An introduction to geometrisation

David Gauld

May 13, 2005

1 Some History and a Definition

> 'Tout polyèdre qui a tous ses nombres de Betti égaux à 1 et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère, [5].

This is the original version of the Poincaré Conjecture. Soon after Poincaré gave a counterexample, Example 2.8 below, and posed the following instead, for a long time known as the Poincaré Conjecture.

> If M^3 is a compact connected manifold with $\pi(M)$ trivial then $M \approx \mathbb{S}^3$.

Definition 1.1 *By a* manifold *is meant a Hausdorff space each point of which has a neighbourhood homeomorphic to* $\mathbb{R}^n$ *for some positive integer* n*.*

Most of the time we will also assume that our manifolds are compact and connected. When we write "M^n is a manifold" or "M is an n-manifold" we will mean that M is a manifold and n the *dimension*, which is the integer of the definition.

Sometimes we will need a *manifold-with-boundary*, which is a Hausdorff space each point of which has a neighbourhood homeomorphic either to $\mathbb{R}^n$ or to the half space $\mathbb{R}^n_+$. If M^n is a manifold-with-boundary then the *boundary of* M consists of those points not having neighbourhoods homeomorphic to $\mathbb{R}^n$ and is denoted by ∂M, an $(n-1)$-manifold.

For the duration of the 20$^{\text{th}}$ Century many mathematicians tried to solve the PC. Sometimes they constructed 'counterexamples' and sometimes 'proofs.' Sometimes they kept their failures to themselves while at other times there was a lot of publicity before errors were found: see [6] and [8] for discussion of one such "solution."

One favourite trick of mathematicians when confronted with a problem is to generalise it. There is an obvious way to generalise the PC to the Generalised Poincaré Conjecture:

> If M^n is a compact connected manifold with $\pi_i(M)$ trivial whenever $i < n$ ($i \leq \frac{n}{2}$ is sufficient) then $M \approx \mathbb{S}^n$.

GPC_n is trivially true when $n = 1$ and is seen to be true for $n = 2$ if one makes use of the classification of compact connected surfaces. Interestingly GPC_n was shown true for $n \geq 5$ by Stephen Smale and others (notably Christopher Zeeman and MHA Newman) in the early 1960s and for $n = 4$ by Mike Freedman in the 1980s. Only in the last couple of years has the case $n = 3$ apparently been resolved.

2 Action of a Group on a Space

Definition 2.1 *A* topological group *is a triple* $(\Gamma, \bullet, \mathcal{T})$*, where* $\bullet$ *is a binary operation on* Γ *so that* $(\Gamma, \bullet)$ *is a group and* $\mathcal{T}$ *is a topology on* Γ *such that the group operations* $\bullet : \Gamma \times \Gamma \to \Gamma$ *and* $^{-1} : \Gamma \to \Gamma$ *are both continuous.*

Example 2.2 *Give any group the discrete topology to get a topological group.*

Example 2.3 *$GL(n)$ denotes the* general linear group *and consists of all non-singular $n \times n$ matrices with real entries; equivalently the collection of all vector space isomorphisms of $\mathbb{R}^n$. Make $GL(n)$ into a group by use of matrix multiplication. $GL(n)$ may be topologised as a subspace of $\mathbb{R}^{n^2}$. There are two important topological subgroups of $GL(n)$, viz $O(n)$, the* orthogonal group *consisting of orthogonal matrices, and the* special orthogonal group, *$SO(n)$ consisting of matrices of determinant 1.*

In passing we notice an exact sequence $\{I\} \to SO(n) \to O(n) \stackrel{\det}{\to} \mathbb{Z}_2 \to \{1\}$, where I is the identity matrix and $\mathbb{Z}_2$ is the multiplicative group $\{-1, 1\}$.

Definition 2.4 *A topological group* Γ acts on a space X *if there is a continuous function $\alpha : \Gamma \times X \to X$ such that the following two diagrams commute:*

$$\text{(i)}\quad \begin{tikzcd} X \arrow[r, "{(e,1)}"] \arrow[dr, "1"'] & \Gamma \times X \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} \Gamma \times \Gamma \times X \arrow[r, "1 \times \alpha"] \arrow[d, "\bullet \times 1"] & \Gamma \times X \arrow[d, "\alpha"] \\ \Gamma \times X \arrow[r, "\alpha"] & X, \end{tikzcd}$$

where $(e,1)(x) = (e,x)$ for all $x \in X$ and e is the identity of Γ.

There is an alternative way of looking at group actions. Let $\mathcal{H}(X)$ denote the group of homeomorphisms of X, the group operation being composition. $\mathcal{H}(X)$ may be topologised in a number of ways to make it into a topological group. Then a group action is a continuous homomorphism $\theta : \Gamma \to \mathcal{H}(X)$, though we need to be a bit careful with topologies. We might then think of any $g \in \Gamma$ being a homeomorphism $g : X \to X$.

We will be most interested in the case where Γ is discrete and acts *properly discontinuously* on X, ie

- $\forall x \in X\ \exists N_x$, a neighbourhood of x, such that $\forall g \in \Gamma$ if $g(N_x) \cap N_x \neq \varnothing$ then $g = e$ and
- $\forall x, y \in X$ with $y \notin \Gamma x\ \exists O_x, O_y$, neighbourhoods of x, y, such that $g(O_x) \cap O_y = \varnothing$ for all $g \in \Gamma$.

Here the set Γx is the *orbit of* x and is $\{\alpha(g,x)\ /\ x \in X\}$. Usually $\alpha(g,x)$ is abbreviated to gx or $g(x)$.

There is a natural quotient space sitting in here: decree $x \sim y$ iff $gx = y$ for some $g \in \Gamma$. The quotient space is usually denoted X/Γ and is the *orbit space*. The quotient map $X \to X/\Gamma$ is a covering projection.

The *stabiliser* of $x \in X$ is $\Gamma_x = \{g \in \Gamma\ /\ gx = x\}$, and Γ *acts freely* provided that the stabiliser is always trivial.

A *fundamental region* for Γ is a closed subset $C \subset X$ such that

$$\cup_{g \in \Gamma} gC = X \qquad \text{and} \qquad \mathring{C} \cap g\mathring{C} = \varnothing.$$

Example 2.5 *Think of $\mathbb{S}^1$ as the complex numbers of unit modulus and for any positive integer $n > 1$ let $\mathbb{Z}_n$ be the group of integers mod n. Then $\mathbb{Z}_n$ acts on $\mathbb{S}^1$ by rotation through $\frac{2\pi}{n}$. The quotient $\mathbb{S}^1/\mathbb{Z}_n$ is homeomorphic to $\mathbb{S}^1$.*

Example 2.6 *Let $\mathbb{Z}$ act on $\mathbb{R}$ by the translation $(n,x) \mapsto n + x$. Then $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$.*

Example 2.7 *Doubling the dimension of the previous example is interesting: let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ by translation. Then $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$.*

Example 2.8 *Let Γ be the group of oriented symmetries of a regular dodecahedron. Locating the dodecahedron with its centre at the origin ensures that $\Gamma < SO(3)$. Then Γ acts on $SO(3)$ using the group operation. $SO(3)/\Gamma$ is a counterexample to the statement at the beginning of this note and is called the* Poincaré homology sphere, *P^3. As $SO(3)$ is the quotient space of $\mathbb{S}^3$ by antipodal identification (another group action!) there is another group $\widetilde{\Gamma}$ which is a 2-fold cover of Γ so that P is also $\mathbb{S}^3/\widetilde{\Gamma}$. Covering space theory tells us that $\pi(P) \approx \widetilde{\Gamma}$. Γ is a group of order 60 and $\widetilde{\Gamma}$ a group of order 120, the binary icosahedral group. Thus P is not a counterexample to the Poincaré Conjecture.*

The reason that the original PC failed was that he asked that a 3-manifold with the homology, rather than the homotopy, of a 3-sphere should be a 3-sphere. The difference is superficially small, being in dimension 1, ie the difference between $H_1(M)$ and $\pi_1(M)$. It turns out that $H_1(M)$ is the abelianisation of $\pi_1(M)$. Well, the binary icosahedral group has presentation $\langle a, b, c | a^2 = b^3 = c^5 = abc \rangle$ and it's not too hard to find that the abelianisation is trivial.

3 More on Manifolds

Definition 3.1 *Let M^n be a manifold. A* chart *on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to \mathbb{R}^n$ an embedding. An* atlas *is a collection $\{(U_\alpha, \varphi_\alpha) \ / \ \alpha \in A\}$ of charts so that $\cup_{\alpha \in A} U_\alpha = M$. Given two charts (U, φ) and (V, ψ) the embedding $\psi\varphi^{-1} : \varphi(U \cap V) \to \mathbb{R}^n$ is a* coordinate transformation.

Definition 3.2 *A* differential structure *on a manifold M is an atlas $\mathcal{D}$ so that whenever $(U, \varphi), (V, \psi) \in \mathcal{D}$ the corresponding coordinate transformations are differentiable. Especially if $\mathcal{D}$ is maximal with respect to this property, the pair $(M, \mathcal{D})$ is called a* differentiable or smooth manifold.

Theorem 3.3 (Moise) *Every compact manifold of dimension ≤ 3 supports a differential structure.*

Moise's theorem fails in higher dimensions. Smale's initial result pertaining to GPC$_n$ worked only for smooth manifolds using Morse Theory. There is another related category called piecewise linear and Zeeman solved GPC$_n$ in this category. Newman then proved the result without any such extra structure.

So anything more relating to the structure of manifolds of dimension ≤ 3 may assume the manifold to be smooth. We can go further but that needs some more work.

Let $(M^n, \mathcal{D})$ be a smooth manifold and let $p \in M$. A *tangent vector at p* is a function v with domain $\{(U, \varphi) \ / \ p \in U \text{ and } (U, \varphi) \in \mathcal{D}\}$ and range the collection of $n \times 1$ column matrices such that if $(U, \varphi), (V, \psi) \in \mathcal{D}$ with $p \in U \cap V$ then $v(V, \psi) = D(\psi\varphi^{-1})(\varphi(p)) \times v(U, \varphi)$. Here $D(f)(x)$ denotes the Jacobian matrix of f at x and we are dealing with matrix multiplication.

We denote by TM_p the collection of all tangent vectors at p. TM_p is called the *tangent space* at p. It is a vector space of dimension n. The 'disjoint union' of the spaces TM_p as p ranges through M is called the *tangent manifold* and is denoted TM. The tangent manifold takes on a natural differential structure making it into a $2n$-manifold.

Definition 3.4 *A* Riemannian metric *on a smooth manifold M assigns to each point $p \in M$ an inner product $\langle ., . \rangle_p$ on the tangent space TM_p in such a way that $\langle ., . \rangle_p$ varies smoothly with p. A* Riemannian manifold *is a smooth manifold with a Riemannian metric.*

Note that the inner products give a notion of length to tangent vectors and consequently for smooth paths. In turn this makes M into a metric space by making the distance between two points to be the infimum of the lengths of all paths joining the two points. Shortest paths are called *geodesics*. For most of our purposes this is all we really need from the Riemannian metric, and it may be summarised by a formula for the infinitesimal arc length s. This formula depends on the coordinate chart chosen: for example in euclidean n-space $\mathbb{E}^n$ with the usual coordinates this is $ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2$. The length of a curve γ is $\int_\gamma ds$.

We will want to look at curvature too. A sophisticated study requires more effort than we have time for so we just look at the case of a surface. The basic idea is this. For any point P on a surface look at a small "circle" centred at P, ie the set of points some fixed distance r from P. This gives a curve γ_r. Let $c_r = \oint_{\gamma_r} ds$ be the circumference of γ_r. If the surface is flat then we should have $c_r = 2\pi r$; on the sphere we have $c_r < 2\pi r$; on some surfaces maybe $c_r > 2\pi r$. Looking at the ratio $R_r = \frac{c_r}{2\pi r}$ we compare R_r with 1. The curvature measures how this changes, more precisely the *curvature* at P is $\kappa = -3 \lim_{r \to 0^+} \frac{d^2 R_r}{dr^2}$.

Definition 3.5 *A* geometry *is a simply connected homogeneous unimodular Riemannian manifold X.* Unimodular *means that X admits a discrete group of isometries with compact quotient.*

Definition 3.6 *A compact manifold(-with-boundary) M is called* geometric *if $M - \partial M = X/\Gamma$ has finite volume, where X is a geometry and Γ is a discrete group of isometries of X acting freely on X.*

The covering projection allows us to transfer the geometry from X to M.

On a completely different track, we can 'add' two n-manifolds together.

Definition 3.7 *Let M_1^n and M_2^n be two manifolds. The* connected sum *of M_1 and M_2 is the n-manifold $M_1 \sharp M_2$ obtained as follows. Let $e_i : \mathbb{B}^n \to M_i$ be an embedding. In the disjoint sum $(M_1 - e_1(\mathrm{Int}\mathbb{B}^n)) \cup (M_2 - e_2(\mathrm{Int}\mathbb{B}^n)) \cup (\mathbb{S}^{n-1} \times [1,2])$ declare $e_i(x) \sim (x,i)$ for any $x \in \mathbb{S}^{n-1}$ and $i = 1,2$. Then $M_1 \sharp M_2$ is the quotient space by $\sim$.*

If each M_i is smooth then $M_1 \sharp M_2$ may be made smooth too.

4 Some Geometry in Dimension 2

There is a classification of compact connected surfaces (2-manifolds), there being two families, the orientable surfaces and the non-orientable surfaces.

Orientable Surfaces. Start with $\mathbb{S}^2$ and $\mathbb{T}^2 \approx \mathbb{S}^1 \times \mathbb{S}^1$. The n^{th} member of this sequence is obtained by iterating the connected sum operation on $\mathbb{T}^2$ n times. Alternatively the n^{th} term may be obtained from $\mathbb{S}^2$ by cutting out $2n$ copies of $\mathrm{Int}\mathbb{B}^2$ from $\mathbb{S}^2$ much as in the connected sum operation, pairing off the resulting boundary circles and then attaching a copy of the cylinder $\mathbb{S}^1 \times [0,1]$, whose boundary consists of a pair of circles, again much as in the connected sum operation. This process is called adding a handle. Adding a single handle to $\mathbb{S}^2$ yields $\mathbb{T}^2$.

Non-orientable Surfaces. Start with $\mathbb{S}^2$. For any positive integer n cut out n copies of $\mathrm{Int}\mathbb{B}^2$ from $\mathbb{S}^2$, declare two points equivalent if they are antipodal on the same bounding circle and take the resulting quotient space. This process is called adding a cross-cap. When $n = 1, 2$ we get respectively the projective plane and the Klein bottle.

We will ignore the non-orientable surfaces.

What are the possible geometries in dimension 2? There are 3: $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$. We will look at two aspects, the geodesics and the groups of isometries. Because we are very familiar with euclidean space we look at that first.

The geometry of $\mathbb{E}^2$. As already noted, $ds^2 = dx^2 + dy^2$. The geodesics are straight lines. It is trivial to check that the curvature is constantly 0. While $\mathbb{E}^2$ is not compact we saw in Example 2.7 that it has quotient $\mathbb{T}^2$ by $\mathbb{Z}^2$. The group of all isometries of $\mathbb{E}^2$, denoted $\mathrm{Isom}(\mathbb{E}^2)$, consists of translations, rotations, reflections and glide-reflections: the last of these is the composition of reflection in a line l followed by translation along that line. $\mathrm{Isom}(\mathbb{E}^2)$ is generated by reflections: reflect in two parallel lines to get a translation and reflect in two intersecting lines to get a rotation. To construct a geometric surface from $\mathbb{E}^2$ we need to take a subgroup $\Gamma < \mathrm{Isom}(\mathbb{E}^2)$ consisting only of translations so will effectively be a discrete subgroup of $\mathbb{R}^2$. To ensure compactness, it will need to be generated by two translations and the resulting quotient will be homeomorphic to the torus $\mathbb{T}^2$. When $\mathbb{T}^2$ is given the geometry as a quotient of $\mathbb{E}^2$ its curvature is also constant: in our usual model of $\mathbb{T}^2$ this is **not** the case.

The geometry of $\mathbb{S}^2$. The geometry of $\mathbb{S}^2$ is inherited from $\mathbb{E}^3$, $\mathbb{S}^2$ being a subspace. Now $ds^2 = dx^2 + dy^2 + dz^2$. The geodesics are great circles. In contrast to the case of $\mathbb{E}^2$, there is no such thing as parallel geodesics: any two geodesics meet. Furthermore two points no longer necessarily determine a unique geodesic: indeed, antipodal points may be joined by infinitely many geodesics. Another major difference is that the area of a triangle is determined completely by its angles! One can check that if the angles are α, β and γ then the area of the resulting geodesic triangle is $\alpha + \beta + \gamma - \pi$. In particular any triangle on $\mathbb{S}^2$ has angle sum greater than π. In fact the angle sum can lie anywhere in the interval $(\pi, 5\pi)$. One can verify that the curvature is constantly 1.

The group of all isometries of $\mathbb{S}^2$ is $O(3)$, which is also the group of isometries of $\mathbb{E}^3$ fixing the origin, 0. An element of $SO(3)$ is a rotation of $\mathbb{E}^3$ about some line through 0. Hence every orientation-preserving isometry of $\mathbb{S}^2$ is a rotation and a rotation of $\mathbb{S}^2$ fixes two antipodal points of $\mathbb{S}^2$. A reflection of $\mathbb{E}^3$ through a plane containing 0 restricts to a reflection of $\mathbb{S}^2$ through the great circle which is the intersection of the plane with $\mathbb{S}^2$. $O(3)$ is generated by reflections. The only discrete subgroups of $O(3)$ are finite. The only surfaces of the form $\mathbb{S}^2/\Gamma$, for Γ discrete(=finite), are $\mathbb{S}^2$ itself and $\mathbb{P}^2$. The latter, the projective plane, is not orientable.

The geometry of $\mathbb{H}^2$. The underlying set can be described in two standard ways: either as the open upper half of $\mathbb{R}^2_+$ or $\mathrm{Int}\mathbb{B}^2$. The respective formulae for the infinitesimal distance, using the usual coordinates inherited from $\mathbb{R}^2$, are

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2\Big) \qquad \text{and} \qquad ds^2 = \frac{4}{\Big(1-(x^2+y^2)\Big)^2}\Big(dx^2+dy^2\Big).$$

In either case the geodesics are circles which meet the boundary ($\mathbb{S}^1$ or the x-axis) at right angles, where in either case circles include also straight lines. Why? Let's just concentrate on the first model for now.

Suppose that γ is a vertical segment in $\mathbb{R}^2_+$ joining points with y-coordinates y_0 and y_1 with $y_0 < y_1$. The length of γ is $\int_\gamma ds = \int_{y_0}^{y_1} \frac{1}{y} dy = \ln \frac{y_1}{y_0}$. If δ is another path joining those two points then we may parametrise δ by t so that it has length $\int_{t_0}^{t_1} \frac{1}{y}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{\frac{1}{2}} dt$. As $\frac{dx}{dt} \neq 0$ it follows that δ has length greater than that of γ.

To get some more geodesics we identify $\mathrm{Isom}(\mathbb{H}^2)$, for any isometry takes one geodesic to another. The simplest are reflections of $\mathbb{R}^2_+$ in vertical lines; as we already saw, the composition of two of those gives a translation. More interesting is reflection of $\mathbb{R}^2_+$ in a (semi)circle of radius r centred at a point $(a,0)$: denote this by ρ. Suppose that $\rho(x,y) = (u,v)$. We show that $\frac{1}{v^2}(du^2+dv^2) = \frac{1}{y^2}(dx^2+dy^2)$, which will then confirm that ρ really is an isometry. It is easier to use the complex structure of $\mathbb{C}$, so write $z = x+iy$ and $w = u+iv$. Then the reflection is given by $(w-a)(\overline{z}-a) = r^2$, so $w = a + \frac{r^2}{\overline{z}-a}$. From this we get $dw = \frac{r^2}{(\overline{z}-a)^2}\overline{dz}$ with a similar expression for $\overline{dw}$ and hence

$$\frac{dw\overline{dw}}{dz\overline{dz}} = \frac{r^4}{(z-a)^2(\overline{z}-a)^2} = \frac{(\overline{w}-w)^2}{(\overline{z}-z)^2} \qquad \text{and} \qquad \frac{dw\overline{dw}}{(w-\overline{w})^2} = \frac{dz\overline{dz}}{(z-\overline{z})^2}.$$

Now unravel this in terms of u, v, x, y to find that $\frac{1}{v^2}(du^2+dv^2) = \frac{1}{y^2}(dx^2+dy^2)$ as claimed.

The reflection ρ takes the known geodesic which is the vertical line through $(a-r,0)$ to the semicircle through which we reflected. As reflections preserve angles and send circles and straight lines to circles or straight lines, it follows that all of the curves we claimed to be geodesics really are. There are no more! Indeed, suppose γ is a geodesic not of the form discovered already. By taking a subcurve if necessary we can assume that γ is a shortest (possibly equal) path between two points P and Q. Choose the semicircle or vertical straight line containing P and Q; call it δ. If δ is not a vertical line then we can reflect to convert it into a vertical line and this reflection will take γ into a geodesic which joins two points with the same x-coordinate but the geodesic, like γ, is not one of the standard ones. So let us assume that P and Q have the same x-coordinate and δ is a vertical line. We have already seen that δ is the unique geodesic joining P to Q. This shows that γ cannot exist and that we have identified all geodesics.

Like $\mathbb{E}^2$, any pair of points of $\mathbb{H}^2$ lie on a unique geodesic and two distinct geodesics meet in at most one point. However given a point P and a geodesic γ not containing it there are infinitely many geodesics passing through P and not meeting γ.

By evaluating the formula $\int\int_\Delta \frac{1}{y^2} dxdy$ for a geodesic triangle Δ in $\mathbb{H}^2$ it may be shown that the area of a triangle with angles α, β, γ is $\pi - (\alpha+\beta+\gamma)$. Try some simplifications such as starting with one angle 0 and reflecting so that two sides of the triangle are vertical segments and the third side lies on the circle $|z| = 1$. Thus all triangles have area less than π. The angle sum for a triangle may be anywhere in the interval $(0, \pi)$. If S is the region bounded by a (geodesic) n-gon having interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ then S has area $(n-2)\pi - (\alpha_1 + \ldots + \alpha_n)$.

It is clear that for any two points $P, Q \in \mathbb{H}^2$ there is a reflection taking P to Q. Thus $\mathbb{H}^2$ is homogeneous, and hence has constant curvature.

We have identified all isometries, viz compositions of reflections in geodesics. Indeed, suppose that $f : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry. We shall construct a sequence of reflections taking f to the identity. Pick any point $P \in \mathbb{H}^2$. If $f(P) \neq P$ then we may compose f with a reflection sending $f(P)$ to P. So now we assume that $f(P) = P$. Let γ be a geodesic through P. If $f(\gamma) \neq \gamma$ then we may compose f with a reflection which carries the semicircle or line $f(\gamma)$ onto γ. So now we assume that $f(\gamma) = \gamma$. If f interchanges the ends of γ then we may follow f by a reflection which sends γ to itself and interchanges the ends of γ. So now we assume that $f(\gamma) = \gamma$ and f preserves the ends of γ. This means that f fixes each point of γ because f fixes P and any other point is at some particular distance from P which is preserved by f as f is an isometry. Note that γ splits $\mathbb{H}^2$ into two pieces. If f does not send each piece to itself then following f by a reflection will achieve this. It remains to show that if f is an isometry of $\mathbb{H}^2$ which fixes the geodesic γ pointwise and sends each complementary component of $\mathbb{H}^2 - \gamma$ to itself then f is the identity. Suppose $Q \in \mathbb{H}^2 - \gamma$: to show $f(Q) = Q$. Let δ be the unique geodesic through P and Q. As isometries preserve angles it follows that $f(\delta) = \delta$, hence that $f(Q) = Q$ as f preserves the distance from P to Q.

The description in the previous paragraph shows that any isometry of $\mathbb{H}^2$ is a composition of at most four reflections. However the description was wasteful in the sense that when we reached the situation where $f(P) = P$ we then used two more reflections to ensure that γ was fixed pointwise; this could have been achieved by a single reflection. So an isometry is a composition of at most three reflections in a geodesic. As reflections reverse orientation it follows that the orientation-preserving isometries consist only of the double reflections. One can check that such maps are of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Such a map is called a Möbuis transformation (even when we relax the requirements that $a, b, c, d \in \mathbb{R}$ to $a, b, c, d \in \mathbb{C}$ and $ad - bc > 0$ to $ad - bc \neq 0$ but if we did then the transformation would not send $\mathbb{R}^2_+$ to itself). Furthermore such a Möbius transformation is a double reflection. So we can identify the orientation-preserving isometries by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;/\; ad - bc > 0 \right\}$ but this does not give a complete description as the two matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ give rise to the same Möbius transformation. The quotient group $SL(2, \mathbb{R})/\pm$ is the projective version of $SL(2, \mathbb{R})$ and is denoted $PSL(2, \mathbb{R})$.

The group $PSL(2, \mathbb{R})$ is a subgroup of the group $PSL(2, \mathbb{C})$ of all Möbius transformations of $\mathbb{C}$. In that group lies the transformation $z \mapsto \frac{z-i}{z+i}$ and this takes $\mathbb{R}^2_+$ to $\mathrm{Int}\mathbb{B}^2$, giving us the link to the second model of $\mathbb{H}^2$.

Using the second model it is fairly easy to verify that the curvature is -1 everywhere. Using homogeneity it is only necessary to look at the curvature at one point: the origin is easiest as circles centred there are Euclidean circles but of a different radius. A circle of hyperbolic radius r has circumference $2\pi \sinh r$.

We can find groups acting freely on $\mathbb{H}^2$ by starting with fundamental regions. Use the second model of $\mathbb{H}^2$ and consider a regular octagon centred at the origin. If the octagon is small then its angle sum is almost 6π so each angle is almost $\frac{3\pi}{4}$. On the other hand if the octagon is large with the vertices near the euclidean circle bounding $\mathbb{H}^2$ then each angle is near 0. Somewhere between there is an octagon, call it O, whose angles are each $\frac{\pi}{4}$. Denote the vertices of O by $A, \ldots H$. Define $\alpha, \beta, \gamma, \delta \in \mathrm{Isom}(\mathbb{H}^2)$ by declaring $\alpha(AB) = DC$, $\beta(BC) = ED$, $\gamma(EF) = HG$ and $\delta(FG) = AH$, with each of $\alpha, \beta, \gamma, \delta$ taking $\mathring{O}$ to a set disjoint from O. Letting Γ be the group of isometries generated by $\alpha, \beta, \gamma, \delta$, we see that O is a fundamental region for Γ. The quotient $\mathbb{H}^2/\Gamma$ is a surface because the 8 vertices $A, \ldots, H$ are identified to a single point at which the angle sum is $8\frac{\pi}{4} = 2\pi$. This surface is $\mathbb{S}^2$ with two handles, and we have shown that this surface carries a hyperbolic structure.

For any $n \geq 2$ we could repeat the exercise of the previous paragraph. Replace O by a regular $4n$-gon with angles $\frac{\pi}{2n}$. We now need $2n$ generators for the group and we obtain a surface which is $\mathbb{S}^2$ with n handles carrying a hyperbolic structure.

To summarise: We have shown that all (compact, connected,) orientable surfaces carry a geometric structure. $\mathbb{S}^2$ is the only one having spherical geometry, $\mathbb{T}^2$ is the only one having euclidean geometry and all the rest have hyperbolic geometry in which each surface has constant curvature -1. The separation is closely related to the Euler character of the surface as in the following table.

Euler character	$\chi > 0$	$\chi = 0$	$\chi < 0$
Geometry	$\mathbb{S}^2$	$\mathbb{E}^2$	$\mathbb{H}^2$

5 Geometry in Dimension 3

Now there are eight relevant geometries. Some are easy to understand, being closely related to those we have seen in dimension 2, some less so. The eight are: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}$, $\widetilde{SL_2(\mathbb{R})}$, *Nil* and *Sol*.

The geometry of $\mathbb{E}^3$. We have $ds^2 = dx^2 + dy^2 + dz^2$ and again the geodesics are straight lines and the curvature 0. The group of isometries is interesting.

An isometry of $\mathbb{E}^3$ is of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ where A is an orthogonal 3×3 matrix and $\mathbf{b} \in \mathbb{E}^3$. Thus we obtain an exact sequence:

$$1 \to \mathbb{R}^3 \to \text{Isom}(\mathbb{E}^3) \to O(3) \to 1.$$

To prove that every isometry is of the form above we prove each of the following:

1. Every translation of $\mathbb{E}^3$ is an isometry.
2. Every orthogonal transformation of $\mathbb{E}^3$ is an isometry.
3. Any isometry of $\mathbb{E}^3$ fixing the origin is an orthogonal transformation.
4. Given any isometry of $\mathbb{E}^3$ there are unique orthogonal A and $\mathbf{b} \in \mathbb{E}^3$ such that the isometry is of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$.

Note that an orientation-preserving orthogonal transformation of $\mathbb{E}^3$ is just a rotation about some line through the origin.

We can go further: any orientation-preserving isometry of $\mathbb{E}^3$ is a *screw motion* consisting of a rotation about and translation along a fixed line. The proof of this starts at point 4 above, which, if we choose the axes carefully (so that the x_3-axis coincides with the axis of the rotation part), tells us that an orientation-preserving isometry has the form:

$$\begin{aligned} x_1 &\mapsto x_1 \cos\alpha - x_2 \sin\alpha + b_1 \\ x_2 &\mapsto x_1 \sin\alpha + x_2 \cos\alpha + b_2 \\ x_3 &\mapsto x_3 + b_3. \end{aligned}$$

Then proceed as follows:

5. If the isometry fixes some point then $b_3 = 0$ so the isometry is the same rotation in each plane $x_3 =$ const, so a rotation about some line perpendicular to the plane $x_3 = 0$.
6. If the isometry does not fix any point then consider two cases:

   (a) $b_3 = 0$ so the equations

   $$\begin{aligned} x_1 &= x_1 \cos\alpha - x_2 \sin\alpha + b_1 \\ x_2 &= x_1 \sin\alpha + x_2 \cos\alpha + b_2. \end{aligned}$$

   must have no solution from which $\cos\alpha = 1$ and $\sin\alpha = 0$ and we have a translation;

(b) $b_3 \neq 0$ leading to a rotation about a line perpendicular to $x_3 = 0$ followed by the translation $x_3 \mapsto x_3 + b_3$.

The geometry of $\mathbb{S}^3$. $\mathbb{S}^3$ can be viewed in three relevant ways:

- as the subspace of $\mathbb{E}^4$ consisting of points whose distance from the origin is 1;
- as $\{(z, w) \in \mathbb{C}^2 \;/\; |z|^2 + |w|^2 = 1\}$;
- as the unit quaternions.

Successively identify $(x, y, u, v) \in \mathbb{E}^4$ with $(z, w) = (x+iy, u+iv) \in \mathbb{C}^2$ with $z+wj = x+iy+ju+kv \in \mathbb{Q}$ (which now represents the quaternions rather than the rational numbers). The last view gives $\mathbb{S}^3$ the structure of a group, analogously with the real group structure on $\mathbb{S}^0$ and the complex group structure on $\mathbb{S}^1$.

The Riemannian metric comes from $ds^2 = dw^2 + dx^2 + dy^2 + dz^2$, the group of isometries is $O(4)$ and the curvature is 1. There is a nice way to describe the geodesics. Suppose that $P, Q \in \mathbb{S}^3$ are close enough together that there is a unique geodesic, γ, joining them. Let $\pi \subset \mathbb{E}^4$ be the 2-plane determined by 0, P and Q. Then $\gamma \subset \pi \cap \mathbb{S}^3$ and since the latter is a circle it follows that all geodesics are circles. To see why $\gamma \subset \pi \cap \mathbb{S}^3$, let $\Sigma \subset \mathbb{E}^4$ be any 3-hyperplane containing π. Reflection of $\mathbb{E}^4$ in Σ gives an isometry of $\mathbb{S}^3$ fixing P and Q and hence all of γ. Thus $\gamma \subset \Sigma$. As this holds for all 3-hyperplanes Σ containing π it follows that γ lies in the intersection of those hyperplanes, viz π.

Going off on a normal mention is made here of the *Hopf fibration.* Think of $\mathbb{S}^3$ as $\{(z, w) \in \mathbb{C}^2 \;/\; |z|^2 + |w|^2 = 1\}$ and $\mathbb{S}^2$ as $\mathbb{C} \cup \{\infty\}$. The Hopf map is $h : \mathbb{S}^3 \to \mathbb{S}^2$ given by $h(z, w) = \frac{z}{w}$. Note that for any $\zeta \in \mathbb{C} \cup \{\infty\}$ we have $h^{-1}(\zeta) = \{(z, w) \in \mathbb{S}^3 \;/\; z = \zeta w\}$, which is a geodesic circle in $\mathbb{S}^3$. The Hopf fibration is represented by the diagram

$$\begin{array}{ccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^3 \\ & & \Big\downarrow h \\ & & \mathbb{S}^2 \end{array}$$

and any fibration leads to an exact sequence of homotopy groups, in this case

$$\ldots \to \pi_n(\mathbb{S}^1) \to \pi_n(\mathbb{S}^3) \xrightarrow{h_*} \pi_n(\mathbb{S}^2) \to \pi_{n-1}(\mathbb{S}^1) \to \ldots \to \pi_1(\mathbb{S}^2).$$

We have $\pi_n(\mathbb{S}^n) \approx \mathbb{Z}$ for all n, and $\pi_n(\mathbb{S}^1) \approx 0$ whenever $n > 1$. Taking $n = 3$ in the part of the exact sequence displayed we conclude that $\pi_3(\mathbb{S}^2) \approx \mathbb{Z}$.

The geometry of $\mathbb{H}^3$. The usual model for $\mathbb{H}^3$ is the upper half 3-space $\mathbb{R}^3_+$ with the Riemannian metric coming from $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$. As before the geodesics are the vertical straight lines and semicircles meeting the xy-plane orthogonally. Now isometries are generated by reflections in vertical planes or euclidean hemispheres centred on the xy-plane. Identifying the xy-plane with the complex plane, we get a "2-sphere at infinity," which is $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then an isometry of $\mathbb{H}^3$ is determined by its 'restriction' to $\hat{\mathbb{C}}$. The restriction of a reflection is just a reflection sending $\hat{\mathbb{C}}$ to itself. Thus $\mathrm{Isom}(\mathbb{H}^3)$ may be identified with the Möbius transformations of $\hat{\mathbb{C}}$, the group $PSL(2, \mathbb{C})$. Identifying the point $(x, y, z) \in \mathbb{R}^3_+$ with the quaternion $x + iy + jz$, the 2×2 complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ gives the isometry $w \mapsto \frac{aw+b}{cw+d}$, where w is a quaternion of the form $x + iy + jz$ with $z > 0$.

All of the geometries we have considered so far are *isotropic*. Recall that a space is *homogeneous* provided that it looks the same at every point; more precisely for any pair of points in the space there is an isometry of the space taking one point to the other: the group of isometries is transitive. Recall that when a group Γ acts on a space X there corresponds to each point $x \in X$ the stabiliser $\Gamma_x = \{g \in \Gamma \;/\; gx = x\}$. This group is also called the *isotropy subgroup* of x. A space is isotropic if the isotropy subgroup of x acts transitively on the frames of TM_x; a frame is an ordered orthonormal base. In an isotropic space the view is the same however we may tilt our heads. The remaining geometries are anisotropic.

The geometry of $\mathbb{S}^2 \times \mathbb{E}$. The isometry group is $O(3) \times \mathbb{E}$. The isotropy groups are isomorphic to $O(2)$, and act as rotations of the spherical factor. The simplest compact manifold supporting this geometry is $\mathbb{S}^2 \times (\mathbb{E}/\mathbb{Z}) \approx \mathbb{S}^2 \times \mathbb{S}^1$, and there are only three others, $\mathbb{P}^2 \times \mathbb{S}^1$, $\mathbb{P}^3 \sharp \mathbb{P}^3$ and a 'twisted product' of $\mathbb{S}^2$ and $\mathbb{S}^1$.

The geometry of $\mathbb{H}^2 \times \mathbb{E}$. This manifold has isometry group $\mathrm{Isom}(\mathbb{H}^2) \times \mathbb{E}$. It also has 1-dimensional isotropy groups. The product of any compact hyperbolic 2-manifold (for example an orientable surface of genus greater than 1) with the circle is an example of a compact manifold modelled on this geometry.

> Recall the Map Lifting Criterion which says that if $p : (E, e_0) \to (X, x_0)$ is a covering projection then any map $f : (Y, y_0) \to (X, x_0)$ lifts over p if and only if $f_*(\pi(Y, y_0)) \subset p_*(\pi(E, e_0))$ provided some basic connectivity assumptions hold. In particular
>
> - if Y is simply connected then such a lifting always exists;
> - if f is a covering projection then so is a lifting $(Y, y_0) \to (E, e_0)$ if it exists.
>
> Recall also uniqueness of any lifting. Combining these we conclude that if (X, x_0) has a covering projection $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ for which $\tilde{X}$ is simply connected then $(\tilde{X}, \tilde{x}_0)$ is unique and all other covering spaces lie 'below' it. We may call such a space the *universal cover* of X. Universal covers do not always exist: as $\tilde{X}$ looks locally like X all small loops in X must be able to be shrunk in X but there are spaces in which this fails. A space X in which each point x has a neighbourhood U such that each loop in U based at x can be shrunk in X to x is called semi-locally simply connected. Every semi-locally simply connected, locally path connected, connected space has a universal cover. In particular every connected manifold has a universal cover which is also a manifold. $\mathbb{R}$ is the universal cover of $\mathbb{S}^1$ and $\mathbb{S}^n$ the universal cover of $\mathbb{P}^n$ (when $n > 1$).

The geometry of $\widetilde{SL_2(\mathbb{R})}$. Begin with $SL_2(\mathbb{R})$, the (3-dimensional Lie) group of all 2×2 real matrices with determinant 1. The space $\widetilde{SL_2(\mathbb{R})}$ is the universal cover. There is another way to look at it.

Start with any Riemannian n-manifold M and extend the Riemannian metric to TM, a $2n$-manifold, as follows. Let $x \in M$ and let $v \in T_xM$: to describe an inner product on the tangent space T_vTM. There are horizontal and vertical subspaces, H and V respectively, of T_vTM. V is naturally identified with T_xM by the exponential map, so inherits an inner product. The natural projection $p : TM \to M$ identifies H with T_xM isomorphically, so induces an inner product. Finally it is stipulated that H and V should be orthogonal. For this Riemannian metric, if $f : M \to M$ is an isometry then so is $df : TM \to TM$. This property descends with the restriction of the Riemannian metric to the unit sphere bundle UM: this consists of all vectors of TM of length 1.

Apply this analysis to the case $M = \mathbb{H}^2$. The group of orientation preserving isometries of $\mathbb{H}^2$ was identified as $PSL(2, \mathbb{R})$ but we could have normalised further and asked for the determinant to be 1; call that group $PSL_2(\mathbb{R})$. It acts transitively on $U\mathbb{H}^2$ with trivial stabiliser, so there is a natural identification of $U\mathbb{H}^2$ with $PSL_2(\mathbb{R})$. It follows that there is an induced metric on $\widetilde{SL_2(\mathbb{R})}$ invariant under the action of $PSL_2(\mathbb{R})$, and hence under the action of $\widetilde{SL_2(\mathbb{R})}$.

As for $\mathrm{Isom}(\widetilde{SL_2(\mathbb{R})})$, there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(\widetilde{SL_2(\mathbb{R})}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2) \longrightarrow 1.$$

The last two geometries possess a special property (as do $\mathbb{E}^3$, $\mathbb{S}^3$ and $\widetilde{SL_2(\mathbb{R})}$)). A *Lie group* is a topological group $(M, \bullet, \mathcal{T})$ such that M is a smooth manifold with respect to which the group operations are smooth. Of course a Lie group is furnished with a group action, viz the group operation itself. This often gives a good way of specifying a Riemannian metric: specify it at a single point and then declare the group action to consist of isometries which will therefore carry the metric to all other points. Examples of Lie groups include $\mathbb{E}^n$, the unit reals, complex numbers and quaternions (respectively $\mathbb{S}^0$, $\mathbb{S}^1$ and $\mathbb{S}^3$) and the general linear group.

The geometry of $\boldsymbol{Nil}$**.** Nil is the Heisenberg group, ie the group of all matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ with $x, y, z \in \mathbb{R}$. It is a Lie group under matrix multiplication so we may specify the metric at a single point and then translate it to all other points. Note that

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \xi & \zeta \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+\xi & z+\zeta+x\eta \\ 0 & 1 & y+\eta \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that the matrix and the entries 0 and 1 are somewhat superfluous; we could think of this as a new multiplication on $\mathbb{R}^3$, viz

$$(x, y, z)(\xi, \eta, \zeta) = (x+\xi, y+\eta, z+\zeta+x\eta).$$

Taking $ds^2 = du^2 + dv^2 + dw^2$ at the origin we can then apply the multiplication to get

$$dx = du, \qquad dy = dv \qquad \text{and} \qquad dz = dw + xdv = dw + xdy,$$

Hence

$$\begin{aligned} ds^2 &= du^2 + dv^2 + dw^2 \\ &= dx^2 + dy^2 + (dz - xdy)^2. \end{aligned}$$

As for $\mathrm{Isom}(Nil)$, there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(Nil) \longrightarrow \mathrm{Isom}(\mathbb{E}^2) \longrightarrow 1.$$

$\mathrm{Isom}(Nil)$ has two components.

The geometry of $\boldsymbol{Sol}$**.** This geometry is determined by another Lie group structure on $\mathbb{R}^3$ given by

$$(x, y, z)(\xi, \eta, \zeta) = (x + e^{-z}\xi, y + e^{z}\eta, z + \zeta).$$

A similar calculation as for Nil gives $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$.

$\mathrm{Isom}(Sol)$ has eight components; the identity component being Sol itself acting on the left.

6 Thurston's Geometrisation Conjecture

In the 1980s Thurston proposed the ambitious Geometrisation Conjecture.

> Let M be an orientable prime closed 3-manifold. Then there is a disjoint union of 2-tori and Klein bottles in M such that every component of the complement is geometric.

The connected sum operation $\sharp$ may be used to decompose a manifold into simpler parts. This decomposition terminates; more precisely given any closed connected 3-manifold M there is a collection $\{M_1, \ldots, M_n\}$ of closed connected 3-manifolds such that

- $M \approx M_1 \sharp M_2 \sharp \ldots \sharp M_n$;
- for each i if $M_i \approx M_i' \sharp M_i''$ then either M_i' or M_i'' is homeomorphic to $\mathbb{S}^3$.

Note that $\mathbb{S}^3$ is the identity for $\sharp$. Manifolds of the sort M_i above are called *prime*. Kneser showed that each 3-manifold has a prime decomposition and Milnor that the prime decomposition is unique.

This Geometrisation Conjecture has lots of consequences for the classification of 3-manifolds. In particular it has as a corollary the Poincaré Conjecture. The idea of the proof of the corollary is this. Suppose that M is a closed simply connected 3-manifold. Decompose M into the connected sum of prime components $M_1, \ldots, M_n$. Each M_i is closed and simply connected. As $\pi(M_i) = 0$ there are no incompressible tori or Klein bottles so by the Geometrisation Conjecture M_i is already geometric. The only compact 3-dimensional geometry is $\mathbb{S}^3$, so $M \approx \mathbb{S}^3$ and hence $M \approx \mathbb{S}^3$ as well.

How does Perelman prove the Geometrisation Conjecture? Not surprisingly it is a long story beginning with a proposal in [2] involving a concept called Ricci flow. The article [3] gives a quick look at that.

The website [10] is a good place to start looking at what is going on.

Exercises

1. Show that the connected sum of two n-manifolds is an n-manifold.
2. Show that the area of a geodesic triangle in $\mathbb{S}^2$ with angles α, β and γ is $\alpha + \beta + \gamma - \pi$.
3. Use the formula $\kappa = -3 \lim_{r \to 0^+} \frac{d^2 R_r}{dr^2}$ to show that the curvature of $\mathbb{S}^2$ is 1.
4. Let $A \in O(3)$. Show that there is $x \in \mathbb{S}^2$ so that either $Ax = x$ or $Ax = -x$.
5. Show that the area of a geodesic triangle in $\mathbb{H}^2$ with angles α, β and γ is $\pi - (\alpha + \beta + \gamma)$.
6. Use the formula $\kappa = -3 \lim_{r \to 0^+} \frac{d^2 R_r}{dr^2}$ to show that the curvature of $\mathbb{H}^2$ is 1.
7. Identify a regular octagon in $\mathbb{H}^2$ each angle of which is $\frac{\pi}{4}$.
8. Given $A \in SO(3)$ and $\mathbf{b} \in \mathbb{E}^3$ identify the axis of the screw motion which $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ determines.
9. Prove that the quaternion $\frac{aw+b}{cw+d}$ really has no term in k when w does not and $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc = 1$. In other words show that the action of $PSL(2, \mathbb{C})$ on $\mathbb{H}^3$ by the Möbius transformation really does send $\mathbb{H}^3$ to $\mathbb{H}^3$. [Hint. It seems convenient to express elements of $\mathbb{Q}$ in the standard form $a + bj$ with $a, b \in \mathbb{C}$. Expansion of $(a + bj)(\bar{a} - bj)$ leads to a simple formula for $\frac{1}{a+bj}$. This allows $\frac{aw+b}{cw+d}$ to be expressed in the standard form with the coefficient of j being real. Be careful when commuting elements of $\mathbb{Q}$!]

References

[1] Alan Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, 91, Springer-Verlag, 1983.

[2] Richard Hamilton, *Three-manifolds with positive Ricci curvature*, Journal of Differential Geometry, 17 (1982), 255–306.

[3] Michael Kapovich, *Geometrization conjecture and the Ricci flow*, http://www.math.ucdavis.edu/%7Ekapovich/eprints.html.

[4] John Morgan, *Recent progress on the Poincaré conjecture and the classification of 3-manifolds*, Bull. Amer. Math. Soc., 42 (2005), 57–78.

[5] Henri Poincaré, *Second complément a l'analysis situs*, Proc. London Math. Soc., 32 (14/6/1900), 277–308.

[6] Colin Rourke and Ian Stewart, *Poincaré's perplexing problem*, New Scientist, 4/9/1986, 41–45.

[7] Peter Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc., 15 (1983), 401–487.

[8] Gary Taubes, *What happens when hubris meets nemesis*, Discover, July 1987, 66–77.

[9] William Thurston (ed. by Silvio Levy), *Three-dimensional geometry and topology*, Princeton Math. Series 35, Princeton University Press, 1997.

[10] *Notes and commentary on Perelman's Ricci flow papers*, http://www.math.lsa.umich.edu/research/ricciflow/perelman.html

[11] Various people, *Search the web for Poincare Conjecture or Geometrization Conjecture or Geometrisation Conjecture*, www.