1. Suppose $I \subset \mathbb{R}$ is a interval and $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Fix $M > 0$. Suppose that for all $t \in I$ the function $x \mapsto F(t,x)$ is a C^1-function and all partial derivatives are bounded by M. Let $t_0 \in I$ and $y_0 \in \mathbb{R}^n$. Prove that there exists a unique C^1-function $y\colon I \to \mathbb{R}^n$ such that

$$\left[\begin{array}{l} y'(t) = F(t,y(t)) \quad \text{for all } t \in I, \\ y(t_0) = y_0. \end{array}\right.$$

2. Let $I \subset \mathbb{R}$ an interval and $k \in \mathbb{N}$. Let $F\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^k-function and suppose it satisfies the global Lipschitz condition

$$\exists_{L>0} \forall_{t \in I} \forall_{y,z \in \mathbb{R}} \Big[\|F(t,y) - F(t,z)\| \leq L \, \|y - z\| \Big].$$

Let $t_0 \in I$ and $y_0 \in \mathbb{R}^n$. Prove that there exists a unique C^{k+1}-function $y\colon I \to \mathbb{R}^n$ such that

$$\left[\begin{array}{l} y'(t) = F(t,y(t)) \quad \text{for all } t \in I, \\ y(t_0) = y_0. \end{array}\right.$$

3. (a) Prove that there exists a unique continuously differentiable function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = 1$ and $f'(x) = \frac{1}{2} f(1-x)$ for all $x \in [0,1]$.

(b) Prove that there exists a unique infinite differentiable function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) = 1$ and $f'(x) = \frac{1}{2} f(1-x)$ for all $x \in [0,1]$.

4. In this question consider the complete metric space $C[0,2]$ of all continuous functions from $[0,2]$ into $\mathbb{R}$ and metric $d(f,g) = \max\limits_{x \in [0,2]} |f(x) - g(x)|$.

(a) Let $f \in C[0,2]$ and $x, x_1, x_2, \ldots \in [0,2]$. Suppose that $\lim\limits_{n \to \infty} x_n = x$. For all $n \in \mathbb{N}$ define $g_n\colon [0,1] \to \mathbb{R}$ by

$$g_n(t) = f(t\,x_n) \quad (t \in [0,1]).$$

Moreover, define $g\colon [0,1] \to \mathbb{R}$ by $g(t) = f(t\,x)$. Prove that $\lim\limits_{n \to \infty} g_n = g$ uniformly on $[0,1]$.
(Hint. Use that f is uniformly continuous.)

(b) Let $f \in C[0,2]$. Define $F\colon [0,2] \to \mathbb{R}$ by

$$F(x) = \int_0^1 f(t\,x)\,dt \quad (x \in [0,2]).$$

Prove that $F \in C[0,2]$.
(Hint. Use part (a).)

(c) Prove that there exists precisely one $f \in C[0,2]$ such that

$$f(x) = 37e^{x^2} + \tfrac{1}{2} \int_0^1 f(t\,x)\,dt$$

for all $x \in [0,2]$.