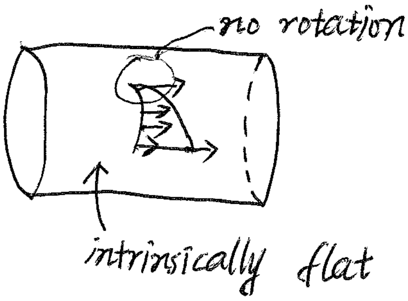
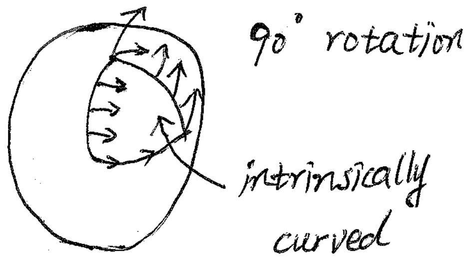
28/11/05 Mon

in E^3 $(R^3, \cdot)$

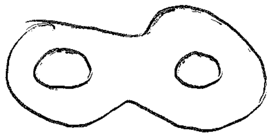
What's the difference between a cylinder and a sphere?



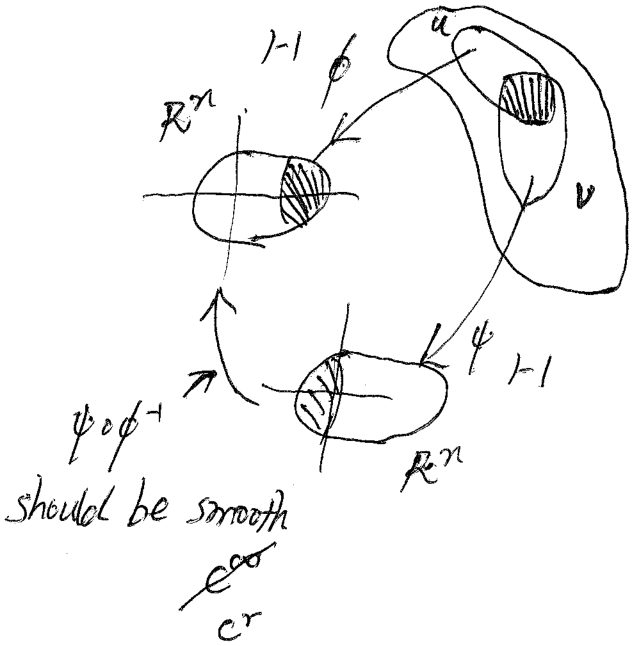


The cylinder has non-vanishing smooth vector v-field

Differential topology in R^3



and higher dim generlisations - manifold



(set M) + atlas = smooth manifold

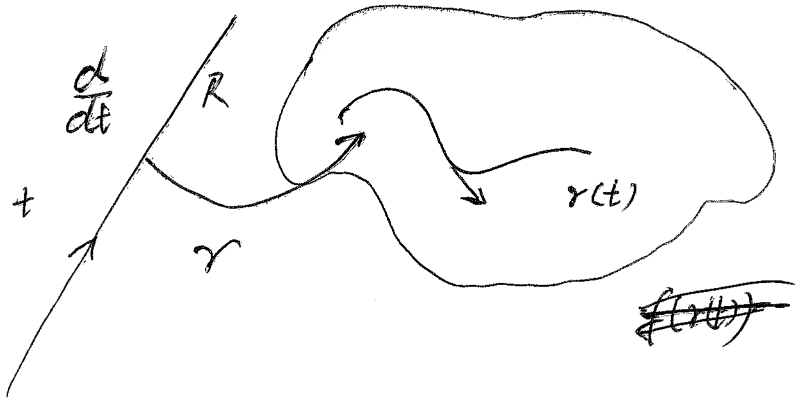
↳ collection of charts (U, ϕ)

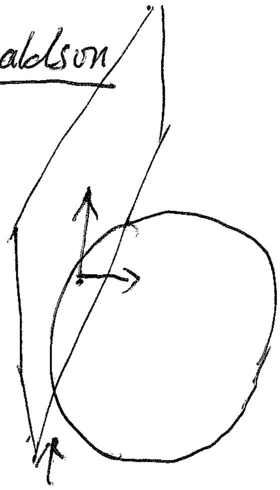
All notion of topology and smoothness determined by the atlas

- 1950's $\exists$ smooth 7 manifolds homeo to std S^7 but not diffeomorphic
- $\exists$ topological manifolds $\bar{w}$ no smooth structure
- 1980's Freechram classified simply curiecte cpt topological ~~on~~ 4 manifolds

— smooth structures on M^4 is not well understood. Donaldson

First





T_pM – a plane in E^3 that osculates at p

M – ~~can~~ not embedded

$r_* \frac{d}{dt}$ is the derivation along $f(r(t))$ trace of r given by $\frac{d}{dt} f(r(t))$

i.e. an operator D s.t. if f, g are $f^n{}_s$ on M

$$D(fg) = f\,Dg + g\,Df$$

In E^n – vector $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$$\boxed{\underset{\text{grad}}{v^i \nabla_i f} \quad — \quad v^i \frac{\partial}{\partial x^i} f}$$

ex rotate to above

D Geometry – M smooth mfld + local structure

An example of a D.G. is a Riemanian mfld, a manifold M equipped with a metric g meaning a symmetric bilinear 2-form on M that is +ve definite

$$g : T_pM \times T_pM \to R$$

$$g(V, V) \geqslant 0 \qquad = 0 \Rightarrow V_p = 0$$

In Euclidean space we know how to differentiate vector fields.

— Just do it componentwise

$$\nabla_{\delta} V = \begin{pmatrix} \frac{dV^1}{dt} \\ \vdots \\ \frac{dV^n}{dt} \end{pmatrix} = \lim_{\delta \to \infty} \frac{V(t+\delta) - V(t)}{\delta}$$

Evidently Euclidean space has canonical connection

A connection ∇,

u, v, w — vector fields on M.

a map $(u, v) \longmapsto \nabla_u V$ — v-field

(i) $\nabla_{fu+v} W = f \nabla_u W + \nabla_v W$

linear over ring of $f \in C^\infty(M)$

(ii) $\nabla_u (fv) = f \nabla_u V + (df)(u) \cdot V$

Riemanian manifolds have a canonical connection

proof (first note: parallel transport of vectors along curves $\Leftrightarrow$ a connection)

$\Rightarrow$ obvious a v-field

$\Leftarrow$ we say that (v is ~~parallel~~ parallel along $\gamma(t)$ if $\nabla_{\dot\gamma} \cdot v = 0$)

If ∇ a connection: define T by $T(u,v) = \nabla_u v - \nabla_v u - [u,v]$ for smooth v-fields u, v.

$$\underset{\searrow}{C^\infty(M)}$$

$$[u,v]f = uvf - vuf$$

<u>Ex</u> T is a <u>tensor</u> i.e. $T(fu+v, w) = fT(u,w) + T(v,w)$

Let ∇ be the connection. Require ∇ preserve the ? g

i.e. $ug(u,w) = g(\nabla_u v, w) + g(v, \nabla_u w)$ $(*)$

$T^\nabla = 0$ — torsion free.

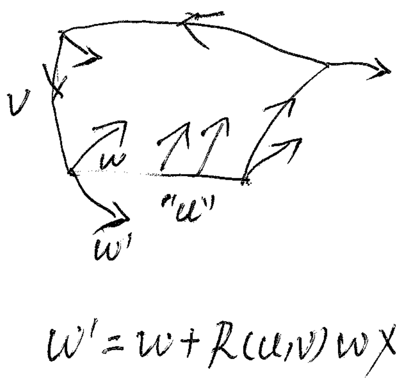
<u>Ex</u> Then we can solve $(*)$ to get a formula for ∇ from g

<u>curvature</u> : Given connection ∇, its curvature R^∇ is defined by

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

<u>Ex</u> this is a tensor

<u>measures</u> in the limit. parallel transport around small loops.



$w' = w + R(u,v)w \times$ length of ?