1. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x\,y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is not differentiable.

2. Give an example of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is not continuous at $(0,0)$, but for which the directional derivative $(D_v f)(0,0)$ exists for all $v \in \mathbb{R}^2$.

3. (a) Let $a \in \mathbb{R}^n$, $\delta > 0$ and $f\colon B_\delta(a) \to \mathbb{R}$. Suppose the partial derivatives of f exist and $D_j f = 0$ for all $j \in \{1,\ldots,n\}$. Prove that f is constant.

(b) Let $U \subset \mathbb{R}^n$ open and connected. Let $f\colon U \to \mathbb{R}$. Suppose the partial derivatives of f exist and $D_j f = 0$ for all $j \in \{1,\ldots,n\}$. Prove that f is constant.

(c) Give an example of an open set $U \subset \mathbb{R}^n$ and a non-constant C^∞-function $f\colon U \to \mathbb{R}$ such that $D_j f = 0$ for all $j \in \{1,\ldots,n\}$.

4. Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$ and $f\colon U \to \mathbb{R}$ a function which is differentiable at x_0. Suppose that f has a maximum at x_0. Prove that $f'(x_0) = 0$ and $(\nabla f)(x_0) = 0$.

5. (a) Does there exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $(D_1 f)(x,y) = x^2$ and $(D_2 f)(x,y) = y^2$ for all $(x,y) \in \mathbb{R}^2$?

(b) Does there exists a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $(D_1 f)(x,y) = y^2$ and $(D_2 f)(x,y) = x^2$ for all $(x,y) \in \mathbb{R}^2$?

6. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x\,y\,\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that the partial derivatives $(D_1 D_2 f)(0,0)$ and $(D_2 D_1 f)(0,0)$ exist and calculate them.