We discuss a representation of real numbers (which was known to 6th century Indian mathematicians) and apply it to two questions:

(1) Given an irrational number $\alpha \in \mathbb{R}$ how do you find fractions a/b which are "close" to α?
(2) Given a positive integer d, how do you find a non-trivial integer solution to the equation $x^2 - dy^2 = 1$?

20.1 Continued Fractions Let $\alpha \in \mathbb{R}$.

(1) Let $a_0 = \lfloor \alpha \rfloor$ (this is clearly a good first approximation to α).
(2) Let $\beta_0 = \alpha - a_0$. Then $0 \leq \beta_0 < 1$. If $\beta_0 = 0$ then stop.
(3) Then $1/\beta_0 \in \mathbb{R}$ and can be approximated by $a_1 = \lfloor 1/\beta_0 \rfloor$. Repeat.

In general we have the formulae

$$a_{i+1} = \lfloor 1/\beta_i \rfloor \quad \text{and} \quad \beta_{i+1} = 1/\beta_i - a_{i+1}.$$

The process can be continued so long as $\beta_i \neq 0$. We can write the resulting fraction as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{a_i + \beta_i}}}}$$

This is called the *simple continued fraction* expansion of α (we usually drop the word "simple" as we will never meet the more general ones in the course). We will show later that it makes sense to say this is equal to α.
If $\beta_m = 0$ for some i then the sequence a_i stops at a_m and it is called a *finite (simple) continued fraction.*

Example 20.2 Take $\alpha = 3/7$.
Example 20.3 Take $\alpha = \sqrt{2}$. Then

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Exercise 20.4. Compute the continued fraction expansions of $11/13$, $\sqrt{6}$, $\sqrt{7}$ and the *golden ratio* $(1 + \sqrt{5})/2$.

Warning: Find the continued fraction of a quadratic irrational using algebra and not a calculator if you want to get the full marks!

Notation 20.5 The fraction notation for continued fractions is messy. So we denote them by

$$[a_0; a_1, a_2, a_3, \dots].$$

We write finite continued fractions as a finite sequence, and periodic sequences with an overline. For example, $\sqrt{2} = [1; 2, 2, 2, 2, \dots] = [1; \overline{2}]$.

Definition 20.6. The fractions $h_0/k_0, h_1/k_1, \dots$ which are defined by truncating the continued fraction expansion are called the *convergents.*

For example, the convergents of $\sqrt{2}$ are

$$1, \ \ 1 + \frac{1}{2} = \frac{3}{2}, \ \ 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}$$

and $17/12, 41/29, 99/70, \ldots$.

Compare their decimal expansions to 6 figures:

1	3/2	7/5	17/12	41/29	99/70
1.00000	1.50000	1.40000	1.41666	1.41379	1.41429

Note that $99/70 = 297/210$ and A4 paper is 297mm by 210mm in size.

Lemma 20.7. Let $a_0, a_1, \ldots$ be a sequence of rational numbers. Define the sequences h_m, k_m by

$$\begin{aligned} h_0 &= a_0, & k_0 &= 1, \\ h_1 &= a_1 a_0 + 1, & k_1 &= a_1, \\ h_{m+1} &= a_{m+1} h_m + h_{m-1} \\ k_{m+1} &= a_{m+1} k_m + k_{m-1}. \end{aligned}$$

Then the finite fraction $[a_0; a_1, \ldots, a_m]$ be equal to h_m/k_m.

Note 20.8. If $a_i \in \mathbb{Z}$ for all $i \geq 0$ then $h_m, k_m \in \mathbb{Z}$ for all m.

Lemma 20.9. For $m \geq 1$

$$h_m k_{m-1} - h_{m-1} k_m = (-1)^{m-1}.$$

Corollary 20.10. If $a_i \in \mathbb{Z}$ for all $i \geq 0$ then $\gcd(h_m, k_m) = 1$ for all $m \geq 0$ and so h_m/k_m is the rational number $[a_0; a_1, \ldots, a_m] = h_m/k_m$ in reduced form.

Exercise 20.11. Let $[a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha \in \mathbb{R}$. Prove that if $i \geq 1$ then $a_i \geq 1$.

Note 20.12. Finite continued fractions are not quite unique, since

$$\frac{1}{a} = \frac{1}{a - 1 + \frac{1}{1}}.$$

For example, $3/7 = [0; 2, 3]$ and also $3/7 = [0; 2, 2, 1]$.

Theorem 20.10. Let $\alpha, \beta \in \mathbb{R}$. Suppose they have continued fraction expansions $\alpha = [a_0; a_1, a_2, \ldots]$ and $\beta = [b_0; b_1, b_2, \ldots]$.

(1) If $\alpha \neq \beta$ then there is some integer i such that $a_i \neq b_i$.
(2) Suppose there is some integer i such that $a_i \neq b_i$. Then either $\alpha \neq \beta$ or both continued fractions are finite.