

1. Show that every Cauchy sequence in a discrete metric space has a constant tail, and hence converges. Conclude that all discrete metric spaces are complete.
2. Show that a metric space (X, d) is **Hausdorff**, i.e., for all distinct points x and y in X there exist open sets U_x, U_y with

$$x \in U_x, y \in U_y, \quad U_x \cap U_y = \{\}.$$

3. In \mathbb{R} and \mathbb{R}^2 give examples of subsets which are
 - (a) open and closed (clopen)
 - (b) open and (not closed)
 - (c) (not open) and closed
 - (d) (not open) and (not closed)
4. Prove De Morgan's laws for a collection of subsets $\{U_\alpha\}_{\alpha \in \Lambda}$ of a set X

$$X \setminus \bigcup_{\alpha} U_{\alpha} = \bigcap_{\alpha} (X \setminus U_{\alpha}), \quad X \setminus \bigcap_{\alpha} U_{\alpha} = \bigcup_{\alpha} (X \setminus U_{\alpha}).$$

Use this to show that an arbitrary intersection of closed sets is closed.

5. Show that if a is an isolated point of a set $U \subset X = \mathbb{R}^n$ (with the Euclidean metric), then it is a boundary point of U . Is this the case in a general metric space?
6. Show that a set is clopen if and only if its boundary is empty.
7. Show that for any $V \subset X$

$$X \setminus \text{int}(V) = \text{cl}(X \setminus V).$$

8. Show that for $A, B \subset X$

$$\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B), \quad \text{cl}(A \cap B) \subset \text{cl}(A) \cap \text{cl}(B).$$

9. Show that every finite set in a metric space is closed by
 - (a) Using the fact finite unions of closed sets are closed
 - (b) The definition of a closed set is one having an open complement
 - (c) It contains its cluster (limit) points
 - (d) It contains its boundary