Let $d > 0$ be a square-free integer. We want to solve the equation $x^2 - dy^2 = \pm 1$ over $\mathbb{Z}$. A solution (x, y) is called positive if $x > 0$ and $y > 0$.

Lemma 23.1. Let the convergents to $\sqrt{d}$ be h_i/k_i for $i \geq 0$. Let q_i be as in Lemma 22.3. Then

$$h_i^2 - dk_i^2 = (-1)^{i+1} q_{i+1}$$

for all $i \geq 0$.

Lemma 23.2. Let r be the period of the continued fraction expansion of $\sqrt{d}$. Then $q_{ir} = 1$ for all $i \in \mathbb{N}$.

Theorem 23.3. Let the convergents to $\sqrt{d}$ be h_i/k_i. Let $N \in \mathbb{Z}$ be such that $|N| < \sqrt{d}$. Suppose $x, y \in \mathbb{N}$ satisfy

$$x^2 - dy^2 = N$$

and are such that $\gcd(x, y) = 1$. Then $(x, y) = (h_i, k_i)$ for some $i \geq 0$.

Theorem 23.4. Let $d > 0$ be a square-free integer. Let r be the period of the continued fraction expansion of $\sqrt{d}$.
If r is even then $x^2 - dy^2 = -1$ has no solutions and all positive solutions to $x^2 - dy^2 = 1$ are given by $(x, y) = (h_{ir-1}, k_{ir-1})$ for $i \in \mathbb{N}$.
If r is odd the all positive solutions to $x^2 - dy^2 = -1$ are given by $(x, y) = (h_{ir-1}, k_{ir-1})$ for odd $i \in \mathbb{N}$ and all positive solutions to $x^2 - dy^2 = 1$ are given by $(x, y) = (h_{ir-1}, k_{ir-1})$ for even $i \in \mathbb{N}$.

Example 23.5. Solve $x^2 - 2y^2 = 1$.
Since $\sqrt{2} = [1; \overline{2}]$ has period equal to one (odd) we must consider the convergents h_{2l-1}/k_{2l-1}. The positive solutions are

$$(x, y) = (3, 2), (17, 12), (99, 70), \ldots$$

Example 23.6. Solve $x^2 - 3y^2 = 1$.
The continued fraction expansion of $\sqrt{3}$ is $[1; \overline{1, 2}]$ which has period 2 (even). Hence the solutions come from the convergents h_{2l-1}/k_{2l-1}. The positive solutions are

$$(x, y) = (2, 1), (7, 4), (26, 15), \ldots$$

Exercise 23.7. Compute the first three positive solutions to $x^2 - 2y^2 = -1$.

Exercise 23.7. Compute the first positive solutions to $x^2 - 14y^2 = 1$ and $x^2 - 29y^2 = -1$.

Exercise 23.8. Let $p \equiv 1 \mod 4$ be prime. Show that if $u^2 - pv^2 = 1$ then v is even and that $v = 2xy$ where $2x^2 \mid (u + 1)$ and $2y^2 \mid (u - 1)$. Letting (u, v) be such that $v > 0$ is minimal deduce that $x^2 - py^2 = -1$ has an integer solution. Hence deduce that the period of the continued fraction of $\sqrt{p}$ is odd.

Lemma 23.9. Let $d \in \mathbb{N}$ be square-free. Let (u, v) be the smallest positive solution to Pell's equation $x^2 - dy^2 = 1$. Suppose $x_0^2 - dy_0^2 = N$ for some $N \in \mathbb{Z}$. Then there are infinitely many solutions (x_i, y_i) to $x^2 - dy^2 = N$.