

1. Show that if a norm is given by an inner product, i.e.,

$$\|f\| := \sqrt{\langle f, f \rangle},$$

then it satisfies the *triangle inequality*

$$\|f + g\| \leq \|f\| + \|g\|,$$

and the *parallelogram law*

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

2. We proved the Cauchy–Schwarz inequality by expanding $\|x - \alpha y\|^2 \geq 0$, where

$$\alpha y := \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$

is the orthogonal projection of x onto y . Determine when there is equality in Cauchy–Schwarz.

3. Using the triangle inequality for \mathbb{R} , show that the norms on $C[a, b]$ given by

$$\|f\|_1 := \int_a^b |f(t)| dt, \quad \|f\|_\infty := \max_{t \in [a, b]} |f(t)|$$

satisfy the triangle inequality. Do either of these norms come from an inner product?

4. The L_2 -norm on $C[a, b]$

$$\|f\|_2 := \left(\int_a^b |f(t)|^2 dt \right)^{\frac{1}{2}}$$

comes from an inner product. Use the polarisation identity

$$\operatorname{Re}\langle f, g \rangle = \frac{1}{4}(\|f + g\|^2 - \|f - g\|^2)$$

to determine what this inner product is.

5. Let (X, d) be a discrete metric space. Identify the Cauchy sequences in X .
6. Let $a \neq b$ be points in a metric space. Show that the sequence (a, b, a, b, \dots) does not converge.