1. Show that the ball $B_r(a)$ in $\mathbb{R}^n$ with the Euclidean metric contains the cube

$$[a_1 - \varepsilon, a_1 + \varepsilon] \times \cdots \times [a_n - \varepsilon, a_n + \varepsilon], \qquad \varepsilon < \frac{r}{\sqrt{n}},$$

and hence show that bounded subsets of Euclidean space are totally bounded.

2. Let $(X, \|\cdot\|)$ be a real or complex finite dimensional normed linear space. A linear map

$$\lambda : X \to \mathbb{R} \quad \text{(or to } \mathbb{C}\text{)}$$

is called a linear functional.

(a) Show that if $f_n \to f$ in norm metric, then $\lambda(f_n) \to \lambda(f)$ for all linear functionals on X.

(b) Let $\lambda_1, \ldots, \lambda_d$ be the coordinate functionals for some basis $\{v_1, \ldots, v_d\}$ of X, i.e.,

$$f = \lambda_1(f)v_1 + \cdots \lambda_d(f)v_d, \qquad \forall f \in X.$$

Show that if $\lambda_j(f_n) \to \lambda_j(f)$ for all j, then $f_n \to f$.

Remark. It can be shown that the vector space X' consisting of all linear functionals on X has dimension d, with a basis given by the coordinate functionals of some basis for X, and moreover any basis of X' is the coordinate functionals of some basis for X.

3. Let Y be finite dimensional subspace of a normed linear space X, and for a nonempty subset $A \subset X$ define

$$\operatorname{dist}(x, A) := \inf_{a \in A} \|x - a\|, \qquad x \in X.$$

Fix $f \in X$.

(a) Show there exists a closed ball K in Y such that

$$\operatorname{dist}(f, Y) = \operatorname{dist}(f, K).$$

(b) Show there exists a $y_0 \in Y$, such that

$$\operatorname{dist}(f, Y) = \|f - y_0\|.$$

(c) Let Π_n be the vector space of all polynomials of degree $\leq n$. Show that there is a polynomial $p_n \in \Pi_n$ which best approximates $f \in C[a, b]$ (in the uniform norm).

Remark. It can be shown that $\operatorname{dist}(f, \Pi_n) \to 0$, which is known as the Weierstrass density theorem.

(d) Find all best approximations to $(2, 0)$ from the closed unit ball in $(\mathbb{R}^2, \|\cdot\|_2)$ and in $(\mathbb{R}^2, \|\cdot\|_\infty)$.