Please deposit your solutions in the appropriate box in the basement of the Maths/Physics building **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a Mathematics Department cover sheet available from outside the Resource Centre.

PLEASE SHOW ALL WORKING. I.e. explain carefully what you are doing.

1. (a) Prove that there exist an open neighbourhood U of $(0,1)$ and C^3-functions $u, v\colon U \to \mathbb{R}$ such that $u(0,1) = 0$, $v(0,1) = \pi$ and

$$\begin{cases} \Big(u(x,y)\Big)^2 + 3\sin(v(x,y)) = x, \\ 2e^{u(x,y)} - \cos\Big(u(x,y)\,v(x,y)\Big) = y, \end{cases} \quad \text{for all } (x,y) \in U.$$

(b) Calculate the (first-order) partial derivatives of u and v at $(0,1)$, where u, v are as in Part (a).

(c) Let u, v be as in Part (a). Do there exist an open neighbourhood V of $(0,1)$ and C^3-functions $f, g\colon V \to \mathbb{R}$ such that

$$\begin{cases} \Big(f(x,y)\Big)^2 + 3\sin(g(x,y)) = x, \\ 2e^{f(x,y)} - \cos\Big(f(x,y)\,g(x,y)\Big) = y, \end{cases} \quad \text{for all } (x,y) \in V$$

AND, in addition, $(u,v)|_{U\cap V} \neq (f,g)|_{U\cap V}$?
(Thus the functions f and g are not identically equal to u and v on their common domain.)

Answer.

(a) Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$F(p,q) = (p^2 + 3\sin q, 2e^p - \cos(pq)).$$

Then F is a C^3-function and $F(0,\pi) = (0,1)$. Moreover,

$$(DF)(p,q) = \begin{pmatrix} 2p & 3\cos q \\ 2e^p + q\sin(pq) & p\sin(pq) \end{pmatrix}.$$

So

$$(DF)(0,\pi) = \begin{pmatrix} 0 & -3 \\ 2 & 0 \end{pmatrix}$$

is invertible. So by the inverse function theorem there exist open $U, V \subset \mathbb{R}^2$ and a C^3-diffeomorphism $G\colon U \to V$ such that $(0,1) \in U$, $(0,\pi) \in V$ and $G^{-1} = F|_V$. Write $G(x,y) = (u(x,y), v(x,y))$. Then $u, v\colon U \to \mathbb{R}$ are C^3-functions. Since $F(u(x,y), v(x,y)) = (F \circ G)(x,y) = (x,y)$ for all $(x,y) \in U$ the statement follows.

(b)

$$\begin{pmatrix} (D_1u)(0,1) & (D_2u)(0,1) \\ (D_1v)(0,1) & (D_2v)(0,1) \end{pmatrix} = (DG)(0,1) = ((DF)(0,\pi))^{-1} = \tfrac{1}{6}\begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}.$$

So $(D_1u)(0,1) = 0$, $(D_2u)(0,1) = \frac{1}{2}$, $(D_1v)(0,1) = -\frac{1}{3}$ and $(D_2v)(0,1) = 0$.

(c) They do exist. Note that $F(0,2\pi) = (0,1)$. Arguing as in Part (a) there exist an open neighbourhood V of $(0,1)$ and C^3-functions $f, g\colon V \to \mathbb{R}$ such that $f(0,1) = 0$, $g(0,1) = 2\pi$ and

$$\begin{cases} \Big(f(x,y)\Big)^2 + 3\sin(g(x,y)) = x, \\ 2e^{f(x,y)} - \cos\Big(f(x,y)\,g(x,y)\Big) = y, \end{cases} \qquad \text{for all } (x,y) \in V.$$

Since $v(0,1) \neq g(0,1)$ the existence is now clear.

2. (a) Prove that there exists an open set $\Omega \subset \mathbb{R}^2$ and C^1-functions $u, v\colon \Omega \to \mathbb{R}$ such that $(1,3) \in \Omega$, $u(1,3) = 0$, $v(1,3) = 2$ and

$$\begin{cases} u(x,y)\, e^{x\,u(x,y)} + y\, v(x,y) = 6 \\ x + y + \big(u(x,y)\big)^5 + \big(v(x,y)\big)^2 = 8 \end{cases} \qquad \text{for all } (x,y) \in \Omega.$$

(b) Determine the (first-order) partial derivatives of the function u at $(1,3)$.

Answer.

(a) Define $f\colon \mathbb{R}^4 \to \mathbb{R}^2$ by

$$f(x,y,p,q) = (pe^{xp} + yq - 6, x + y + p^5 + q^2 - 8).$$

Then f is a C^1-function and $f(1,3,0,2) = (0,0)$. Moreover,

$$(Df)(x,y,p,q) = \begin{pmatrix} p^2e^{xp} & q & e^{xp} + xpe^{xp} & y \\ 1 & 1 & 5p^4 & 2q \end{pmatrix}$$

and

$$(Df)(1,3,0,2) = \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & 0 & 4 \end{pmatrix}.$$

Since $\det \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \neq 0$ it follows from the implicit function theorem that there exist open $U \subset \mathbb{R}^4$, open $\Omega \subset \mathbb{R}^2$ and a C^1-function $g\colon \Omega \to \mathbb{R}^2$ such that

$$U \cap \{(x,y,p,q) \in \mathbb{R}^4 : f(x,y,p,q) = 0\} = \{(x,y,g(x,y)) : (x,y) \in \Omega\}.$$

Write $g(x,y) = (u(x,y), v(x,y))$. Then the the statement follows from the fact that $f(x,y,g(x,y)) = 0$ for all $(x,y) \in \Omega$.

(b) Define $h\colon \Omega \to \mathbb{R}^4$ by $h(x,y) = (x,y,g(x,y))$. Then $f \circ h = 0$. So $(Df) \circ (Dh) = 0$. This gives

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} (Dg)(1,3) = 0$$

and $(Dg)(1,3) = -\frac{1}{4} \begin{pmatrix} -3 & 5 \\ * & * \end{pmatrix}$. Hence $(D_1 u)(1,3) = \frac{3}{4}$ and $(D_2 u)(1,3) = -\frac{5}{4}$.

3. Let $n \in \mathbb{N}$ and let A be a real symmetric $n \times n$ matrix. Define

$$S = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

and $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \langle Ax, x \rangle$.

(a) Show that $f|_S$ has a maximum.

(b) Let $v \in S$ and suppose that $f|_S$ has a maximum at v. Use the Lagrange multiplier theorem to show that v is an eigenvector of A.

Answer.

(a) Since S is compact and f is continuous, the set $f(S)$ is compact in $\mathbb{R}$. Hence it has a maximum.

(b) Define $g\colon \mathbb{R}^n \to \mathbb{R}$ by $g(x) = \|x\|^2 - 1$. Then f and g are C^1-functions. Moreover, $(\nabla f)(x) = 2Ax$ and $(\nabla g)(x) = 2x$ for all $x \in \mathbb{R}^n$. Since $v \neq 0$ it follows that $(\nabla g)(v) \neq 0$. By Lagrange $(\nabla f)(v)$ and $(\nabla g)(v)$ are linearly dependent. So there exists a $\lambda \in \mathbb{R}$ such that $(\nabla f)(v) = \lambda(\nabla g)(v)$. Then $2Av = 2\lambda v$ and $Av = \lambda v$. Hence v is an eigenvector of A.

4. Prove that there exists a unique $f \in C[0,1]$ such that

$$2f(x) = f(1-x) + \tfrac{1}{2} f(\sin x) + 3\cos x$$

for all $x \in [0,1]$.

Answer.

Note that $C[0,1]$ is a Banach space, so in particular it is a **complete** metric space with metric $d(f,g) = \max\{|f(t) - g(t)| : t \in [0,1]\}$. Define $\Phi\colon C[0,1] \to C[0,1]$ by

$$\Big[\Phi(f)\Big](x) = \tfrac{1}{2} f(1-x) + \tfrac{1}{4} f(\sin x) + \tfrac{3}{2}\cos x \quad (f \in C[0,1],\ x \in [0,1]).$$

Observe that $\Phi(f)$ is indeed a continuous function.

Let $f, g \in C[0,1]$. Then for all $x \in [0,1]$ one has

$$\begin{aligned} \Big|\Big[\Phi(f)\Big](x) - \Big[\Phi(g)\Big](x)\Big| &= |\tfrac{1}{2}(f(1-x) - g(1-x)) + \tfrac{1}{4}(f(\sin x) - g(\sin x))| \\ &\leq \tfrac{1}{2}|f(1-x) - g(1-x)| + \tfrac{1}{4}|f(\sin x) - g(\sin x)| \\ &\leq \tfrac{1}{2} d(f,g) + \tfrac{1}{4} d(f,g) \\ &= \tfrac{3}{4} d(f,g). \end{aligned}$$

This is for all $x \in [0,1]$. So

$$d(\Phi(f), \Phi(g)) = \sup_{x \in [0,1]} \Big|\Big[\Phi(f)\Big](x) - \Big[\Phi(g)\Big](x)\Big| \leq \tfrac{3}{4} d(f,g)$$

and Φ is a contraction. By the Banach fixed point theorem the function Φ has a unique fixed point. This proves the statement.