Maths 333 Solutions to Assignment 2 Due: Wednesday August 25, 2010

1. Let U be a subset of $\mathbb{R}$ (with the Euclidean metric). Recall an interval I in $\mathbb{R}$ is a subset with the property that if $a, b \in I$, then $[a, b] \subset I$, where $[a, b]$ denotes the closed interval with endpoints a and b whether or not $a < b$.

(a) We say $a, b \in U$ are related if $[a, b] \subset U$. Check that this gives an equivalence relation on U.

(b) Show that the equivalence classes in (a) are intervals.

(c) Show that if U is open, then the equivalence classes in (a) are open.

(d) Prove that an open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

Solution:

(a) The reflexivity and symmetry are immediate from the definition, and the transitivity follows by checking $[a, c] \subset [a, b] \cup [b, c]$ (for the various cases).

(b) If a, b are any two points in a equivalence class, then so is the interval with endpoints a, b (definition of the equivalence), this is the defining property of an interval, so the equivalence class is an interval.

(c) If U is open, then for each $a \in U$ there exists $r = r_a > 0$ with $I_a = B_r(a) = (a - r, a + r) \subset U$. All the points of I_a are equivalent, so I_a is contained in the equivalence class of a, which is therefore open.

(d) Let U be an open subset of $\mathbb{R}$. By (c), it may be written as a disjoint union of nonempty open intervals $U = \cup_\alpha I_\alpha$. Choose a rational $q_\alpha \in I_\alpha$ in each of the intervals. Then these rationals are distinct (since the intervals are disjoint). Since the rationals are countable, the subset $\{q_\alpha\}$, and hence $\{I_\alpha\}$ (which is in 1–1 correspondence with it) is countable.

2. Recall that the closure of a set A in a metric space X is (by definition) the smallest closed set containing A.

(a) Use the definiton of the closure to show

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B).$$

(b) Let (A_j) be a collection of subsets of X. Does it follow that $\mathrm{cl}(\cup_j A_j) = \cup_j \mathrm{cl}(A_j)$?

(c) Use De Morgan's law, and the facts

$$X = \mathrm{int}(A) \cup \partial A \cup \mathrm{ext}(A) \quad \text{(disjoint union)}, \qquad \mathrm{cl}(A) = A \cup \partial A.$$

to deduce the analogue of (a) for the $\mathrm{ext}(A)$ the exterior of a set A.

(d) Use $\mathrm{int(A)} = \mathrm{ext}(X \setminus A)$ to find the analogue of your result in (c) for the interior.

Solution:

(a) Let K be a closed set containing $A \cup B$. Since K is closed and contains A and B, we have

$$A \subset \mathrm{cl}(A) \subset K, \quad B \subset \mathrm{cl}(B) \subset K \quad \Longrightarrow \quad A \cup B \subset \mathrm{cl}(A) \cup \mathrm{cl}(B) \subset K.$$

Since the union of two closed sets is closed, $\mathrm{cl}(A)\cup\mathrm{cl}(B)$ is the smallest closed set containing $A\cup B$, i.e., $\mathrm{cl}(A\cup B)=\mathrm{cl}(A)\cup\mathrm{cl}(B)$.

(b) No. In $\mathbb{R}$ with the Euclidean metric, let $A_k=\{q_k\}$ (which is closed), where (q_k) is the set of rationals arranged as a sequence, then $\mathrm{cl}(\cup_j A_j)=\mathrm{cl}(\mathbb{Q})=\mathbb{R}\neq\cup_j\mathrm{cl}(A_j)=\mathbb{Q}$.

(c) Since $A\subset\mathrm{int}(A)\cup\partial(A)$, we have $\mathrm{cl}(A)=\mathrm{int}(A)\cup\partial(A)$, so that $X\setminus\mathrm{cl}(A)=\mathrm{ext}(A)$. Hence by De Morgan

$$\mathrm{ext}(A\cup B)=X\setminus\mathrm{cl}(A\cup B)=X\setminus(\mathrm{cl}(A)\cup\mathrm{cl}(B))=(X\setminus\mathrm{cl}(A))\cap(X\setminus\mathrm{cl}(B))=\mathrm{ext}(A)\cap\mathrm{ext}(B).$$

(d) By De Morgan, and part (c),

$$\mathrm{int}(A\cap B)=\mathrm{ext}(X\setminus(A\cap B))=\mathrm{ext}((X\setminus A)\cup(X\setminus B))=\mathrm{ext}(X\setminus A)\cap\mathrm{ext}(X\setminus B)=\mathrm{int}(A)\cap\mathrm{int}(B).$$

3. Recall the norm of a linear map $L:X\to Y$ between normed linear spaces is

$$\|L\|:=\sup_{\substack{x\in X\\x\neq 0}}\frac{\|Lx\|}{\|x\|}=\sup_{\substack{x\in X\\\|x\|=1}}\|Lx\|.$$

(a) We have seen that if L is bounded, i.e., $\|L\|<\infty$, then it is Lipschitz continuous. Show that if L is a continous linear map, then it is a bounded linear map.

(b) Show that the composition $S\circ T$ of bounded linear maps $T:X\to Y$ and $S:Y\to Z$ is a bounded linear map (denoted ST), which satisfies

$$\|ST\|\le\|S\|\|T\|.$$

(c) Let $A\in\mathbb{C}^{m\times n}$ be a matrix. Find a formula for the norm of the linear map $L:\mathbb{C}^n\to\mathbb{C}^m$ represented by A (with respect to the standard basis) where $\mathbb{C}^n$ and $\mathbb{C}^m$ have the max norm.

Solution:

(a) Suppose that L is continuous at some point $x_0\in X$, i.e., $\forall\varepsilon>0$, $\exists\delta>0$ so that

$$\|x-x_0\|\le\delta\quad\Longrightarrow\quad\|Lx-Lx_0\|<\varepsilon.$$

For $y\in X$, $y\neq 0$, define x by $x-x_0=\frac{y}{\|y\|}\delta$. Then

$$\|x-x_0\|=\delta\quad\Longrightarrow\quad\|Lx-Lx_0\|<\varepsilon.$$

Using the homogeniety of the norm this gives

$$\|Lx-Lx_0\|=\|L(x-x_0)\|=\|L(\frac{y}{\|y\|}\delta)\|=\frac{\delta}{\|y\|}\|Ly\|<\varepsilon\quad\Longrightarrow\quad\|Ly\|<\frac{\varepsilon}{\delta}\|y\|,$$

so that L is bounded with norm $\|L\|\le\frac{\varepsilon}{\delta}$.

(b) For $x\in X$

$$\|STx\|=\|S(Tx)\|\le\|S\|\|Tx\|\le\|S\|\|T\|\|x\|,$$

so that ST is bounded with norm $\le\|S\|\|T\|$.

(c) Suppose wlog $A \neq 0$. Since $(Ax)_j = \sum_{k=1}^n a_{jk}x_k$,

$$\|Ax\|_\infty = \max_j |\sum_k a_{jk}x_k| \le \max_j \sum_k |a_{jk}x_k| \le \max_j \sum_k |a_{jk}|\,\|x\|_\infty = (\max_j \sum_k |a_{jk}|)\,\|x\|_\infty,$$

so that A is bounded with norm

$$\|A\| \le \max_j \sum_k |a_{jk}|.$$

Choose j for which the maximum above is attained, and let $x_k := \operatorname{sign}(a_{jk})$, where the sign is

$$\operatorname{sign}(z) := \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

Then

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \ge \frac{|(Ax)_j|}{1} = \sum_k |a_{jk}| = \max_j \sum_k |a_{jk}| \quad \Longrightarrow \quad \|A\| = \max_j \sum_k |a_{jk}|.$$

4. If $f : X \to Y$ is a continuous map and $E \subset X$, prove that

$$f(\overline{E}) \subset \overline{f(E)},$$

with $\overline{V}$ the closure of V, and give an example where the inclusion is strict.

Solution:

Suppose $f(x) \in f(\overline{E})$, so there is a sequence x_n in E with $x_n \to x$ (take a constant sequence if $x \in E$ or a sequence of distinct points of E if x is a limit point of E). Since f is continuous, $f(x_n) \to f(x)$, and hence $f(x) \in \overline{f(E)}$. The inclusion can be strict, e.g., take $X = E = (0,\infty)$, $f : X \to \mathbb{R} : x \mapsto \frac{1}{x}$, then $f(\overline{E}) = f((0,\infty)) = (0,\infty)$, $\overline{f(E)} = [0,\infty)$, or take $f = \tan^{-1}$. Another example is $f : \mathbb{Q} \to \mathbb{R} : x \mapsto x$ (which is continuous since it is a restriction of a continuous map), and $E = \mathbb{Q}$ (which is closed in $\mathbb{Q}$) giving $f(\overline{E}) = \mathbb{Q} \subset \mathbb{R} = \overline{f(E)}$.

5. Let U, V be subsets of a metric space (X, d) with a nonempty intersection.

(a) Show if U and V are connected then so is $U \cup V$.

(b) Show if U and V are path connected then so is $U \cup V$.

(c) If U and V are both connected, but with empty intersection, is it possible that $U \cup V$ be path connected.

Solution:

(a) Let $f : U \cup V \to Z$ be a continuous map to a 2–point space Z. The restrictions $f|_U, f|_V$ are continuous maps from connected sets to a 2–point space, and so must be constant. But these maps are defined on a common point, and so must take the same constant value, i.e., f is constant, and therefore $U \cup V$ is connected.

(b) Since U and V are path connected, to show their union is path connected it suffices to show there is a curve from $u \in U$ to $v \in V$. Let $z \in U \cup V$, and $\gamma_1 : [0,1] \to X$ be a curve from u to z, and $\gamma_2 : [1,2] \to X$ be a curve from z to v. Then $\gamma : [0,2] \to X$ defined by

$$\gamma(t) := \begin{cases} \gamma_1(t), & 0 \le t \le 1; \\ \gamma_2(t), & 1 < t \le 1 \end{cases}$$

is a curve from u to v. This follows since $\gamma(0)=\gamma_1(0)=u$, $\gamma(2)=\gamma_2(2)=v$, and γ is continuous at all points $t\neq 1$ by the continuity of γ_1 and γ_2, and continuous at $t=1$ since

$$\lim_{t\to 1^-}\gamma(t)=\lim_{t\to 1^-}\gamma_1(t)=\gamma_1(1)=z, \qquad \lim_{t\to 1^+}\gamma(t)=\lim_{t\to 1^+}\gamma_2(t)=\gamma_2(1)=z.$$

(c) Let $U=[-1,0]$, $V=(0,1]$, which are disjoint and both (path) connected. Their union is the interval $[-1,1]$, which is path connected.

6. Let (X,d) be a metric space, K a compact subset of X. Show that if $f:X\to\mathbb{R}$ is continuous, and

$$f(x)>0, \qquad \forall x\in K,$$

then there exists a $\delta>0$ for which

$$f(x)\geq\delta, \qquad \forall x\in K.$$

Solution:

The set of reals $f(K)$ is the continuous image of a compact set, and so is compact. Let $\delta:=\inf f(K)$. By the definition of the infimum, there is a sequence $f(x_n)\to\delta$. But compact spaces are complete, so that $\delta=f(x_0)>0$ for some $x_0\in K$, and hence $f(x)\geq\inf f(K)=\delta>0$, $\forall x\in K$.