**1.** Let $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } |x_2| \leq x_1^2\}$. Define $f\colon U \to \mathbb{R}$ by $f(x_1, x_2) = 0$. Define $x_0 = (0, 0)$. Define the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}$ by $L(h_1, h_2) = h_2$.
Prove that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $h \in U$ with $0 < \|h\| < \delta$ it follows that

$$\frac{|f(x_0 + h) - f(x_0) - L(h)|}{\|h\|} < \varepsilon.$$

**2.** Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$ and $f, g\colon U \to \mathbb{R}^m$ differentiable at $x_0$. Define $F\colon U \to \mathbb{R}$ by

$$F(x) = \langle f(x), g(x) \rangle.$$

Prove that $F$ is differentiable at $x_0$.

**3.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in \mathbb{R}^n$. Prove that $f$ is constant.
(Remark and warning: there is no mean value theorem for functions of two or more variables.)

**4.** Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \dfrac{x\,y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that for every differentiable function $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ the composition $f \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is differentiable.
(Remark. Next week you have to show that $f$ is *not* differentiable.)

**5.** For fun.

(a) Let $a_1, a_2, \ldots \in \mathbb{R}^n$, $a \in \mathbb{R}^n$ and suppose that $\lim_{k \to \infty} a_k = a$. Show that there exists a continuous map $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(\frac{1}{k}) = a_k$ for all $k \in \mathbb{N}$.

(b) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function. Prove that $f$ is continuous if and only if for every continuous $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ the composition $f \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is continuous.