compactness of a product of locally compact spaces?

12. Let X be a σ-compact, locally compact Hausdorff space. Prove that X is paracompact. [Hint: start off by finding sequences of compacta $\langle K_n \rangle$ and open sets $\langle U_n \rangle$ so that $\cup K_n = X$, $K_n \subset U_n$ and $U_i \cap U_j \neq \varnothing$ only if $|i - j| \leq 1$.]

13. Let X be a space and let $\mathcal{F}$ be a locally finite family of subsets of X. Prove that $\overline{\cup\{F \in \mathcal{F}\}} = \cup\{\bar{F} \ / \ F \in \mathcal{F}\}$.

14. Let X be as in Problem 18 of section 1. Prove that X is not Lindelöf. What other topological properties does X enjoy or fail to enjoy?

15. Prove that any locally compact, paracompact, connected space is Lindelöf. Hence, using Problem 14 of section 3, prove that every locally connected, locally compact, paracompact, separable space is Lindel$f$.

16. Prove that every metacompact, separable space is Lindelöf. (Thus the last part of Problem 15 can be improved considerably.)