1. Find all integer solutions of the equation $987x + 567y = 63$.

$$\begin{array}{rrr} 987 & 1 & 0 \\ 567 & 0 & 1 \\ 420 & 1 & -1 \\ 147 & -1 & 2 \\ 126 & 3 & -5 \\ 21 & -4 & 7 \\ 0 & 27 & -47 \end{array}$$

We see that $(u, v) = (-4, 7)$ is a solution of $987u + 567v = 21 (= \gcd(987, 567))$. Multiplying by 3, we find that $(x_0, y_0) = (-12, 21)$ is a solution of our equation, and all solutions are given by

$$(x, y) = (-12, 21) + (27, -47)t, \quad t \in \mathbb{Z}.$$

2. Use the fact that $1001 = 7 * 11 * 13$ to state a criterion for divisibility by 7, or 11, or 13 (similar to a well-known criterion for divisibility by 3 or 9).

Let $n \in \mathbb{N}$, and write n in base 1000 (i.e. break the decimal representation of n into parcels of 3 digits, starting from the right):

$$n = a_0 1000^0 + a_1 1000^1 + a_2 1000^2 + \cdots .$$

Since $1000 \equiv -1 \mod k$ (where $k = 7, 11, 13$), we see that $n \equiv a_0 - a_1 + a_2 - a_3 \pm \cdots$, and in particular n is divisible by k if and only if this alternating sum is divisible by k.

E.g. $n = 22123453$ is not divisible by 7 or 13, but divisible by 11 (look at $123 - 453 + 22 = 352$).

3. Find all solutions of the congruence $x^3 + x + 2 \equiv 0 \mod 140$.

Factorize: $140 = 4 * 5 * 7$. Now find all x which satisfy all three congruences: $x^3 + x + 2 \equiv 0 \mod 4$, $x^3 + x + 2 \equiv 0 \mod 5$, $x^3 + x + 2 \equiv 0 \mod 7$.

We find $x \equiv 1, 2, 3 \mod 4$, $x \equiv 4 \mod 5$, $x \equiv 4, 6 \mod 7$. Using the Chinese Remainder Theorem we find $x \equiv 34, 39, 69, 74, 109, 139 \mod 140$.

4. Let $p > 4$ be a prime, $n \in \mathbb{N}$, and assume that $\frac{2}{3}n < p \leq n$. Prove that p does not divide the binomial coefficient $\binom{2n}{n}$.

For prime numbers in the interval $(\frac{2}{3}n, p]$, we have $n < 2p \leq 2n$, showing that such prime factor occurs exactly once in the numerator and in the denominator. Consequently $\binom{2n}{n}$ is not divisible by p.

5. Find the number of positive divisors of 20! (that's 20 factorial).

Calculate the sums $\sum_{i \in \mathbb{N}} \lfloor \frac{20}{p^i} \rfloor$ for all primes (well, need only look at the primes less than 20, and of those only the ones less than 10 need a calculation):
the highest power of 2 dividing 20 is $10 + 5 + \lfloor 2.5 \rfloor + \lfloor 1.25 \rfloor = 18$;
the highest power of 3 dividing 20 is $\lfloor 6.7 \rfloor + \lfloor 2.3 \rfloor = 8$;

the highest power of 5 dividing 20 is 4;
the highest power of 7 dividing 20 is 2;
the highest power of 11, 13, 17, 19 dividing 20 is 1.

Therefore 20 is a product of 8 distinct primes raised to powers 18, 8, 4, 2, 1, 1, 1, 1 respectively. The number of positive divisors of 20 is thus $19 \cdot 9 \cdot 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 41040$.

$(20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 11 \cdot 13 \cdot 17 \cdot 19.)$

6. Let n be a perfect number, i.e. n is a positive integer such that the sum of all positive divisors of n equals $2n$. Prove that

$$\sum_{d|n} \frac{1}{d} = 2.$$

Note that $\sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} \frac{1}{d}$. It follows that $\sum_{d|n} = 2n/n = 2$.

7. From Euclid's proof that there are infinitely many primes, deduce that the n-th prime is less than 2^{2^n} (induction might help).

The statement is obviously true for the first prime ($2 \leq 2^2$). If p_k denotes the k-th prime, and we assume the result for $n = k$, then Euclid's idea says that the product $u = p_1 p_2 \cdot \cdots \cdot p_k + 1$ contains a further prime factor, and therefore $p_{k+1} \leq u < 2 \cdot 2^2 \cdot \cdots \cdot 2^k + 1 = 2^{2+2^2+\cdots+2^k} + 1 = 2^{2^{k+1}-1} + 1 < 2^{2^{k+1}}$. So the statement is also true for $n = k + 1$.

8. For $n \in \mathbb{N}$, define $d(n)$ to be the number of positive divisors of n. Prove that d is a multiplicative function, i.e. if $\gcd(m, n) = 1$ then $d(mn) = d(m)d(n)$.

If m is a product of r distinct primes p_i, each raised to power α_i, then the number of positive divisors of m is $d(m) = \prod(\alpha_i + 1)$. If n is a product of s distinct primes q_i, each raised to power β_i, then the number of positive divisors of n is $d(n) = \prod(\beta_i + 1)$. If m and n are relatively prime, then the prime factors of m and n are distinct, so mn is a product of $r + s$ distinct primes p_i or q_i, each raised to power α_i or β_i, and the number of positive divisors of mn is $d(mn) = \prod(\alpha_i + 1) \prod(\beta_i + 1) = d(m)d(n)$.

9. Show that there are infinitely primes of the form $4n + 3$ ($n \in \mathbb{N}$). (Dirichlet's theorem guarantees that — but find a more elementary approach.)

(A variation on Euclid's proof that there are infinitely many primes.) Assume that p is the largest prime of the form $4n+3$ and form the product of all primes up to p, then put $N = 4 \cdot 3 \cdot 5 \cdot \cdots \cdot p - 1$. Then $N \equiv 3 \pmod 4$, and at least one of the prime factors of N, say q, must be $\equiv 3 \pmod 4$ (since the product of numbers congruent to 1 mod 4 is again congruent to 1 mod 4). q cannot be any of the primes $\leq p$, so there is a prime of the required form which is $> p$, a contradiction.

10. True or false: $x^2 - x + 41$ is prime for all $x \in \mathbb{N}$.

(Just a joke exercise. This polynomial is associated with the name of Euler, producing prime numbers for all values of x that are < 41.)

False: e.g. try $x = 41$, then $41 | x^2 - x + 41$.