Please submit your solutions in class.

Please show all working.

1. Find all integer solutions of the equation $987x + 567y = 63$.

2. Use the fact that $1001 = 7 * 11 * 13$ to state a criterion for divisibility by 7, or 11, or 13 (similar to a well-known criterion for divisibility by 3 or 9).

3. Find all solutions of the congruence $x^3 + x + 2 \equiv 0 \mod 140$.

4. Let $p > 4$ be a prime, $n \in \mathbb{N}$, and assume that $\frac{2}{3}n < p \leq n$. Prove that p does not divide the binomial coefficient $\binom{2n}{n}$.

5. Find the number of positive divisors of 20! (that's 20 factorial).

6. Let n be a perfect number, i.e. n is a positive integer such that the sum of all positive divisors of n equals $2n$. Prove that

$$\sum_{d|n} \frac{1}{d} = 2.$$

7. From Euclid's proof that there are infinitely many primes, deduce that the n-th prime is less than 2^{2^n} (induction might help).

8. For $n \in \mathbb{N}$, define $d(n)$ to be the number of positive divisors of n. Prove that d is a multiplicative function, i.e. if $\gcd(m, n) = 1$ then $d(mn) = d(m)d(n)$.

9. Show that there are infinitely primes of the form $4n + 3$ ($n \in \mathbb{N}$). (Dirichlet's theorem guarantees that — but find a more elementary approach.)

10. True or false: $x^2 - x + 41$ is prime for all $x \in \mathbb{N}$.