Please deposit your solutions in the appropriate box in the basement of the Maths/Physics building **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a Mathematics Department cover sheet available from outside the Resource Centre.

PLEASE SHOW ALL WORKING. I.e. explain carefully what you are doing.

1. (a) Prove that there exist an open neighbourhood U of $(0,1)$ and C^3-functions $u, v\colon U \to \mathbb{R}$ such that $u(0,1) = 0$, $v(0,1) = \pi$ and

$$\begin{cases} \Big(u(x,y)\Big)^2 + 3\sin(v(x,y)) = x, \\ 2e^{u(x,y)} - \cos\Big(u(x,y)\,v(x,y)\Big) = y, \end{cases} \quad \text{for all } (x,y) \in U.$$

(b) Calculate the (first-order) partial derivatives of u and v at $(0,1)$, where u, v are as in Part (a).

(c) Let u, v be as in Part (a). Do there exist an open neighbourhood V of $(0,1)$ and C^3-functions $f, g\colon V \to \mathbb{R}$ such that

$$\begin{cases} \Big(f(x,y)\Big)^2 + 3\sin(g(x,y)) = x, \\ 2e^{f(x,y)} - \cos\Big(f(x,y)\,g(x,y)\Big) = y, \end{cases} \quad \text{for all } (x,y) \in V$$

AND, in addition, $(u,v)|_{U\cap V} \neq (f,g)|_{U\cap V}$?
(Thus the functions f and g are not identically equal to u and v on their common domain.)

2. (a) Prove that there exists an open set $\Omega \subset \mathbb{R}^2$ and C^1-functions $u, v\colon \Omega \to \mathbb{R}$ such that $(1,3) \in \Omega$, $u(1,3) = 0$, $v(1,3) = 2$ and

$$\begin{cases} u(x,y)\,e^{x\,u(x,y)} + y\,v(x,y) = 6 \\ x + y + \big(u(x,y)\big)^5 + \big(v(x,y)\big)^2 = 8 \end{cases} \quad \text{for all } (x,y) \in \Omega.$$

(b) Determine the (first-order) partial derivatives of the function u at $(1,3)$.

3. Let $n \in \mathbb{N}$ and let A be a real symmetric $n \times n$ matrix. Define

$$S = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

and $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \langle Ax, x\rangle$.

(a) Show that $f|_S$ has a maximum.

(b) Let $v \in S$ and suppose that $f|_S$ has a maximum at v. Use the Lagrange multiplier theorem to show that v is an eigenvector of A.

4. Prove that there exists a unique $f \in C[0,1]$ such that

$$2f(x) = f(1-x) + \tfrac{1}{2}f(\sin x) + 3\cos x$$

for all $x \in [0,1]$.