$\pi(x)$ is the number of primes $\leq x$.

The celebrated *Prime Number Theorem* states that $\pi(x)$ grows asymptotically exactly as fast as $f(x) = x/\log x$, that is, $\pi(x)/f(x) \to 1$ when $x \to \infty$. (Conjectured by Gauss, proved independently by J. Hadamard and C. de la Vallée Poussin in 1896, 'elementary' proof given independently by P. Erdös and A. Selberg in 1948.) We will prove a weaker form of the Prime Number Theorem, due to Chebychev.

Lemma 1. *For any prime p, the exponent of p in the prime factorisation of n! is* $\sum_{j\geq 1}\lfloor n/p^j \rfloor$.

Lemma 2. *For any* $n \geq 1$ *we have* $2^n \leq \binom{2n}{n} < 2^{2n}$.

Lemma 3. *Let* $f(x) = x/\log x$. *Then,*

(a) $f(x)$ *is increasing for* $x \geq e$,

(b) $f(x-2) > f(x)/2$ *for* $x \geq 4$,

(c) $f((x+2)/2) < (15/16) \cdot f(x)$ *for* $x \geq 8$.

Theorem 4 (Chebyshev)**.** *If* $x \geq 8$, *then* $c_1 x/\log x < \pi(x) < c_2 x/\log x$ *where* $c_1 = (\log 2)/4$ *and* $c_2 = 30(\log 2)$.

Proof. For each prime p let e_p be defined by $p^{e_p} \leq 2n < p^{e_p+1}$. Let $Q_n = \prod_p p^{e_p}$. By Lemma 1, the exponent of a prime p in the prime factorisation of $B_n = \binom{2n}{n}$ is equal to $\sum_{j=1}^{e_p}(\lfloor 2n/p^j \rfloor - 2\lfloor n/p^j \rfloor)$. Since $0 \leq \lfloor 2y \rfloor - 2\lfloor y \rfloor \leq 1$ for any positive real y, the exponent of p in the prime factorisation of B_n is $\leq e_p$. It follows that $B_n | Q_n$, and hence, using Lemma 2, we have $2^n \leq B_n \leq Q_n \leq (2n)^{\pi(2n)}$. Taking logarithms and letting $c_1 = (\log 2)/4$, we arrive at

$$n \log 2 \leq \pi(2n) \log (2n), \qquad \text{that is,} \qquad \pi(2n) \geq 2c_1 \cdot f(2n).$$

Let $x \geq 5$ and let $n = \lfloor x/2 \rfloor$, so that $x \geq 2n > x - 2$. With the help of (a) and (b) of Lemma 3, the above inequality yields

$$\pi(x) \geq \pi(2n) \geq 2c_1 \cdot f(2n) > 2c_1 \cdot f(x-2) > c_1 \cdot f(x) = c_1 \cdot x/\log x \ .$$

For the other part, let P_n be the product of all primes p such that $n < p \leq 2n$. Then, $P_n | B_n$ and, by Lemma 2, $n^{\pi(2n)-\pi(n)} < P_n < B_n < 2^{2n}$. Taking logarithms and letting $C = 32 \log 2$, we obtain

$$(\pi(2n) - \pi(n)) \log n < 2n \log 2 \ , \quad \text{that is,} \quad \pi(2n) < (C/16) \cdot f(n) + \pi(n) \ .$$

We prove by induction that $\pi(2n) < C \cdot f(n)$ for $n \geq 2$. Induction step: Assume that this holds for integers $< n$, where $n \geq 8$, and let $k = \lfloor (n+2)/2 \rfloor$. Then, $n < 2k$, and using parts (a) and (c) of Lemma 3 we obtain $\pi(n) \leq \pi(2k) < C \cdot f(k) \leq C \cdot f((n+2)/2) < (15C/16) \cdot f(n)$. The following computation completes the induction step:

$$\pi(2n) < (C/16) \cdot f(n) + \pi(n) < (C/16) \cdot f(n) + (15C/16) \cdot f(n) = C \cdot f(n) \ .$$

Finally, for any $x \geq 8$, from Lemma 3 and from $\pi(2n) < Cf(n)$ applied to $n = \lfloor x/2 \rfloor + 1$ we have

$$\pi(x) < \pi(2n) < C \cdot f(n) \leq C \cdot f((x+2)/2) < (15C/16) \cdot f(x) = c_2 \cdot f(x) = c_2 \cdot x/\log x.$$

□

Exercise 1. Prove that there exists a prime between x and $125x$ for any sufficiently large x.