1. Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^n$ a C^1-function such that $f'(x)$ is invertible for all $x \in U$. Show that $x \mapsto \|f(x)\|$ does not have a maximum.

2. Let $U, V \subset \mathbb{R}^n$ open and $f\colon U \to V$ a bijection.

(a) Prove or disprove: If f is differentiable then $f'(x)$ is invertible for all $x \in U$.

(b) Prove or disprove: If both f and its inverse are differentiable then $f'(x)$ is invertible for all $x \in U$.

3. (a) Prove that there exists an open set $U \subset \mathbb{R}^2$ with $(0,1) \in U$, and C^2-functions $u, v\colon U \to \mathbb{R}$, such that $u(0,1) = v(0,1) = 0$ and

$$\begin{cases} u(x,y)\,v(x,y) + 2\sin u(x,y) = x, \\ 1 - 3v(x,y)\cos u(x,y) = y, \end{cases} \qquad \text{for all } (x,y) \in U.$$

(Hint. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(p,q) = (p\,q + 2\sin p, 1 - 3q\cos p)$.)

(b) Calculate the partial derivatives $(D_1 u)(0,1)$ and $(D_1 v)(0,1)$.

(c) For fun. Calculate the second-order partial derivative $(D_1 D_2 u)(0,1)$.

4. One might hope that one can weaken the assumptions in the inverse function theorem by requiring that the function f is merely differentiable instead of C^1 (and then that the inverse function g is also differentiable). Show that such a theorem does not exist.
(Hint. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and note that $f'(0) \neq 0$. Show that for all $\varepsilon > 0$ the restriction of f to the interval $(-\varepsilon, \varepsilon)$ is not injective. Thus f does not have a local inverse.)