

1. Here we consider a continuous version of Hölder's and Minkowski's inequalities.

(a) Using the generalised geometric mean inequality

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \quad \alpha, \beta \geq 0, \quad 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

prove Hölder's inequality for  $C[a, b]$ , i.e.,

$$\int_a^b |f(t)g(t)| dt \leq \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \left( \int_a^b |g(t)|^q dt \right)^{\frac{1}{q}}.$$

(b) Using (a), prove that

$$\|f\|_p := \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \in C[a, b]$$

satisfies the triangle inequality.

*Hint:* Consider the corresponding proof for  $\ell_p(\mathbb{N})$  given in class.

**Solution:**

(a) If  $f$  or  $g$  is zero, there is nothing to prove. Suppose first that  $\|f\|_p = \|g\|_q = 1$ , then integrating  $|f(t)g(t)| \leq \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$  gives

$$\int_a^b |f(t)g(t)| dt \leq \frac{1}{p} \int_a^b |f(t)|^p dt + \frac{1}{q} \int_a^b |g(t)|^q dt = 1.$$

Finally, for  $f$  and  $g$  nonzero, substitute  $f/\|f\|_p$  and  $g/\|g\|_q$  (which have unit  $p$  and  $q$  norms) into the above inequality, and multiply both sides by  $\|f\|_p\|g\|_q$ .

(b) If  $\|f + g\|_p = 0$ , the triangle inequality holds, so suppose wlog (without loss of generality) that  $\|f + g\|_p \neq 0$ . Then  $|f(t) + g(t)| \leq |f(t)| + |g(t)|$  gives

$$(\|f + g\|_p)^p = \int_a^b |f(t) + g(t)|^p dt \leq \int_a^b |f(t)| |f(t) + g(t)|^{p-1} dt + \int_a^b |g(t)| |f(t) + g(t)|^{p-1} dt.$$

Hence, by Hölder's inequality, and  $p = (p-1)q$ ,

$$(\|f + g\|_p)^p \leq (\|f\|_p + \|g\|_p) \left( \int_a^b |f(t) + g(t)|^{(p-1)q} dt \right)^{\frac{1}{q}} = (\|f\|_p + \|g\|_p) (\|f + g\|_p)^{\frac{p}{q}}.$$

But  $p - \frac{p}{q} = 1$ , and so dividing by  $(\|f + g\|_p)^{\frac{p}{q}}$  gives the triangle inequality.

2. Here we consider  $\ell_p = \ell_p(\mathbb{N})$ ,  $1 \leq p \leq \infty$ .

(a) Show that  $\ell_p \subset \ell_r$ ,  $1 \leq p \leq r < \infty$ .

(b) Find a sequence which is in  $\ell_r$  but not  $\ell_p$ ,  $1 \leq p < r < \infty$ .

(c) If  $x \in \ell_p$ , for some  $1 \leq p < \infty$ , then is  $x \in \ell_\infty$ ? If  $x \in \ell_\infty$ , then is  $x \in \ell_p$  for some  $1 \leq p < \infty$ ?

(d) For which values of  $p$  is  $\ell_p$  an inner product space?

**Solution:**

(a) Suppose that  $x \in \ell_p$ , i.e.,  $\sum_{n=1}^{\infty} |x_n|^p < \infty$ . Since  $|x_n|^p \rightarrow 0$ ,  $\exists N$  with  $|x_n|^p < 1$ ,  $\forall n > N$ . But  $|x_n|^p < 1$  implies  $|x_n| < 1$  and hence  $|x_n|^r \leq |x_n|^p$ , so that

$$\sum_{n=1}^{\infty} |x_n|^r = \sum_{n=1}^N |x_n|^r + \sum_{n=N+1}^{\infty} |x_n|^r \leq \sum_{n=1}^N |x_n|^r + \sum_{n=N+1}^{\infty} |x_n|^p < \infty \implies x \in \ell_r.$$

*Remark.* By considering the sequence  $n \mapsto |x_n|^p / \|x\|_p \leq 1$ , one can show  $\|x\|_r \leq \|x\|_p$ .

(b) Recall that  $\sum \frac{1}{n^\alpha}$  converges for  $\alpha > 1$  and diverges for  $\alpha = 1$  (the harmonic series). Hence  $x = (\frac{1}{n^{1/p}})_{n=1}^{\infty}$  belongs to  $\ell_r$ ,  $r > p$ , but not  $\ell_p$ .

(c) If  $x \in \ell_p$ , then by the estimate of (a),  $\|x\|_\infty \leq \max\{|x_1|, \dots, |x_N|, 1\}$ , so  $x \in \ell_\infty$ . Any nonzero constant sequence, e.g.,  $x = (1, 1, \dots)$  is in  $\ell_\infty$ , but not in any  $\ell_p$ ,  $1 \leq p < \infty$ .

(d) For  $x = e_1 = (1, 0, 0, \dots)$ ,  $y = e_2 = (0, 1, 0, \dots)$  the parallelogram law fails for  $p \neq 2$

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2^{2/p} + 2^{2/p} \neq 2(1 + 1) = 2(\|x\|_p^2 + \|y\|_p^2),$$

and so  $\ell_p$ ,  $p \neq 2$  is not an inner product space. For  $p = 2$  the parallelogram law holds

$$\begin{aligned} \|x + y\|_2^2 + \|x - y\|_2^2 &= \sum_n |x_n + y_n|^2 + \sum_n |x_n - y_n|^2 \\ &= \sum_n (|x_n|^2 + x_n \bar{y}_n + \bar{x}_n y_n + |y_n|^2) + \sum_n (|x_n|^2 - x_n \bar{y}_n - \bar{x}_n y_n + |y_n|^2) \\ &= 2 \sum_n |x_n|^2 + 2 \sum_n |y_n|^2 = 2(\|x\|_2^2 + \|y\|_2^2), \end{aligned}$$

and so  $\ell_2$  is an inner product space. To justify the above, either first consider partial sums, or use Hölder's inequality to conclude that all the above sequences are absolutely convergent.

3. Here we show that  $C[a, b]$  with the uniform norm is complete. Let  $(f_n)$  be Cauchy in  $(C[a, b], \|\cdot\|_\infty)$ .

(a) Show that the complex sequences  $(f_n(x))$  are uniformly Cauchy in  $x$ , i.e.,  $\forall \varepsilon > 0$ ,  $\exists N$  such that

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m > N \text{ and } \forall x \in [a, b].$$

(b) Let  $f(x)$  be the limit of the above sequence. Show that  $f_n \rightarrow f$  in the uniform norm as  $n \rightarrow \infty$ .

(c) Verify that convergence in the uniform norm is the same as uniform convergence, and so that  $f$  as the uniform limit of continuous functions is continuous, and hence  $(C[a, b], \|\cdot\|_\infty)$  is complete.

(d) Is  $(C[a, b], \|\cdot\|_\infty)$  a Hilbert space?

*Hint:*  $|f(x)| \leq \|f\|_\infty, \forall f \in C[a, b], \forall x \in [a, b]$ .

**Solution:**

(a) Since  $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty, \forall x \in [a, b]$  and  $(f_n)$  is Cauchy, we have  $\forall \varepsilon > 0, \exists N$  such that

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon, \quad \forall n, m > N \text{ and } \forall x \in [a, b].$$

(b) In the above estimate, we may take the limit as  $m \rightarrow \infty$ , to obtain

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \varepsilon, \quad \forall n > N \text{ and } \forall x \in [a, b].$$

Thus  $\|f_n - f\|_\infty \leq \varepsilon, \forall n > N$ , i.e.,  $f_n \rightarrow f$  in the uniform norm.

(c) This follows, since by (b)

$$|f_n(x) - f(x)| \leq \sup_{x \in [a, b]} |f_n(x) - f(x)| = \|f_n - f\| \leq \varepsilon, \quad \forall n > N \text{ and } \forall x \in [a, b],$$

and conversely take the supremum over  $x \in [a, b]$  of the estimate for uniform convergence.

(d)  $(C[a, b], \|\cdot\|_\infty)$  is not a Hilbert space since it fails the parallelogram law, e.g., take  $f, g$  nonzero with disjoint supports, then

$$\|f + g\|_\infty^2 + \|f - g\|_\infty^2 = 2(\|f\|_\infty + \|g\|_\infty)^2 \neq 2(\|f\|_\infty^2 + \|g\|_\infty^2).$$

4. Here we show that  $C[-1, 1]$  with the  $L_p$ -norm,  $1 \leq p < \infty$  is not complete.

Let  $f_n : [-1, 1] \rightarrow \mathbb{R}$  be the sequence of continuous functions given by

$$f_n(x) = \begin{cases} 1, & -1 \leq x \leq 0; \\ 1 - nx, & 0 < x < \frac{1}{n}; \\ 0, & \frac{1}{n} \leq x \leq 1. \end{cases}$$

(a) Show that  $(f_n)$  is Cauchy in the  $L_p$ -norm.

(b) Show that if  $f_n$  has an  $L_p$ -limit  $f$ , then  $f = 1$  on  $[-1, 0]$  and  $f = 0$  on  $[\varepsilon, 1], \forall 0 < \varepsilon \leq 1$ .

(c) Show that  $C[-1, 1]$  with the  $L_p$ -norm is not a Banach space.

*Remark:* The completion of  $C[a, b]$  in the  $L_p$ -norm is studied Maths 730. It is called the Lebesgue space, and is denoted by  $L_p[a, b]$ .

**Solution:**

(a) For  $n \geq m$ ,  $f_n - f_m$  is zero outside  $[0, \frac{1}{m}]$ , so that

$$\|f_n - f_m\|_p = \left( \int_{-1}^1 |f_n(t) - f_m(t)|^p dt \right)^{\frac{1}{p}} \leq \left( \int_0^{\frac{1}{m}} 1 dt \right)^{\frac{1}{p}} = \left( \frac{1}{m} \right)^{\frac{1}{p}} \rightarrow 0, \quad m \rightarrow \infty,$$

and so  $\{f_n\}$  is Cauchy (in  $L_p$ ).

(b) Suppose that  $f \in C[-1, 1]$  is the  $L_p$ -limit of  $\{f_n\}$ . Then

$$0 \leq \int_{-1}^0 |1 - f(t)|^p dt = \int_{-1}^0 |f_n(t) - f(t)|^p dt \leq \int_{-1}^1 |f_n(t) - f(t)|^p dt = (\|f_n - f\|_p)^p,$$

and

$$0 \leq \int_{\varepsilon}^1 |0 - f(t)|^p dt = \int_{\varepsilon}^1 |f_n(t) - f(t)|^p dt \leq (\|f_n - f\|_p)^p, \quad (n > \frac{1}{\varepsilon}).$$

Since  $(\|f_n - f\|_p)^p \rightarrow 0$  as  $n \rightarrow \infty$ , the Sandwich theorem applied to the above inequalities gives

$$\int_{-1}^0 |1 - f(t)|^p dt = 0, \quad \int_{\varepsilon}^1 |0 - f(t)|^p dt = 0 \implies f = 1 \text{ on } [-1, 0], f = 0 \text{ on } [\varepsilon, 1].$$

(Since the Riemann integral of a nonnegative continuous function is zero if and only if it is the zero function). Thus the limit  $f$  (should it exist) is 1 on  $[-1, 0]$  and 0 on  $(0, 1]$ .

(c) Such a function  $f$  cannot be continuous (consider the right and left hand limits at 0), and so  $C[-1, 1]$  is not complete in the  $L_p$ -norm ( $1 \leq p < \infty$ ), i.e., is not a Banach space.

5. Let  $\hat{B}_r(a)$  be the **closed ball**

$$\hat{B}_r(a) := \{x \in X : d(x, a) \leq r\}, \quad r > 0.$$

(a) Show that  $\hat{B}_r(a)$  is a closed set

(b) Show that every open set can be written as a union of closed balls.

(c) Find the interior, boundary and exterior of the closed ball  $\hat{B}_r(a)$ .

(d) Show that the boundary of the open ball  $B_r(a)$  may or may not be the sphere

$$S_r(a) := \{x \in X : d(x, a) = r\}.$$

(e) Prove or disprove. The closure of the open ball  $B_r(a)$  equals  $\hat{B}_r(a)$ .

**Solution:**

(a) Let  $c \in X \setminus \hat{B}_r(a)$ , so that  $d(a, c) > r$ . With  $\delta := d(a, c) - r > 0$ , we have

$$B_\delta(c) = \{x \in X : d(x, c) < \delta\} = \{x : d(a, c) - d(x, c) > r\} \subset \{x : d(a, x) > r\} = X \setminus \hat{B}_r(a),$$

since  $d(a, x) \geq d(a, c) - d(x, c)$  by the triangle inequality. Hence  $X \setminus \hat{B}_r(a)$  is open, and so  $\hat{B}_r(a)$  is closed.

(b) The open ball  $B_r(a)$  can be written as a union of closed balls

$$B_r(a) = \bigcup_{\delta < r} \hat{B}_\delta(a),$$

since

$$\begin{aligned} x \in B_r(a), x \neq a &\implies \delta_x := d(x, a) < r \implies x \in \hat{B}_{\delta_x}(a) \subset \bigcup_{\delta < r} \hat{B}_\delta(a), \\ x \in \bigcup_{\delta < r} \hat{B}_\delta(a) &\implies x \in \hat{B}_\delta(a), \exists \delta < r \implies d(x, a) \leq \delta < r \implies x \in B_r(a). \end{aligned}$$

Since every open set can be written as a union of open balls, each of which is a union of closed balls, every open set is a union of closed balls.

Alternatively, if  $U$  is open, then centred at each point  $a \in U$  there is an open ball (and hence a closed ball of smaller radius, say half) which is contained in  $U$ ; take the union of these closed balls.

(c) Let  $V = \hat{B}_r(a)$  and  $x \in X$ . Similarly, to (a)

$$d(x, a) < r \implies B_\delta(x) \subset V, \delta := r - d(x, a),$$

$$d(x, a) > r \implies B_\delta(x) \subset X \setminus V, \delta := d(x, a) - r,$$

If  $d(x, a) = r$ , then  $x \in B_\delta(x) \cap V$ , whilst  $B_\delta(x) \cap (X \setminus V)$  may or may not be empty for all  $\delta > 0$ . Let

$$D := \{x \in X : d(x, a) = r, B_\delta(x) \cap (X \setminus V) \neq \emptyset, \forall \delta > 0\}.$$

Then

$$\text{int}(V) = B_r(a) \cup (S_r(a) \setminus D), \quad \partial V = D, \quad \text{ext}(V) = X \setminus \hat{B}_r(a),$$

(d) The boundary of the open (or closed ball) may equal  $S_r(a)$ , e.g., in  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , or not, e.g.,  $X = (-\infty, 0] \cup \{1\}$  (with the Euclidean metric)

$$B_1(0) = (-1, 0], \quad \hat{B}_1(0) = [-1, 0] \cup \{1\}, \quad S_1(0) = \{-1, 1\},$$

$$\text{int}(B_1(0)) = (-1, 0], \quad \partial B_1(0) = \{-1\}, \quad \text{ext}(B_1(0)) = (-\infty, -1) \cup \{1\}.$$

$$\text{int}(\hat{B}_1(0)) = (-1, 0] \cup \{1\}, \quad \partial \hat{B}_1(0) = \{-1\}, \quad \text{ext}(\hat{B}_1(0)) = (-\infty, -1).$$

(e) Since the closure of set is the set union its boundary, the example in (d) shows that the open ball  $B_r(a)$  may or may not equal the closed ball  $\hat{B}_r(a)$ .