1. Let U be a subset of $\mathbb{R}$ (with the Euclidean metric). Recall an interval I in $\mathbb{R}$ is a subset with the property that if $a, b \in I$, then $[a, b] \subset I$, where $[a, b]$ denotes the closed interval with endpoints a and b whether or not $a < b$.

(a) We say $a, b \in U$ are related if $[a, b] \subset U$. Check that this gives an equivalence relation on U.

(b) Show that the equivalence classes in (a) are intervals.

(c) Show that if U is open, then the equivalence classes in (a) are open.

(d) Prove that an open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

2. Recall that the closure of a set A in a metric space X is (by definition) the smallest closed set containing A.

(a) Use the definiton of the closure to show

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B).$$

(b) Let (A_j) be a collection of subsets of X. Does it follow that $\mathrm{cl}(\cup_j A_j) = \cup_j \mathrm{cl}(A_j)$?

(c) Use De Morgan's law, and the facts

$$X = \mathrm{int}(A) \cup \partial A \cup \mathrm{ext}(A) \quad \text{(disjoint union)}, \qquad \mathrm{cl}(A) = A \cup \partial A.$$

to deduce the analogue of (a) for the $\mathrm{ext}(A)$ the exterior of a set A.

(d) Use $\mathrm{int(A)} = \mathrm{ext}(X \setminus A)$ to find the analogue of your result in (c) for the interior.

3. Recall the norm of a linear map $L : X \to Y$ between normed linear spaces is

$$\|L\| := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Lx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} \|Lx\|.$$

(a) We have seen that if L is bounded, i.e., $\|L\| < \infty$, then it is Lipschitz continuous. Show that if L is a continous linear map, then it is a bounded linear map.

(b) Show that the composition $S \circ T$ of bounded linear maps $T : X \to Y$ and $S : Y \to Z$ is a bounded linear map (denoted ST), which satisfies

$$\|ST\| \leq \|S\|\|T\|.$$

(c) Let $A \in \mathbb{C}^{m \times n}$ be a matrix. Find a formula for the norm of the linear map $L : \mathbb{C}^n \to \mathbb{C}^m$ represented by A (with respect to the standard basis) where $\mathbb{C}^n$ and $\mathbb{C}^m$ have the max norm.

4. If $f : X \to Y$ is a continuous map and $E \subset X$, prove that

$$f(\overline{E}) \subset \overline{f(E)},$$

with $\overline{V}$ the closure of V, and give an example where the inclusion is strict.

5. Let U, V be subsets of a metric space (X, d) with a nonempty intersection.

(a) Show if U and V are connected then so is $U \cup V$.

(b) Show if U and V are path connected then so is $U \cup V$.

(c) If U and V are both connected, but with empty intersection, is it possible that $U \cup V$ be path connected.

6. Let (X, d) be a metric space, K a compact subset of X. Show that if $f : X \to \mathbb{R}$ is continuous, and

$$f(x) > 0, \qquad \forall x \in K,$$

then there exists a $\delta > 0$ for which

$$f(x) \geq \delta, \qquad \forall x \in K.$$