MATHS750
Riemann integration from a topologist's viewpoint

First a bit about filters, which generalise sequences. A *filter* on a set X is a family $\mathcal{F}$ of subsets of X satisfying:

1. $\mathcal{F} \neq \varnothing$;
2. $\varnothing \notin \mathcal{F}$;
3. if $F, G \in \mathcal{F}$ then $F \cap G \in \mathcal{F}$;
4. if $F \in \mathcal{F}$ and $F \subset G \subset X$ then $G \in \mathcal{F}$.

Some examples of filters include

- $\{x\}^{\uparrow} = \{S \subset X \ / \ x \in S\}$ (the *principal filter* at x);
- $\{N \ / \ N$ is a neighbourhood of $x\}$ (the *neighbourhood filter* at x);
- $\{F \subset X \ / $ there is n_0 such that $x_n \in F$ for each $n \geq n_0\}$ (the *cofinite filter* of the sequence $\langle x_n \rangle$)

for $x \in X$ and a sequence $\langle x_n \rangle$ of points of X, with X requiring a topology in the second example.

A filter $\mathcal{F}$ *converges to* a point x of a topological space provided that $\mathcal{F}$ contains all neighbourhoods of x; write $\mathcal{F} \to x$. Note that the cofinite filter of a sequence $\langle x_n \rangle$ converges to x iff $x_n \to x$.

Often it is convenient to specify a filter by a base. A collection $\mathcal{B}$ of subsets of X is a *filterbase* provided:

1. $\mathcal{B} \neq \varnothing$;
2. $\varnothing \notin \mathcal{B}$;
3. if $B, C \in \mathcal{B}$ then there is $D \in \mathcal{B}$ such that $D \subset B \cap C$.

Given a filterbase $\mathcal{B}$ we can create the associated filter: $\{F \subset X \ / $ there is $B \in \mathcal{B}$ such that $B \subset F\}$.

The three filters described above have respective bases $\{\{x\}\}$, any neighbourhood base and $\{\{x_n \ / \ n \geq m\} \ / \ m$ is a positive integer$\}$.

Let $[a, b] \subset \mathbb{R}$. A *partition of* $[a, b]$ is a subset $P = \{x_0, \ldots, x_n\} \subset [a, b]$ such that $a = x_0 < \ldots < x_n = b$. Write $P = \{x_0 < \ldots < x_n\}$. Denote by $\mathcal{P}$ the collection of all partitions on $[a, b]$.

- Inclusion $\subset$ defines a partial order on $\mathcal{P}$.
- If $P, Q \in \mathcal{P}$ then $P \cup Q \in \mathcal{P}$ and $P \subset P \cup Q$ and $Q \subset P \cup Q$.

These two points together tell us that $(\mathcal{P}, \subset)$ is what is called a *directed set*: a partially ordered set in which each two elements are bounded above by a common element. The most commonly used directed set is the set of integers with the usual order and it gives rise to sequences. Directed sets give rise to a generalisation of sequences called *nets*. Convergence in a general topological space may be handled equally satisfactorily by filters or nets: I'm afraid that patriotism has got in the way a bit!

Given a partition $P = \{x_0 < \ldots < x_n\}$ of $[a, b]$, a *selection for* P is an ordered n-tuple $\langle \xi_1, \ldots \xi_n \rangle$ such that $\xi_i \in [x_{i-1}, x_i]$ for each i.

Now suppose that $f : [a, b] \to \mathbb{R}$. Given a selection $S = \langle \xi_1, \ldots \xi_n \rangle$ for a partition P of $[a, b]$ let $r_f(P, S) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, the Riemann sum for f, P and S. Again for $P \in \mathcal{P}$ let

$$F_f(P) = \{r_f(Q, T) \ / \ Q \in \mathcal{P}, P \subset Q \text{ and } T \text{ is a selection for } Q\}.$$

Note that if $P \subset Q$ then $F_f(Q) \subset F_f(P)$. From this it follows that $\{F_f(P) \ / \ P \in \mathcal{P}\}$ is a filterbase, say for the filter $\mathcal{F}_f$ on $\mathbb{R}$.

The function f is *Riemann integrable on* $[a, b]$ iff the filter $\mathcal{F}_f$ converges. Further, $\mathcal{F}_f \to \int_a^b f$.