32.1. Recall: Let $d \in \mathbb{Z}$ be a square-free integer. Let $F = \mathbb{Q}(\sqrt{d})$ with discriminant Δ and ring of integers D. The ideal class group C_F is the multiplicative group of non-zero fractional ideals modulo principal fractional ideals. The class number $h_d = h_F$ is the order of C_F.

Lemma 32.2. Let I and J be ideals. Then $N(IJ) = N(I)N(J)$. (See Stewart and Tall pages 116-118.)

Theorem 32.3. (Minkowski) Every ideal class in F contains an ideal I such that

$$N(I) \leq c\sqrt{|\Delta|}$$

where $c = 1/2$ if $d > 0$ and $c = 2/\pi$ if $d < 0$.

Example 32.4. The class number of $\mathbb{Q}(\sqrt{-7})$ is 1.
Minkowski implies every ideal class contains an ideal I such that $N(I) \leq 2\sqrt{7}/\pi \approx 1.684$. Since $N_F(I) = 1$ implies $I = (1)$ it follows that the ideal class group is trivial.

Exercise 32.5. The class number of $\mathbb{Q}(i)$ is 1.

Example 32.6. The class number of $\mathbb{Q}(\sqrt{-5})$ is 2.
Every ideal class contains an ideal I such that $N(I) \leq 2\sqrt{20}/\pi \approx 2.85$. So we need to check whether there are any ideals of norm 2. Indeed, by Dedekind, since $x^2 + 5 \equiv (x+1)^2 \mod 2$ there is a unique ideal $I = (2, 1 + \sqrt{-5})$ with $N(I) = 2$. Now I cannot be principal since the norm equation $N(u + v\sqrt{-5}) = u^2 + 5v^2 = 2$ has no solution for $u, v \in \mathbb{Z}$ (by considering the equation modulo 5). Hence there are two ideal classes.

Exercise 32.7. Determine the class number of $\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-10}), \mathbb{Q}(\sqrt{10})$.

Gauss conjecture:

(1) As $d \to -\infty$, $\liminf h_d = \infty$. (Proved by Heilbronn)
(2) There are infinitely many square-free $d > 0$ such that $\mathbb{Q}(\sqrt{d})$ has class number one. (Still open)

Theorem: (Heegner, Stark, Baker) Let $d < 0$ be square free. Then $\mathbb{Q}(\sqrt{d})$ has class number one if and only if $d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$.

Example 32.8. The only integer solutions to $x^3 = y^2 + 54$ are $(x, y) = (7, \pm 17)$.
Factor the right hand side as $(y + 3\sqrt{-6})(y - 3\sqrt{-6})$. The first step is to show that any solution $x, y \in \mathbb{Z}$ has $2 \nmid xy$ and $3 \nmid xy$. Then show that the two factors are coprime in $\mathbb{Z}[\sqrt{-6}]$.
The crucial step is to consider the equation $(x)^3 = (y+3\sqrt{-6})(y-3\sqrt{-6})$ as ideals. It follows that $(y + 3\sqrt{-6}) = I^3$ for some ideal I of $\mathbb{Z}[\sqrt{-6}]$. In other words, I has order dividing 3 in the ideal class group. But the ideal class group has order 2, hence I must be principal.
Finally, since $I = (a + b\sqrt{-6})$ it follows that $y + 3\sqrt{-6} = \pm((a^3 - 18ab^2) + \sqrt{-6}(3a^2 b - 6b^3))$. Equating coefficients gives the result.

Example 32.9. Show that $x^3 = y^2 + 10$ has no solutions $x, y \in \mathbb{Z}$.