

Please submit your solutions in class.

Please show all working.

1. You have seen in Lecture 13 that the ring  $\mathbb{Z}[i]$  is a principal ideal domain (PID). (Equivalently the class number is 1). For each of the ideals  $I = (2, 1 + i)$ ,  $(111, 5 + 7i)$ ,  $(3, 1 + i)$  find an  $\alpha \in \mathbb{Z}[i]$  such that  $I = (\alpha)$ .
2. The ring  $R = \mathbb{Z}[\sqrt{-17}]$  is not a PID. Let  $I = (2, 1 + \sqrt{-17})$  and  $J = (3, 1 + \sqrt{-17})$  be ideals. Show that  $I^2$  is a principal ideal and  $J^2$  is not a principal ideal. (In the language of ideal class groups: the class number of  $\mathbb{Q}(\sqrt{-17})$  is 4,  $I$  has order 2,  $J$  has order 4.)
3. Compute the continued fraction expansion of  $\sqrt{41}$ . Hence determine the smallest positive solutions to  $x^2 - 41y^2 = -1$  and  $x^2 - 41y^2 = 1$ .
4. Prove that the convergents of the continued fraction expansion of  $[1; \bar{1}]$  are ratios of successive Fibonacci numbers  $F_n$ . (It therefore follows that  $\lim_{n \rightarrow \infty} F_{n+1}/F_n$  is the golden ratio.)
5. Let  $n \in \mathbb{N}$ . Prove that the continued fraction expansion of  $\sqrt{n^2 + 1}$  is  $[n; \overline{2n}]$ . Hence give a general formula for the first two non-trivial solutions to  $x^2 - (n^2 + 1)y^2 = 1$ .
6. Prove that the rational solutions to the equation  $x^2 - dy^2 = 1$  are given by

$$\left( \pm \frac{1 + dt^2}{1 - dt^2}, \pm \frac{2t}{1 - dt^2} \right)$$

where  $t \in \mathbb{Q}$ .

[Hint: Consider a line of slope  $t \in \mathbb{Q}$  through  $(1, 0)$ . Also, don't forget to prove that every rational solution arises in this way.]

7. We all know what a square number is:  $n^2$ . There are also *triangular numbers*:  $\frac{1}{2}m(m + 1)$ . These are just the sum of all the positive integers up to  $m$  (and if you write down all the numbers in terms of dots you will see why these might have been called triangular numbers).

Show that there are infinitely many square numbers which are also triangular numbers. Find the first three positive "square and triangular" numbers

[Hint: Transform the problem into a Pell equation. The convergents of this Pell equation have been computed in the notes, so you do not need to recompute them.]