Please deposit your solutions in the appropriate box in the basement of the Maths/Physics building **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a Mathematics Department cover sheet available from outside the Resource Centre.

PLEASE SHOW ALL WORKING. I.e. explain carefully what you are doing.

**1.** Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \|x\|.$$

Determine all $x_0 \in \mathbb{R}^n$ such that $f$ is differentiable at $x_0$. What is then the differential?

**2.** Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$ and $f\colon U \to \mathbb{R}$ a function. Prove that $f$ is differentiable at $x_0$ if and only if there exists a function $g\colon U \to \mathbb{R}^n$ such that $g$ is continuous at $x_0$ and

$$f(x) - f(x_0) = \langle x - x_0, g(x) \rangle$$

for all $x \in U$. As usual, $\langle\ ,\ \rangle$ denotes the inner product on $\mathbb{R}^n$.

**3.** Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^3\,y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Prove that $f$ is continuous at $(0,0)$.

(b) Determine whether the partial derivatives exist at $(0,0)$.

(c) Determine whether $f$ is differentiable at $(0,0)$.

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**4.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$-function. Let $a, b \in \mathbb{R}$ with $a < b$. Define $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_a^b f(x, y)\, dy.$$

Fix $x_0 \in \mathbb{R}$. Define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x, y) = (D_1 f)(x, y) - (D_1 f)(x_0, y).$$

Let $\varepsilon > 0$.

(a) Show that there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ and $y \in [a, b]$ one has $|g(x, y)| < \varepsilon$.
(Hint. For each $y \in [a, b]$ the function $g$ is continuous at $(x_0, y)$. Then use that $[a, b]$ is compact.)

(b) Let $\delta > 0$ be as in Part (a). Show that

$$\left| \frac{f(x, y) - f(x_0, y)}{x - x_0} - (D_1 f)(x_0, y) \right| < \varepsilon$$

for all $x \in \mathbb{R}$ and $y \in [a, b]$ with $0 < |x - x_0| < \delta$.
(Hint. Apply the mean value theorem to the function $t \mapsto f(t, y)$.)

(c) Let $\delta > 0$ be as in Part (a). Show that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - \int_a^b (D_1 f)(x_0, y)\, dy \right| < \varepsilon$$

for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$.

(d) Show that $F$ is differentiable.

**5.** Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$, $k \in \mathbb{N}$ and $f\colon U \to \mathbb{R}$ a $C^k$-function. Let $p$ be the $k$-th Taylor polynomial for $f$ about $x_0$. Let $q\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $k$. Suppose that

$$\lim_{x \to x_0} \frac{|f(x) - q(x)|}{\|x - x_0\|^k} = 0.$$

Prove that $p = q$.
(Hint. Wlog $x_0 = 0$. Write $p - q = \sum_{m=0}^k p_m$ with $p_m$ a polynomial which is homogeneous of degree $m$. For all $y \in \mathbb{R}^n \setminus \{0\}$ consider $\lim_{t \to 0} \frac{|(p-q)(ty)|}{\|ty\|^k}$.)