

Please submit your solutions in class.

Please show all working.

1. Calculate the Legendre symbol  $\left(\frac{1234}{89}\right)$ .
2. (a) Use Euler's criterion to check whether 7 is a quadratic residue modulo 23.  
(b) Use Gauss's Lemma to check whether 7 is a quadratic residue modulo 23.  
(c) Use quadratic reciprocity to calculate  $\left(\frac{7}{23}\right)$ .
3. For which odd primes  $p$  is 7 a quadratic residue mod  $p$ .
4. Find the square roots of 7 modulo 513.
5. (a) Show that for an odd prime  $p$  there are as many square residue classes modulo  $p$  as non-square residue classes in  $U(\mathbb{Z}_{p^e})$ ,  $p \geq 1$ .  
(b) How many squares and how many non-squares are there in  $U(\mathbb{Z}_{2^e})$ ?
6. Let  $p > 3$  be a prime. Prove that the product of all primitive roots mod  $p$  is congruent to 1 modulo  $p$ .
7. (a) Prove that 2 is irreducible in the ring of integers of the quadratic field  $\mathbb{Q}(\sqrt{-5})$ .  
(b) Prove that  $1 + \sqrt{-5}$  is irreducible in the ring of integers of the quadratic field  $\mathbb{Q}(\sqrt{-5})$ .
8. Prove that for any  $n \geq 3$  the numbers  $\alpha_n = \cos(2\pi/n)$  are algebraic numbers but not algebraic integers. Determine the smallest  $b$  for which  $b\alpha_n$  are algebraic integers for all  $n \geq 3$ .