1. Here we consider a continuous version of Hölder's and Minkowskii's inequalities.

(a) Using the generalised geometric mean inequality

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \qquad \alpha, \beta \ge 0, \quad 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

prove Hölder's inequality for $C[a, b]$, i.e.,

$$\int_a^b |f(t)g(t)|\, dt \le \Big(\int_a^b |f(t)|^p\, dt\Big)^{\frac{1}{p}} \Big(\int_a^b |g(t)|^q\, dt\Big)^{\frac{1}{q}}.$$

(b) Using (a), prove that

$$\|f\|_p := \Big(\int_a^b |f(t)|^p\, dt\Big)^{\frac{1}{p}}, \qquad f \in C[a, b]$$

satisfies the triangle inequality.

Hint: Consider the corresponding proof for $\ell_p(\mathbb{N})$ given in class.

2. Here we consider $\ell_p = \ell_p(\mathbb{N})$, $1 \le p \le \infty$.

(a) Show that $\ell_p \subset \ell_r$, $1 \le p \le r < \infty$.

(b) Find a sequence which is in ℓ_r but not ℓ_p, $1 \le p < r < \infty$.

(c) If $x \in \ell_p$, for some $1 \le p < \infty$, then is $x \in \ell_\infty$? If $x \in \ell_\infty$, then is $x \in \ell_p$ for some $1 \le p < \infty$?

(d) For which values of p is ℓ_p an inner product space?

3. Here we show that $C[a, b]$ with the uniform norm is complete. Let (f_n) be Cauchy in $(C[a, b], \|\cdot\|_\infty)$.

(a) Show that the complex sequences $(f_n(x))$ are uniformly Cauchy in x, i.e., $\forall \varepsilon > 0$, $\exists N$ such that

$$|f_n(x) - f_m(x)| < \varepsilon, \qquad \forall n, m > N \text{ and } \forall x \in [a, b].$$

(b) Let $f(x)$ be the limit of the above sequence. Show that $f_n \to f$ in the uniform norm as $n \to \infty$.

(c) Verify that convergence in the uniform norm is the same as uniform convergence, and so that f as the uniform limit of continuous functions is continuous, and hence $(C[a, b], \|\cdot\|_\infty)$ is complete.

(d) Is $(C[a, b], \|\cdot\|_\infty)$ a Hilbert space?

Hint: $|f(x)| \le \|f\|_\infty$, $\forall f \in C[a, b]$, $\forall x \in [a, b]$.

4. Here we show that $C[-1,1]$ with the L_p–norm, $1 \le p < \infty$ is not complete.
Let $f_n : [-1,1] \to \mathbb{R}$ be the sequence of continuous functions given by

$$f_n(x) = \begin{cases} 1, & -1 \le x \le 0; \\ 1 - nx, & 0 < x < \frac{1}{n}; \\ 0, & \frac{1}{n} \le x \le 1. \end{cases}$$

(a) Show that (f_n) is Cauchy in the L_p–norm.

(b) Show that if f_n has an L_p–limit f, then $f = 1$ on $[-1,0]$ and $f = 0$ on $[\varepsilon, 1]$, $\forall 0 < \varepsilon \le 1$.

(c) Show that $C[-1,1]$ with the L_p–norm is not a Banach space.

Remark: The completion of $C[a,b]$ in the L_p–norm is studied Maths 730. It is called the Lebesgue space, and is denoted by $L_p[a,b]$.

5. Let $\hat{B}_r(a)$ be the **closed ball**

$$\hat{B}_r(a) := \{x \in X : d(x,a) \le r\}, \qquad r > 0.$$

(a) Show that $\hat{B}_r(a)$ is a closed set

(b) Show that every open set can be written as a union of closed balls.

(c) Find the interior, boundary and exterior of the closed ball $\hat{B}_r(a)$.

(d) Show that the boundary of the open ball $B_r(a)$ may or may not be the sphere

$$S_r(a) := \{x \in X : d(x,a) = r\}.$$

(e) Prove or disprove. The closure of the open ball $B_r(a)$ equals $\hat{B}_r(a)$.