

Please deposit your solutions in the appropriate box in the basement of the Maths/Physics building **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a Mathematics Department cover sheet available from outside the Resource Centre.

PLEASE SHOW ALL WORKING. I.e. explain carefully what you are doing.

1. Define $f: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f(x) = \|x\|.$$

Determine all $x_0 \in \mathbb{R}^n$ such that f is differentiable at x_0 . What is then the differential?

Answer.

If $x_0 = 0$ then $\frac{f(x_0 + t\epsilon_1) - f(x_0)}{t} = \frac{|t|}{t}$ for all $t \in \mathbb{R} \setminus \{0\}$. So the first partial derivative of f does not exist. Hence f is not differentiable.

If $x_0 \neq 0$ define $f_1: \mathbb{R}^n \rightarrow \mathbb{R}$ by $f_1(x) = \|x\|^2$ and $f_2: (0, \infty) \rightarrow \mathbb{R}$ by $f_2(t) = \sqrt{t}$. Then $f|_{\mathbb{R}^n \setminus \{0\}} = f_2 \circ f_1$. Both f_1 is differentiable at x_0 and f_2 is differentiable at $f_1(x_0)$. By the chain rule one deduces that f is differentiable at x_0 and

$$\left(f'(x_0)\right)(h) = \left(f_2'(f_1(x_0)) \circ f_1'(x_0)\right)(h) = \frac{1}{2\sqrt{f_1(x_0)}} \left(f_1'(x_0)\right)(h) = \frac{1}{2\|x_0\|} \cdot 2\langle x_0, h \rangle = \frac{\langle x_0, h \rangle}{\|x_0\|}$$

for all $h \in \mathbb{R}^n$.

2. Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$ and $f: U \rightarrow \mathbb{R}$ a function. Prove that f is differentiable at x_0 if and only if there exists a function $g: U \rightarrow \mathbb{R}^n$ such that g is continuous at x_0 and

$$f(x) - f(x_0) = \langle x - x_0, g(x) \rangle$$

for all $x \in U$. As usual, $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^n .

Answer.

' \Rightarrow '. Suppose f is differentiable at x_0 . Define $\rho: U \rightarrow \mathbb{R}$ by

$$\rho(x) = f(x) - f(x_0) - \left(f'(x_0)\right)(x - x_0).$$

Then $\lim_{x \rightarrow x_0} \frac{\rho(x)}{\|x - x_0\|} = 0$. Define $g: U \rightarrow \mathbb{R}^n$ by

$$g(x) = \begin{cases} (\nabla f)(x_0) + \frac{\rho(x)}{\|x - x_0\|^2} (x - x_0) & \text{if } x \neq x_0, \\ (\nabla f)(x_0) & \text{if } x = x_0. \end{cases}$$

Then $f(x) - f(x_0) = \langle x - x_0, g(x) \rangle$ for all $x \in U$. Moreover,

$$\left\| \frac{\rho(x)}{\|x - x_0\|^2} (x - x_0) \right\| = \frac{|\rho(x)|}{\|x - x_0\|}$$

for all $x \in U \setminus \{x_0\}$. So $\lim_{x \rightarrow x_0} g(x) = (\nabla f)(x_0) = g(x_0)$ by the squeeze theorem. Hence g is continuous.

' \Leftarrow '. Define $L: \mathbb{R}^n \rightarrow \mathbb{R}$ by $L(h) = \langle h, g(x_0) \rangle$. Then L is linear. Let $\delta > 0$ be such that $B_\delta(x_0) \subset U$. Then for all $h \in B_\delta(0)$ one has

$$|f(x_0 + h) - f(x_0) - L(h)| = |\langle h, g(x_0 + h) - g(x_0) \rangle| \leq \|h\| \|g(x_0 + h) - g(x_0)\|.$$

So $\frac{|f(x_0+h)-f(x_0)-L(h)|}{\|h\|} \leq \|g(x_0+h)-g(x_0)\|$ for all $h \in B_\delta(0) \setminus \{0\}$. Since $\lim_{h \rightarrow 0} \|g(x_0+h)-g(x_0)\| = 0$ it follows from the squeeze theorem that $\lim_{h \rightarrow 0} \frac{|f(x_0+h)-f(x_0)-L(h)|}{\|h\|} = 0$. Hence f is differentiable at x_0 .

3. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- Prove that f is continuous at $(0, 0)$.
- Determine whether the partial derivatives exist at $(0, 0)$.
- Determine whether f is differentiable at $(0, 0)$.

Answer.

(a) Let $\varepsilon > 0$. Choose $\delta = 2\varepsilon$. Then for all $(x, y) \in \mathbb{R}^2$ with $0 < \|(x, y)\| < \delta$ it follows that

$$|x^3 y| = |x| |x^2| |y| \leq |x| \cdot \frac{1}{2}(x^4 + y^2) < \frac{\delta}{2}(x^4 + y^2) = \varepsilon(x^4 + y^2).$$

So $|f(x, y) - f(0, 0)| < \varepsilon$.

- $(D_1 f)(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = \lim_{t \rightarrow 0} 0 = 0$. Similarly $(D_2 f)(0, 0) = 0$.
- The function f is differentiable at 0 if and only if

$$\lim_{h \rightarrow 0} \frac{f(0 + h) - f(0) - (D_1 f)(0)h_1 - (D_2 f)(0)h_2}{\|h\|} = 0.$$

So it is differentiable at 0 if and only if

$$\lim_{h \rightarrow 0} \left| \frac{h_1^3 h_2}{(h_1^4 + h_2^2) \sqrt{h_1^2 + h_2^2}} \right| = 0.$$

Now choose $(h_1, h_2) = (t, t^2)$ and take the limit $t \downarrow 0$. Since

$$\lim_{t \downarrow 0} \left| \frac{t^5}{(t^4 + t^4) \sqrt{t^2 + t^4}} \right| = \frac{1}{2} \neq 0$$

the function f is not differentiable at 0.

4. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 -function. Let $a, b \in \mathbb{R}$ with $a < b$. Define $F: \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \int_a^b f(x, y) dy.$$

Fix $x_0 \in \mathbb{R}$. Define $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(x, y) = (D_1 f)(x, y) - (D_1 f)(x_0, y).$$

Let $\varepsilon > 0$.

(a) Show that there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ and $y \in [a, b]$ one has $|g(x, y)| < \varepsilon$.

(Hint. For each $y \in [a, b]$ the function g is continuous at (x_0, y) . Then use that $[a, b]$ is compact.)

(b) Let $\delta > 0$ be as in Part (a). Show that

$$\left| \frac{f(x, y) - f(x_0, y)}{x - x_0} - (D_1 f)(x_0, y) \right| < \varepsilon$$

for all $x \in \mathbb{R}$ and $y \in [a, b]$ with $0 < |x - x_0| < \delta$.

(Hint. Apply the mean value theorem to the function $t \mapsto f(t, y)$.)

(c) Let $\delta > 0$ be as in Part (a). Show that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - \int_a^b (D_1 f)(x_0, y) dy \right| < (b - a)\varepsilon$$

for all $x \in \mathbb{R}$ with $0 < |x - x_0| < \delta$.

(d) Show that F is differentiable.

Answer.

(a) For all $y \in [a, b]$ the function g is continuous at (x_0, y) and $g(x_0, y) = 0$. Hence there exists a $\delta_y > 0$ such that $|g(x', y') - g(x_0, y)| < \varepsilon$ for all $(x', y') \in B_{2\delta_y}(x_0, y)$. In particular, $|g(x', y')| < \varepsilon$ for all $(x', y') \in (x_0 - \delta_y, x_0 + \delta_y) \times (y - \delta_y, y + \delta_y)$. Now $\left((y - \delta_y, y + \delta_y) \right)_{y \in [a, b]}$ is an open cover of $[a, b]$. By compactness, there are $N \in \mathbb{N}$ and $y_1, \dots, y_N \in [a, b]$ such that

$$[a, b] \subset \bigcup_{n=1}^N (y_n - \delta_{y_n}, y_n + \delta_{y_n}).$$

Let $\delta = \min\{\delta_{y_1}, \dots, \delta_{y_N}\}$. Then $\delta > 0$. Finally, let $x \in (x_0 - \delta, x_0 + \delta)$ and $y \in [a, b]$. Then there exists an $n \in \{1, \dots, N\}$ such that $y \in (y_n - \delta_{y_n}, y_n + \delta_{y_n})$. Then $|x - x_0| < \delta \leq \delta_{y_n}$. So $|g(x, y)| < \varepsilon$.

(b) Define $h: \mathbb{R} \rightarrow \mathbb{R}$ by $h(t) = f(t, y)$. The mean value theorem for h gives that there exists a c between x_0 and x such that $\frac{h(x) - h(x_0)}{x - x_0} = h'(c)$. Then $|c - x_0| < |x - x_0| < \delta$. Note that $h'(c) = (D_1 f)(c, y)$. Then the statement follows immediately from Part (a).

(c)

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - \int_a^b (D_1 f)(x_0, y) dy \right| &= \left| \int_a^b \frac{f(x, y) - f(x_0, y)}{x - x_0} - (D_1 f)(x_0, y) dy \right| \\ &\leq \int_a^b \left| \frac{f(x, y) - f(x_0, y)}{x - x_0} - (D_1 f)(x_0, y) \right| dy \\ &< \int_a^b \varepsilon dy = (b - a)\varepsilon. \end{aligned}$$

(d) This follows from Part (c).

5. Let $U \subset \mathbb{R}^n$ open, $x_0 \in U$, $k \in \mathbb{N}$ and $f: U \rightarrow \mathbb{R}$ a C^k -function. Let p be the k -th Taylor polynomial for f about x_0 . Let $q: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial of degree k . Suppose that

$$\lim_{x \rightarrow x_0} \frac{|f(x) - q(x)|}{\|x - x_0\|^k} = 0.$$

Prove that $p = q$.

(Hint. Wlog $x_0 = 0$. Write $p - q = \sum_{m=0}^k p_m$ with p_m a polynomial which is homogeneous of degree m . For all $y \in \mathbb{R}^n \setminus \{0\}$ consider $\lim_{t \rightarrow 0} \frac{|(p-q)(ty)|}{\|ty\|^k}$.)

Answer.

We know from Taylor's theorem that $\lim_{x \rightarrow x_0} \frac{|f(x) - p(x)|}{\|x - x_0\|^k} = 0$. But

$$\frac{|p(x) - q(x)|}{\|x - x_0\|^k} \leq \frac{|f(x) - q(x)|}{\|x - x_0\|^k} + \frac{|f(x) - p(x)|}{\|x - x_0\|^k}$$

for all $x \in U \setminus \{x_0\}$. Hence by the squeeze theorem one deduces that

$$\lim_{x \rightarrow x_0} \frac{|p(x) - q(x)|}{\|x - x_0\|^k} = 0.$$

Wlog assume for the rest that $x_0 = 0$. Write $p - q = \sum_{m=0}^k p_m$ with p_m a polynomial which is homogeneous of degree m . Suppose $p - q \neq 0$. Then there exists an $m \in \{0, \dots, k\}$ such that $p_m \neq 0$. Let

$$m_0 = \min\{m \in \{0, \dots, k\} : p_m \neq 0\}.$$

Then there exists a $y \in \mathbb{R}^n \setminus \{0\}$ such that $p_{m_0}(y) \neq 0$. Then

$$\begin{aligned} 0 &= \lim_{t \rightarrow 0} \frac{|(p - q)(ty)|}{\|ty\|^k} = \lim_{t \rightarrow 0} \sum_{m=m_0}^k \frac{|t|^m p_m(y)}{|t|^k \|y\|^k} \\ &= \lim_{t \rightarrow 0} \frac{|p_{m_0}(y)|}{\|y\|^k} \frac{1}{|t|^{k-m_0}} \left(1 + \sum_{m=m_0+1}^k \frac{|t|^{m-m_0} |p_m(y)|}{|p_{m_0}(y)|} \right). \end{aligned}$$

Since the limit exists one first has that $k = m_0$ and secondly that $p_{m_0}(y) = 0$. The latter is a contradiction. So $p - q = 0$ and $q = p$.