ROD GOVER: Lecture 2, Taipa preparatory lectures.

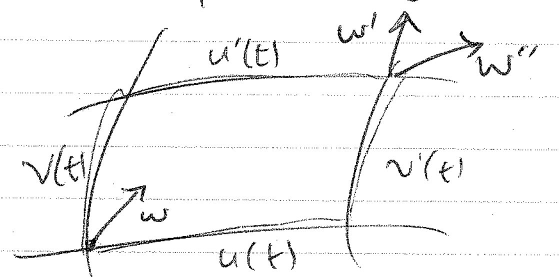
Riemannian mfd. $= (M, g)$

M: smth. mfd.

g: symm. +ve def bilinear form.

Riem. mfd. has preferred way of differentiating vectorfields along curves.

CURVATURE, R, measures the failure of parallel transport to be independent of the route.



$R(u,v)w$ measures the difference between w' & w''.

In terms of local coordinates the metric is determined by a matrix of functions g_{ij}

$$g(u,v) = (u^1, \ldots, u^n) \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

where vectors $u = (u^1, \ldots, u^n)$, $v = (v^1, \ldots, v^n)$ in local coords.

$\Rightarrow$ R is determined by an $n \times n \times n \times n$ 'matrix' of functions.

(Can write down explicitly, in terms of local coords, formula for each entry).

$$R(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w$$

(for any operators A, B, $[A,B] = AB - BA$).

The RICCI TENSOR is a trace of R determined by

$$Ric_{jl}{}^{i} = \sum_{i=1}^{n} R_{ij}{}^{i}{}_{l}$$

RICCI FLOW: Family of t parametrised metrics on mfd. M determined by:

$$g(0) = g_0$$
$$\frac{\partial g_{ij}(t)}{\partial t} = -2\,Ric$$

NORMALISED RICCI FLOW:

$$\cancel{\frac{\partial g(t)}{\partial t} = -2Ric}$$

$$\frac{\partial g_{ij}(t)}{\partial t} = -2Ric_{ij} + \frac{2}{n}\left(\frac{\int_M Sc^g\, dV_g}{\int_M dV}\right) g_{ij}$$

(where $Sc^g = \sum_{i=1}^{n} g^{ij} Ric_{ij}$ is SCALAR CURVATURE)

OBSERVATION ABOUT RICCI TENSOR

Ask that all eigenvalues of the Ricci matrix are the same

i.e. $\boxed{Ric_{jk} = \lambda g_{jk}}$ (*)

Metrics satisfying this are called EINSTEIN METRICS

[FACT: (~~Example~~ Exercise) (*) $\Rightarrow \lambda$ = constant]

Einstein metrics are optimal in following sense:

Define action $S[g] := \int_M Sc^g\, dV_g$

$$\left(\sqrt{\det g_{ij}}\, dx^1 \ldots dx^n\right)$$

$g_t = g + th$, where h is arbitrary "matrix".

Solve $\frac{d}{dt} S[g_t] = 0$

Get $$\int_M \left(\frac{Sc}{2} g - Ric, h \right) dV_g = 0.$$

$$\Rightarrow \quad Ric = \frac{Sc}{2} g$$

$$\Rightarrow \quad Ric = \lambda g$$

Back to normalised Ricci flow: want something like $\frac{\partial}{\partial t} g = 0$ at "$t = \infty$"

$$\Rightarrow \quad -2Ric + \rho g = 0.$$

$$\Rightarrow \quad g \text{ Einstein metric.}$$

What is special in dimension 3?

FACT/OBSERVATION: 3 is an odd number

On even dimensional manifolds it is "easier" to extract topological information from local geometry.

E.g. $n = 2$ — oriented cpt. surface.

$$\text{"}Sc[g] = \int_M Sc \, dV_g = 4\pi \chi(M) = \cancel{4\pi}(2 - 2\,\text{genus}$$

More generally, for n-even:

$$(\text{const.}) \chi(M) = \int_M Pff \, dV_g$$

where $Pff = \text{"det"}(R_{ij}{}^{k}{}_{\ell})$, the PFFAFIAN.

$$= \sum \epsilon^{ij\ldots mn} \underbrace{\epsilon_{pq\ldots rs}}_{\substack{\text{completely alternating} \\ n\text{-forms}}} R_{ij}{}^{pq} \ldots R_{mn}{}^{rs}$$

This cannot work in odd dimensions since if n odd
$2\,\text{odd} = 4k+2$ indices ~~on~~ on forms $\in$ ad
$4l$ indices on the curvatures.

OBSERVATION: 3 is small odd number:

[TRACE FREE: M_{ij} s.t. $\sum_i M_{ij}\delta^{ij} = 0$] Kronecker δ

Write Ricci:

$$R_{ij}{}^{k}{}_{l} = W_{ij}{}^{k}{}_{l} + P \wedge g$$

P_{ij} trace modification of Ricci.

trace free w.r.t. metric

$(Ric = (n-2)P + Jg)$

$W_{ij}{}^{k}{}_{l}$ is WEYL TENSOR.

FACT: Weyl tensor is <u>CONFORMALLY INVARIANT</u>.

i.e. if $\hat{g} = fg$, $f \in C^\infty(M)$, f non-vanishing
$\Rightarrow W^{\hat{g}} = W^{g}$

But for $n=3$, $W=0$ (Weyl tensor vanishes).

[Then if g Einstein then $Ric = \lambda g$
$\Rightarrow P = \lambda g$
$\Rightarrow R$ constant.]
(curvature)

<u>'Proof'</u> ($W=0$ for $n=3$)

In any n, the Riem. curv. has symmetries.

Recall $R(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w$

$\Rightarrow$ (1) $R_{ij}{}^{k}{}_{l} = R_{ji}{}^{k}{}_{l}$

Also (2) $R_{ij}{}^{k}{}_{l}$

corresponds to $Lie(G) \cong so(n)$.

(3) $R_{ijkl} = -R_{ijlk}$

(4) Jacobi identity Cyclic sum$\left([[A,B],C]\right) = 0$

EXERCISE: Jacobi identity + torsion free
$\Rightarrow$ $R_{ijkl} + R_{kijl} + R_{jkil} = 0$.
— BIANCHI IDENTITY.

FACT: Weyl curvature tensor has same symmetries.

Symmetries + trace free $\Rightarrow$ must be zero if $n=3$ ■

Dimension 3: the COTTON TENSOR, C_{ijk}, is conformally invt

Schouten tensor

$$\boxed{C_{ijk} = \nabla_j P_{ki} - \nabla_k P_{ji}}$$

(This all leads on naturally to CONFORMAL GEOMETRY)