1. Let (X, d) be a metric space. Show that the *open ball* of radius $r > 0$ and centre $a \in X$ defined by

$$B_r(a) := \{x \in X : d(x, a) < r\}$$

is an open set.

2. Let d_1 and d_2 be the metrics on $\mathbb{C}$ and $\mathbb{R}^2$ obtained from the norms

$$\|z\|_1 := |z|,\ (|z|^2 := z\overline{z}) \quad z \in \mathbb{C}, \qquad \|(x, y)\|_2 := \sqrt{x^2 + y^2}, \quad (x, y) \in \mathbb{R}^2.$$

Show that these are the same metric if $\mathbb{C}$ is identified with $\mathbb{R}^2$ via

$$\hat{}: \mathbb{C} \to \mathbb{R}^2 : z = x + iy \mapsto \hat{z} = (\operatorname{Re} z, \operatorname{Im} z) = (x, y).$$

3. What are the open balls in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{C}$ with the Euclidean metric?

4. What are the open balls in a set X with the discrete metric?

5. Show that for a norm metric the open balls have the same shape, in the sense that

$$B_r(a) = rB_1(0) + a, \qquad \forall r > 0, \quad \forall a.$$

6. Let $a, b \in \{0, 1\}^n$ be code words of length n, i.e., sequences of 0's and 1's of length n. The **Hamming metric** is defined by

$$d(a, b) := \text{the number of positions in which } a \text{ and } b \text{ differ.}$$

Show that this defines a metric on the set of 2^n code words of length n.