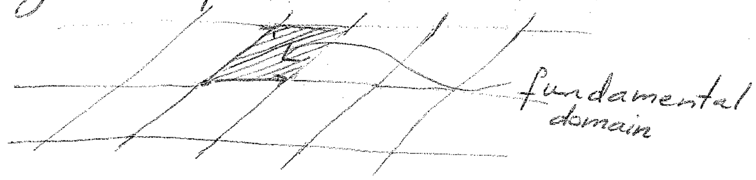
Geometrisation Conjecture:
Given a compact orientable 3-manifold there is a canonical decomposition along 2-spheres and tori and Klein bottles such that each resulting piece is geometric.

A 3-manifold is <u>geometric</u> if each point has a neighbourhood that is isometric to a subset of one of the eight 3-dim. geometries.
This is equivalent to the following:
M^3 is geometric if we can find a group $\Gamma \subset \mathrm{Isom}_+(X_i)$, where X_i is one of the eight geometries, such that $M \overset{\text{homeo}}{\cong} X_i/\Gamma$

e.g. 2-dim T^2 = 

T^2 is $\mathbb{E}^2$-geometric
Find subgroup $\mathbb{Z} \oplus \mathbb{Z} \subset \mathrm{Isom}(\mathbb{E}^2)$ generated by two linearly independent translations.



T^2

If $M = X_i/\Gamma$, X_i geometry, then Γ is isomorphic to the fundamental group $\pi_1(M)$ of M.

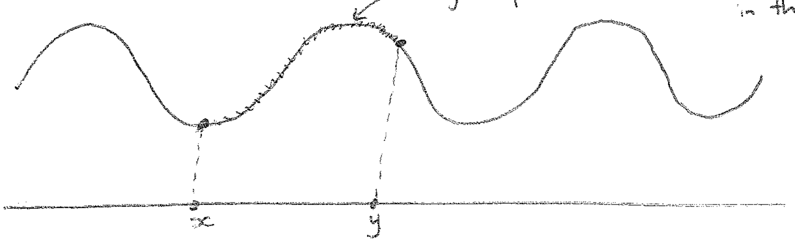
Geom Conj $\Longrightarrow$ Poincaré Conj.
Suppose M^3 is compact 3-manifold such that $\pi_1(M) = 0$. Let $M = M_1 \# M_2 \# \cdots \# M_n$ (using Geom. Conj.) be canonical decomposition along 2-spheres such that each M_i is geometric.
(NOTE: no (interesting) tori or Klein bottles since $\pi_1(M) = 0$)
Now each M_i has $\pi_1(M_i) = 0$ and each M_i is geometric
$\Rightarrow$ each M_i is one of the eight geometries.
Since ~~M~~ each M_i compact
$\Longrightarrow M_i = S^3 \quad \forall i$ (S^3 only compact geometry)
$\Longrightarrow M = S^3 \quad (n=1)$

§ Metrics & Ricci Flow

e.g. 1-dim. Metrics on $\mathbb{R}^1$:

$$ds = g(x)\,dx \quad , \quad g(x) > 0$$

e.g. $ds = (2 + \sin x)\,dx$



Goal of Ricci flow: Start with any metric on manifold M^3. Flow this metric to a 'nice' metric using the Ricci flow.

$$(g'(t) = \mathrm{Ric}(g(t)))$$

$g(t)$ is metric on M at time t.

Observation: if $g(t) = g(0)$ then metric is locally S^3, $\mathbb{E}^3$ or $\mathbb{H}^3$

Hope: Ricci flow will evolve the initial metric $g(0)$ to one of these three metrics.

Problems: in finite time, the metric 'blows up' at some points.

Perelman: topologically, only two ways this can happen. Corresponds to the connected sum (Kneser) decomposition. Perelman shows that places at which metric $\to \infty$ correspond to $S^2 \subset M$ (interesting 2-spheres)