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Community for Undergraduate Learning in the Mathematical Sciences

The CULMS Newsletter

CULMS is the Community for Undergraduate Learning in the Mathematical Sciences.

This newsletter is for mathematical science providers at universities with a focus on teaching and learning.

Each issue will share local and international knowledge and research as well as provide information about resources and conferences.

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Editors: Bill Barton, The University of Auckland
Mike Thomas, The University of Auckland

Associate Editor: Louise Sheryn, The University of Auckland

Published by: The Mathematics Education Unit,
The University of Auckland,
Private Bag 92019
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Correspondence: Louise Sheryn
l.sheryn@math.auckland.ac.nz

Website: www.math.auckland.ac.nz/CULMS

Previous editions of the newsletter can downloaded from our website or for a hard copy please email Louise Sheryn at l.sheryn@math.auckland.ac.nz

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Guest Editorial

This issue's Guest Editorial is by Kevin McLeod, Associate Professor in the Department of Mathematical Sciences, University of Wisconsin-Milwaukee.

What Do We Want Students of Mathematics to Learn?

As mathematicians and mathematics educators, we consider it axiomatic that most people should learn – or at least be exposed to – a significant amount of mathematics during their education. Fortunately for us, our opinion seems to be shared by a majority of educational policymakers and politicians, not to mention the general public. If we were to ask these other groups what it means to learn mathematics, however, or what students should learn from their study of mathematics, we might find that their answers differed somewhat from ours.

In policy circles, the teaching and learning of mathematics is usually justified on the basis of need: it is argued that a flourishing modern economy needs large numbers of technically trained employees, and these potential employees will need strong backgrounds in mathematics. The new Common Core State Standards in the United States, for example, have been largely justified as the mathematics students need to be college or career ready (See their Mission Statement: Common Core State Standards Initiative, 2010). It is hard to disagree with the need for an adequately trained labour force, but I want to argue that an over-emphasis on need as the justification for learning mathematics is potentially both dangerous and limiting for the field of mathematics education.

What is wrong with concentrating on need? It is certainly a quick and easy way for us to justify the place of mathematics in the curriculum, but there are at least three serious drawbacks. In the first place, if we are really honest, there is very little mathematics beyond the middle grades curriculum (perhaps Years 8-10 in New Zealand) that most people can be said to *need.* Secondly, an over-emphasis on need will result in our ignoring a great deal of truly beautiful and interesting mathematics that absolutely nobody could be said to *need*. Thirdly, do we really believe that "You need to know this," or (even worse!) "You will need to know this in next year's math class" is sufficient motivation for the majority of our students?

Underwood Dudley (2010) presents an amusing instance of the first point in a recent article in the Notices of the American Mathematical Society:

> Once, when I was an employee of the Metropolitan Life Insurance Company, I was given an annuity rate to calculate. Back then, insurance

> companies had rate books, but now and then there was need for a rate not in the book. Using my knowledge of the mathematics of life contingencies, I calculated the rate. When I gave it to my supervisor he said, "No, no, that's not right. You have to do it *this* way." "But," I said, "that's three times as much work." Yes, I was told, but that's the way that we calculate rates. My knowledge of life contingencies got in the way of the proper calculation, done the way it had been done before, which any minimally competent employee could have carried out.

(The rest of Dudley's article is also well worth reading.)

I suspect a common response to the second drawback above would be that most people do not find mathematics interesting. I disagree: I think most people are very interested in *interesting* mathematics. Consider, for example, Fermat's Last "Theorem", or the Poincaré Conjecture. The former is a statement that whole numbers do not fit together in certain ways; the latter gives a characterization of spheres – but of 3-dimensional spheres, which can only exist in spaces of 4 or more dimensions, not our "real" 3-dimensional space. Neither is of any apparent use to a modern economy, and yet the recent proofs of the two results, by Andrew Wiles and Grigory Perelman respectively, were headline news in major newspapers around the world. There have also been popular book-length expositions of both results and, in the case of Fermat's Last Theorem, a television documentary, all of which suggest widespread public interest in the results.

So, if we should broaden our emphasis beyond need, what should we be asking? I suggest we should be asking instead: "What do we want our students to learn from their study of mathematics?" I will give my suggestions in the next paragraph, but before reading them, you might want to pause and consider what your own answer(s) would be.

What I would like students to learn is that

1. Mathematics is an *intrinsically interesting subject*, worthy of study in its own right;
2. Mathematics is a *powerful* subject, capable of giving us great insight into the workings of the material world (both natural and man-made); and
3. Mathematics is a *living* subject, with longstanding connections to human culture and history, but with more new mathematics being developed now than at any other time in that history.

Feel free to differ, and to let me know your answers (especially if they are very different), but I hope you would at least acknowledge that these are worthy goals. If you do, then we should proceed to ask how they can be attained.

At the tertiary level, we might acknowledge that if our students have not made any progress towards these goals before they come to us, it may be too late. (We should also acknowledge that we would like students who are not necessarily bound for tertiary education, or who plan to study subjects other than mathematics at tertiary level, to develop the same views.) This is not a question of passing the buck to our colleagues in primary and secondary education (colleagues who were largely trained by us, let's not forget); rather, it is an observation that we may have a responsibility to work with teachers, as well as curriculum and standards writers, to ensure that our desiderata are being met.

When it comes to designing our own courses, keeping these 3 goals in mind may again change our thinking in important ways. A focus on need tends to produce lists of relatively isolated skills. (Do students *need* to know how to diagonalise a matrix by hand? Do they *need* to know one or another technique of integration?) Focusing on whether they learn to see mathematics as intrinsically interesting and powerful should lead to a more holistic approach to course design: after this course, what will the students have come to understand about the beauty, power and history of mathematics? Part of the focus on beauty is surely seeing mathematics as an integrated discipline, which would help to reduce the piecemeal substitution of one "vital" topic for another. Focusing on the power of mathematics would lead to students solving meaningful problems, and hence towards a problem-solving (rather than skills-based) approach to course design. Bringing the history and culture of mathematics into our courses will increase students' awareness of mathematics as a human activity, and possibly console them in their own struggles as they realize that even the greatest mathematicians of the past struggled to formulate and work with concepts that we now take for granted.

So I encourage all of us involved in the teaching of mathematics at any level to move away from asking what our students need to know, and begin a conversation on what we want them to learn. If that conversation results in more students realizing that mathematics is beautiful, powerful and alive, we may find that, after all, they are more motivated to learn the mathematics that we think they need to know!

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Kevin McLeod
kevinm@uwm.edu

Students as Partners in Mathematics Course Design: Some Findings from an Ethnographic Case Study

Francis Duah and Tony Croft
Mathematics Education Centre, Loughborough University

This paper reports some preliminary findings from an ethnographic case study of four undergraduate mathematicians who collaborated with staff as student interns to redesign two second year courses, *Vector Spaces and Complex Variables*, by producing engaging teaching and learning resources for second year students. Data on the students' experiences were collected via diaries kept by the students, self-reflection and evaluation reports produced by the students, participant observation, and field-notes. Members of staff were also interviewed individually in order to collect data to triangulate the students' accounts so as to increase the validity and reliability of the findings. In this paper, we only report on the data collected from the students. Findings from the study showed that the four students were socialised and drawn from the margins of "legitimate peripheral participation" in academic practice into full participation of a community of practising mathematicians. The four student interns were able to play an important role as mathematics course designers, and gained a deeper understanding of the mathematics they worked on.

Introduction

Study Background

In the United Kingdom (UK), studies conducted by Brown, William, Barnard, Rodd and Macrae (2002) have shown that beyond the transition year of undergraduate study of mathematics, some students become disengaged and disillusioned with their studies because of poor performance. For some of these students, the difficulties they experience with their studies may be attributed to the on- or off-campus activities that they engage in, such as employment, which may leave them with little time to devote to their studies. For others, their lack of success and progression may be attributed to the very nature of undergraduate mathematics, which is different from school mathematics, where solutions to mathematics problems are often routine and predictable. The design and the delivery of an undergraduate mathematics course could also impact on student engagement with their study of the course and hence on performance. Whatever the attribution of students' underperformance, students will become dissatisfied with their study of mathematics if they persistently underachieve and consequently this may lead to student attrition. Enhancing the student learning experience and increasing student engagement are now hot topics in higher education discourse, as they are believed to improve performance.

In recent years, there have been calls to the higher education community to involve students in the planning and design of courses (Porter, 2008; Kay, Marshall & Norton, 2007). For example, the 1994 Group of Universities in the UK noted in its policy report entitled Enhancing the Student Experience that member institutions should involve students in the planning and design of courses because students "know how they want to be taught and have ideas about how teaching techniques could be improved." (Kay, Marshall & Norton, 2007, p. 12)

Between March and August 2011, the first author conducted literature searches on direct student involvement in course design and found some examples where students have been involved in the design of non-mathematical sciences courses. However, to the best of our knowledge, there is a dearth of examples where undergraduate mathematicians have been involved in the design of mathematics courses. In a literature review on student involvement in course design, Bovill, Bulley, and Morss (2011) found limited examples of direct student participation in the design of Geography, Education and Environmental Justice courses. In further work on evaluating these examples, Bovill, Cook-Sather and Felten's (2011) note that staff and students stand to benefit from a collaborative approach to course design. Similarly, Hess (2008) provides an account of his own approach to collaborative course design in a graduate law course but his account may be viewed as anecdotal. For the mathematical sciences community, empirical evidence that supports the potential benefits for staff and students in collaborative course design would be informative and increase our knowledge base on tertiary mathematics course design and delivery.

In 2010-2011 academic year, in an effort to enhance the second year mathematics experience so that student engagement and achievement can be increased in two historically problematic second year mathematics courses, the School of Mathematics, at Loughborough University, UK, embarked on a curriculum development project funded by the Higher Education Science, Technology, Engineering and Mathematics (HE-STEM) Programme. The two courses were *Vector Spaces and Complex Variables*. A unique feature of the curriculum development project now called SYMBoL[1], is the recruitment of four undergraduate mathematicians as paid summer interns to collaborate with staff to redesign the two courses by producing engaging teaching and learning resources for students. We designed an ethnographic study to understand the experiences of staff and in particular the four undergraduate

[1] http://sym.lboro.ac.uk

mathematicians. Among a number of research questions the study aimed to answer were:

1. What role are the student interns able to play?
2. What are the outcomes for the student interns?

Methodology

In March 2001, all second year undergraduate mathematicians who had enrolled on and studied *Vector Spaces and Complex Variables* were invited to apply for positions as student interns of which there were four. Eight students out of a cohort of about 100 applied for the positions. All eight students were interviewed by the staff who teach *Vector Spaces and Complex Variables* and an additional member of staff. Four students were successful and commenced their internship in March 2011. They worked part-time for two hours per week between March and June 2011, conducting focus groups to collect the views of their peers about the teaching and learning of *Vector Spaces and Complex Variables* to inform the course design process.

During July and August 2011, the students worked fulltime as student interns for six weeks. They worked closely with staff but with considerable autonomy to design teaching and learning resources. During that period, the first author immersed himself amongst the student interns; sharing an open plan office with them, interacting with them and responding to questions they may have about the use of technology in producing resources. While the student interns worked, Monday to Friday, the first author observed their activities and their interactions with staff and each other and took field-notes. The student interns also kept diaries, which they wrote up daily and sent to the first author at the end of each week. At the end of their internship, the student interns also wrote a self-reflection and evaluation report on their six weeks experience. The qualitative data collected were subjected to thematic analysis (Braun & Clarke, 2011) using NVivo 8 to generate codes which were later categorised into themes, two of which we describe and discuss in the next section.

Findings and Discussions

Role Played by Students

From the observations and field-notes data, we found that the students played two essential roles during their internship; *intermediaries and competent academic apprentices*. These new terms will be discussed in a future publication in *MSOR Connections* and the full research report. However, in this paper, we suggest that the student interns played the role of *intermediaries* between staff and the second year students by soliciting the 'student voice' through focus groups and other informal communication channels. The student voice sought for was more valuable than could be provided by the traditional feedback mechanism, which is perceived to have a different purpose; quality assessment rather than quality enhancement.

The richness and depth of the students' views about the teaching and learning of *Vector Spaces and Complex Variables* would not have been obtained with the traditional quantitative survey on course evaluations.

While working during the six week internship, the student interns had to liaise with the course leaders, produce teaching and learning resources, and seek feedback on the quality and mathematical accuracy of the content of the resources. Samples of resources that one pair of students produced for the *Complex Variables* course are shown in Figures 1 and 2 below.

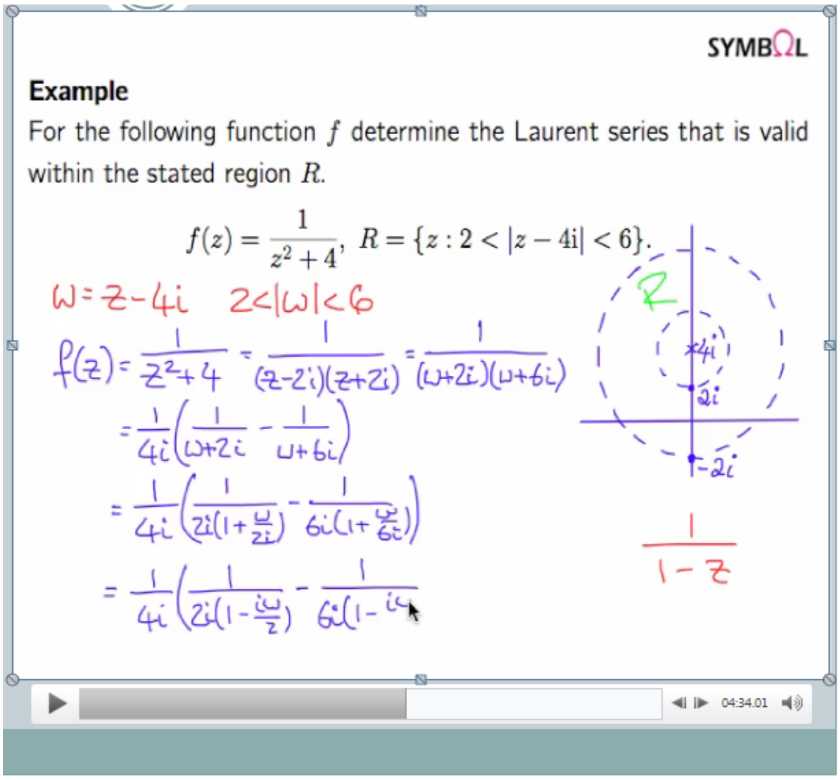

Figure 1. Screenshot of screencast video on Laurent series.

The internship provided the student interns with opportunities to work with the content of *Vector Spaces* and *Complex Variables* as *competent academic apprentices*. Again, although we have not discussed and defined this terminology in this paper, we suggest that the student interns were competent in the content of the mathematics they worked on by the virtue of having taken and passed the examinations. At the start of their internship, three of the student interns, while being competent, showed lack of understanding in some aspects of the content of the courses they were working on.

SYMBΩL

COMPLEX VARIABLES

RESIDUE THEOREM

1 The residue theorem

Suppose that the function f is analytic within and on a positively oriented simple closed contour C except for a finite number of isolated singular points $\{z_j, j = 1, 2, ..., N\}$ *interior to C, then*

$$\int_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=1}^{N} \operatorname*{Res}_{z=z_j} f(z). \tag{1}$$

A proof of this can be found in the lecture notes.
This is a very important result and can help us calculate integrals around contours that would be impossible to do using standard single variable calculus. The residue theorem can even be used when integrating along the real line.

2 Integrals around closed curves

The most obvious way of using this theorem is for finding an integral around a simple closed contour enclosing a finite number of singularities.

2.1 Example

Evaluate

$$I = \int_{C_1(1)} \frac{z}{z^2-1}\,\mathrm{d}z. \tag{2}$$

Solution

By factorizing the denominator of the integrand we get

$$\frac{z}{z^2-1} = \frac{z}{(z-1)(z+1)}.$$

Here we can see that the two poles of this function are at $z = \pm 1$, note that both these poles are simple. Only one of these poles, $z = 1$, is inside the contour, so we need to calculate the residue at this pole

$$\operatorname{Res} f(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z}{(z-1)(z+1)} = \lim_{z\to 1}\frac{z}{(z+1)} = \frac{1}{2}.$$

Now using the residue theorem we evaluate I by multiplying the sum of the residues by $2\pi\mathrm{i}$ to get

$$I = \int_{C_1(1)} \frac{z}{z^2-1}\,\mathrm{d}z = 2\pi\mathrm{i}\frac{1}{2} = \pi\mathrm{i}.$$

Compare this result to example 2.1 of the Cauchy integral formula handout. You will notice that this theorem is just an extension of the formula.

Figure 2. Screenshot of a help sheet on the residue theorem.

Community of Practicing Mathematicians

Community of Practice (Wenger, 1998) was used as an analytical lens to explore the relationship and interactions between staff and the four student interns. We also drew on themes from Bovill, Bulley and Morss's (2011) literature review as well as Bovill, Cook-Sather and Felten's (2011) study on examples of active student involvement in course design to explore the benefits that accrued to the student interns.

In this section we use extracts from transcripts, diaries, self-reflection and evaluation reports, and field-notes to provide evidence in support of answers to the research questions. Extracts attributable to the four student interns are identified as P1, P2, P3, and P4. Each of these identifiers is shown on the right of the related extract.

Staff and students had a *joint enterprise*; that is, to produce engaging teaching and learning resources to enhance the student learning experience. Throughout their internship, the student interns interacted with staff, discussed the mathematical content of the resources they produced, and built equal but professional relationships with staff. Thus, there was a *mutual engagement* amongst staff and the student interns. During their internship, the four student interns had a one-hour tea break each working day when they met in the office of a member of staff who provided refreshments. Through their *mutual engagement*, staff and students engaged in mathematical discourse in ways that lectures and tutorials do not make possible and developed a *shared repertoire* of resources as the following quotations from three of the student interns show:

> Meeting up with some of the staff for tea and biscuits was a good opportunity to get to know people a bit more, and made me feel much more involved and valued as a member of the project. (P2)

> It's good to be able to comfortably talk to lecturers about interesting points in mathematics, it's also interesting to hear what they do as mathematicians and how they work together or alone. (P3)

> I feel Lecturer 1 is more approachable now. (P4)

During the one-hour afternoon tea break, not only were the students acculturated to academic practice as the above quotations indicate, but they also received feedback on the content of the resources they produced. The mathematical accuracy of the resources was of paramount importance since one of the aims of the course redesign process was to make the resources available for use by other institutions. Hence, notwithstanding the autonomy the students had in their role as interns, they felt it was essential that the content of the resources they produced was reviewed by members of staff. Where such feedback was constructive it was often well received and led to revision of the resources as indicated by the following extracts from the diaries of two participants:

> Lecturer 1 has reviewed all of the materials that I have produced and provided feedback for each of them, so I now have to amend these. (P1)

> Got feedback, which I found helpful and constructive. (P4)

Through the process of resource production and feedback, we observed the students received informal training and advice akin to the '*apprenticeship model*' in a work place. Hence our introduction and use of the term *competent academic apprentices* to describe the role played by the student interns.

Although the student interns were enthusiastic about their role and sought and received constructive feedback regularly, our observations and field-notes data indicated that when feedback was perceived to be overtly critical or unrelated to mathematics, such feedback had an unexpected impact on the way the students sought feedback thereafter. For example, one intern hesitated seeking feedback on a very well produced document with a novel approach to solving a problem on Orthogonal Projections because he did not want to receive what he perceived to be critical feedback. Another participant receiving feedback on the use of good grammatical structures of the English language was not amused. For these students, it was the enjoyment of the mathematics that sustained their interest in their role and anything else seen as not mathematically related was not welcomed. This was particularly evident in week 1 when two student interns, identified as P1 and P2, felt that much of what they were doing was administrative duty and not challenging as can be seen from the following two statements made by the two student interns and recorded in the field-notes:

> I am getting bored with this [creating LaTex files] (P1)
>
> I created LaTex files [all day], which I found boring (P3)

Deepening Mathematical Understanding

The internship and the course redesign process provided opportunities for the student interns to gain a much deeper understanding of the course they helped to redesign. Consequently, they gained increased confidence in their abilities as demonstrated through the following quotations:

> My knowledge of Vector Spaces is also improving, as I discovered an application for a Theorem that I had not previously realized was possible. (P1)
>
> I found that as I was creating videos my understanding of the topics is becoming much deeper and I hope these skills will be transferable to other modules I take in the future. (P2)
>
> I feel [that] my knowledge of the Eigenvalue equation has improved a lot. My approach to learning will be very different after this internship. (P4)

Amongst the four student interns, the student identified as P1 was often positioned by the other three as the most able student. He is believed to be on track for a first class degree in Mathematics. However from the field-notes, we note that until the end of the six weeks internship, he did not have secured understanding in all areas of *Vector Spaces*; the module he worked on. He was observed on three occasions using a chalk board to devise a solution to a problem on Orthogonal Projection using a geometric approach and then used his solution to produce a supplementary help sheet for student use. He notes in his diary that his solution to the problem on Orthogonal Projection is different from the way the lecturer had previously explained it in lectures and tutorials. The following extract from the diary of P1 is typical of how the student interns believe that the internship experience has impacted on their mathematical understanding:

> I have had to use the blackboard several times to work through a problem, so that I understand it completely and can convey my understanding through the solutions. This has helped me understand the topics within the module better though, which I believe is very helpful. (P1)

Conclusion

This study showed that students can make a contribution as partners in mathematics course design and they benefit from the experience in several ways including a deeper understanding of the mathematics on which they work. The limitation of the current study, however, is that the four student interns constituted a convenience sample and hence we do not make generalization from the experiences of these four students. Nonetheless, this study appears to support the call for higher education institutions to involve students in shaping their own learning. The full findings of our study including the discussion of students as *intermediaries* and *competent academic apprentices* will be published in due course.

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Authors

Francis Duah and Tony Croft Loughborough University, United Kingdom.
Email: f.k.duah@lboro.ac.uk; a.c.croft@lboro.ac.uk

Comparing University Teachers' Perceptions of Teacher Oral Communication Behaviour in the Calculus, Algebra and Statistics Classes

Charles K. Assuah
Department of Mathematics Education, University of Education, Winneba

This study examined the effect of the subject taught (i.e., calculus, algebra, and statistics) by university teachers on their perceptions of teacher oral communication behaviour in the mathematics classroom. A survey design on perceptions of teacher communication behaviour of 105 university teachers from a western state in the U.S. was carried out using a questionnaire that was developed by She and Fisher in 2002 for data collection. A one-way analysis of variance (ANOVA) was used to investigate whether there were significant differences among university teachers' perceptions of teacher oral communication behaviour. The results indicate that there were no significant differences among the perceptions of the teachers resulting from the subjects they teach. The study concluded that the use of this format by teachers is only good if it positively reflects on students' mathematical understanding, and consequently enables them improve to on their test scores.

Introduction

The achievement of the objectives of school reforms as envisioned by the National Council of Teachers of Mathematics (NCTM) can only become a reality if teachers encourage students to share their ideas and communicate mathematical concepts with their peers. The NCTM (2000) makes a strong case that effective communication enables teachers to create supporting and challenging environments that actively engage students in a rich classroom conversational dialogue, which deepens their mathematical understanding.

Both teachers and students have an active role to play in this endeavour. Teachers have to use a variety of methods to engage students to communicate about mathematics, because students usually need to use words at least 30 times over, to enable them to make it part of their vocabulary (Thompson & Chappell, 2007). To address the issue of communication, Thompson and Chappell (2007) recommend that mathematics literacy should be an integral part of school instruction, because students are absorbed in a world of language that challenges them to speak, write, read, and listen to mathematics. By doing so, students will be able to understand and flexibly work with numbers (NCTM, 2000).

Literature Review

The role of communication in mathematics learning has been identified as a key process in building students' mathematical understanding (Macgregor & Price, 1999; Manouchehri & Enderson, 1999; Warfiel, 2003). Using the vocabulary of mathematics to enhance mathematics teaching and learning is very important (Huang, Normandia, & Greer, 2005). However, a few studies have approached communication in mathematics from a linguistic point of view (Wakefield, 2000). What certainly should concern educators and teachers alike is challenging students to use mathematical language to accomplish social goals that will enable them to select from the set of choices that are available to them in the language system (Christie & Unsworth, 2000).

This calls for a shift in the current curricula that are implemented in schools. Therefore, for successful implementation of school reform, students should learn by participating in communicative activities within classroom discourse communities (Wood & McNeal, 2003). Such communities provide shared responsibilities between teachers and students, both of them identifying and accomplishing respective roles in the mathematics discipline (Boaler, 2003). To achieve this objective, teachers should act as facilitators by building confidence among students to enable them become successful problem solvers (Goos, 2004).

Enhancing Mathematical Discourse

The usefulness of mathematics activities and oral communication to improve teaching and learning has been highlighted by some mathematics educators (Burton & Morgan, 2000). To improve mathematical discourse in schools, instructional design in mathematics education should systematically integrate thinking and oral communication at all levels of the knowledge structure (Huang, Normandia, & Greer, 2005). Teachers, through this process, could play the roles of both mathematicians and mentors by communicating about mathematics for students to cultivate the interest (Huang, Normandia, & Greer, 2005). This goal can be achieved by constructing different knowledge structures or semantic relations associated with the mathematics content (Halliday, 2003). In this regard, teachers could also capitalize on students' mathematical potential in the classroom to determine and develop an appropriate terrain through the reliance of students' oral communication skills, in order to build their mathematical understanding (Cobb, 2001).

Through the facilitation of mathematical discussions by teachers, students actively participate in making conjectures, and provide clear explanations (Pierson, Maldonado, & Pierson, 2008). This has the potential of yielding effective instructional approaches. This study was guided by the following research question: Are there differences among calculus teachers', algebra teachers', and statistics teachers' perceptions of teacher oral communication behaviour in the mathematics classroom?

Method

Sample

A survey design was used to investigate if there were differences between calculus, algebra and statistics teachers' perceptions of teacher oral communication behaviour in the mathematics classroom. Participants for this study were university calculus, algebra, and statistics teachers of a Western State in the U.S. They consisted of 33 calculus, 34 algebra, and 38 statistics teachers, who were purposively sampled from ten universities in order for the sample to fairly reflect demographics such as English Language Learner status, gender, and participant type, the major characteristics of the population. The average age of the participants was 40 years.

Instrument

A modified version of the Teacher Communication Behaviour Questionnaire (TCBQ) that was developed by She and Fisher in 2000, was used for data collection. The modified TCBQ consisted of 32 Likert scale items with 8 in each of the scales: challenging, encouragement and praise, understanding and friendly, and controlling. The non-verbal scale on the original questionnaire, which consisted of 8 items, was excluded from the original questionnaire in order to focus on oral communication. A Likert scale item included the responses: almost never, seldom, sometimes, often, and almost always. Cronbach's alpha coefficient for all the 32 items was 0.94, indicating that the internal consistency and reliability of the modified TCBQ was excellent. Table 1 shows a description of the scales and sample question for each scale of the TCBQ.

Procedure

Letters were initially sent to the principals of each of the 10 schools to seek their approval in allowing their teachers to participate in the study. Prior to that, the teachers had given their consent to participate. Starting Monday to Friday, and during the first three weeks of Spring 2009 semester, the questionnaires were hand delivered to each of the participants in an envelope for them to indicate their responses. The time allotted for the responses was 20 minutes. After the teachers had completed the questionnaires, the questionnaires were put together in a bigger envelope for analysis.

Table 1. Description of Scales and a Sample Question for Each Scale of the TCBQ

Scale Name	Description of Scale	Sample Question
Challenging	Extent to which the teacher uses high-order questions to challenge students in their learning	I ask questions that require students to integrate information that they have learned.
Encouragement and Praise	Extent to which the teacher praises and encourages students	I encourage students to discuss their ideas with other students.
Understanding and Friendly	Extent to which the teacher is understanding and friendly towards the students	If students have something to say, I will listen.
Controlling	Extent to which the teacher controls and manages student behaviour in the classroom	I expert students to obey my instructions.

Data Analysis

A single dependent variable (TOTALTCBQ) for all participants was determined by finding the mean responses of each participant on all 32 Likert scale items. The next dependent variables (Challenging, Encouragement and Praise, Understanding and Friendly, and Controlling) for all participants were determined by finding the mean responses of each participant on the eight Likert scale items under each scale. Table 2 shows the formulae used in calculating the dependent variables where *q1, q2, q3… q32*, are the 32 Likert scale items.

Table 2. Formulae for Dependent Variables

Dependent Variable	Formula
TOTALTCBQ	$\frac{q1+q2+...+q32}{32}$
Challenging	$\frac{q1+q2+...+q8}{8}$
Encouragement and Praise	$\frac{q9+q10+...+q16}{8}$
Understanding and Friendly	$\frac{q17+q18+...+q24}{8}$
Controlling	$\frac{q25+q26+...+q32}{8}$

The distribution of the 32 TotalTCBQ responses was approximately normal with a mean of 3.7 and a standard deviation of 0.645 (Table 3). Skewness and Kurtosis values of –0.582 and 0.659 respectively show that the distribution of responses is approximately symmetrical and matches the

Gaussian distribution. Since the other scales were created out of TOTALTCBQ, each of the scales' responses also matches the Gaussian distribution.

Table 3. Approximate Normal Distribution of TotalTCBQ Scores

Sample Size	Mean	SD	Skewness	Std. Error of Skewness	Kurtosis	Std. Error of Kurtosis
105	3.700	0.645	-0.582	0.120	0.659	0.239

A one-way analysis of variance (ANOVA) was deemed appropriate for the data analysis because, the assumption of homogeneity of variance was tenable, since the Levene's Test of Equality of Error Variance calculated for the dependent variable TOTALTCBQ, with respect to subject (p =.17 > .05). An F-statistic was calculated between the groups to determine the level of significance. ANOVA was appropriate because there was only one independent variable with three levels. The data were analysed as a unit, and then analysed in turn by the various scales i.e., Challenging, Encouragement and Praise, Understanding and Friendly and Controlling. The test of significance was set at the 0.05 alpha level. Where a significant main effect was found, partial η^2 was used to determine the strength of the significant result.

Results

Tables 4 and 5 show the means and standard deviations of teachers' responses for the dependent variables, and one-way ANOVA summary of teachers' responses with respect to subject.

Table 4. Means and Standard Deviations of Teachers' Responses for the Dependent Variables by Subject

Dependent Variable	Calculus (33)		Algebra (34)		Statistics (38)		Total (105)	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
TOTALTCBQ	3.75	0.34	3.91	0.41	3.76	0.27	3.81	0.34
Challenging	3.52	0.46	3.80	0.56	3.51	0.39	3.61	0.47
Encouragement & Praise	3.42	0.41	3.53	0.57	3.41	0.43	3.45	0.47
Understanding & Friendly	3.01	0.38	4.30	0.46	4.20	0.39	3.73	0.41
Controlling	3.90	0.46	4.00	0.58	3.91	0.46	3.94	0.50

Table 5. One-Way ANOVA Summary Table of Teachers' Responses for the Dependent Variables by Subject

Dependent Variable	Source of Variation	*Sum of Squares*	*df*	*MS*	*F*	*P*
TOTALTCBQ	Between Groups	0.228	2	0.114	1.059	0.316
	Within Groups	11.016	102	0.108		
	Total	11.244	104			
Challenging	Between Groups	0.844	2	0.422	2.025	0.170
	Within Groups	21.216	102	0.208		
	Total	22.060	104			
Encouragement & Praise	Between Groups	0.148	2	0.074	0.319	0.578
	Within Groups	23.664	102	0.232		
	Total	23.812	104			
Understanding & Friendly	Between Groups	0.102	2	0.051	0.297	0.592
	Within Groups	17.646	102	0.173		
	Total	17.748	104			
Controlling	Between Groups	0.082	2	0.041	0.157	0.696
	Within Groups	26.316	102	0.258		
	Total	26.398	104			

A one-way ANOVA was conducted to compare the effect of subject (calculus, algebra, and statistics) on TOTALTCBQ. The process was repeated for each of the scales, i.e., Challenging, Encouragement and Praise, Understanding and Friendly and Controlling. As shown in Table 4, there was not a significant effect of subject on TOTALTCBQ, F (2, 104) = 1.06, $p > 0.05$; Challenging, F (2, 104) = 2.03, $p > 0.05$; Encouragement and Praise, F (2, 104) = 0.32, $P > 0.05$; Understanding and Friendly, F (2, 104) = 0.30, $P > 0.05$; Controlling, F (2, 104) = 0.16, $p > 0.05$.

Table 4 shows the means and standard deviations calculated for the dependent variables, TOTALTCBQ (calculus (M = 3.75, SD = 0.34); algebra (M = 3.91, SD = 0.41); statistics (M = 3.76, SD = 0.27)), Challenging (calculus (M = 3.52, SD = 0.46); algebra (M = 3.80, SD = 0.56); statistics (M = 3.51, SD = 0.39)), Encouragement and Praise (calculus (M = 3.42, SD = 0.41); algebra (M = 3.53, SD = 0.57); statistics (M = 3.41, SD = .43)), Understanding and Friendly (calculus (M = 3.01, SD = 0.38); algebra (M = 4.30, SD = 0.46); statistics (M = 4.20, SD = 0.39)), and Controlling (calculus (M = 3.90, SD = 0.46); algebra (M = 4.00, SD = 0.58); statistics (M = 3.91, SD = 0.46)).

Discussion

Teachers' perceptions in this study seem to follow an identical or similar oral communication format across calculus, algebra and statistics. This is to be commended and encouraged if the use of this format by teachers reflects positively on students' mathematical understanding and improves on their test scores. If it does so, the constituents of this format could be studied in greater details for schools to benefit from its usage. If it turns out that the use of this oral communication format does not improve these students' mathematical understanding, then greater efforts could be made by these teachers to revise their communication format. The downside of this approach is that students' mathematical understanding may be linked to other variables other than oral communication.

Implications of the Study

This study provides an insight into the instructional challenges facing calculus, algebra, and statistics teachers. Because teachers' oral communication behaviours are not significantly different, low academic performance of students in this study may be attributable to other variables such as poor instructional methods used by teachers in addition to ineffective oral communication.

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Author

Charles K Assuah Ph.D., University of Education, Winneba, Ghana.
Email: assuahc@gmail.com

Mathematics, Education and Silos

Andy Begg
School of Education, Auckland University of Technology (AUT)

In mathematics education we seem to have developed a silo mentality. This influences the way we look at mathematics itself, its relationship with other subjects, and the way we think about mathematics being learnt and taught. In this presentation my aim is to make four aspects of silo-thinking explicit, to consider these as problematic, and to consider possible changes that could be made to move towards holistic thinking within mathematics education.

What Are We Talking About When We Use The Word Mathematics?

We are all interested in mathematics, but I wonder:

How do we define mathematics for ourselves?
How do we define the subject when teaching it?
How would we describe it to our students?
How might we describe it to people (such as pre-European Maori) who do not partition knowledge into distinct subjects (maths, language, science…)?

These questions imply a need for shared ideas about our subject instead of assuming that we agree with taken-for-granted notions when in fact we may not.

My dictionary defines mathematics this way:

> Mathematics is a group of related sciences including algebra, geometry, and calculus, which use a specialized notation to study number, quantity, space and shape.

But this hardly seems to catch the essence of what mathematics is. Indeed, they assume one has some idea about algebra, geometry, and calculus, (and each of these need defining). The dictionary definition for mathematics provides the first hint of 'silo thinking' because in most educational institutions the topics (arithmetic/number and quantity, algebra, geometry/space/shape, trigonometry, calculus, probability, statistics, and applied mathematics) are usually taught separately – like silos (air-tight and water-tight towers) that allow minimal seepage between one another.

An alternative perspective starts with a different definition, for example,

> Mathematics is the study of relations.

or more fully

> Mathematics is a way of making sense of one's world by considering the relations between the things within both the real world and the abstract world.

and

> A relation is a set of ordered pairs.

With these definitions, relations are the basic, recurring, and unifying theme in mathematics. When asked what do I mean by relations my response is 'connections' and then I illustrate this with some examples such as the following, which include traditional relations and functions, binary operations and numerous everyday verbal relations:

is equal to	is greater than	is congruent to	is parallel to
$f : x \rightarrow y$	$\mathrm{D} : y \rightarrow y'$	$x \rightarrow x!$	$+ : (a, b) \rightarrow a + b$
is the mean of	rotates onto	is the brother of	is the same colour as

… …

(As presented in a GRID Seminar, Mathematics Education Unit, University of Auckland, 14 October 2011.)

About 50 years ago a decision was made (or evolved) in high schools to teach one subject, mathematics, using one textbook. This subject was to replace the topics (arithmetic, algebra, geometry…) that had been taught with different textbooks and different exercise books when I attended high school. However, the change was minimal. Textbook writers (including myself) and curriculum documents (with headings such as number, measurement, geometry, algebra, and statistics) reinforced the partitioning of mathematics into topics and few unifying efforts were evident. Meanwhile, in primary schools the subject did move from arithmetic to include additional topics, although the numeracy emphasis may have partially reversed this.

In universities no such change has been evident. Partitioning seems to have increased with subjects such as statistics and probability breaking away from mathematics in some universities, and topics such as geometry sometimes being neglected completely.

It seems to me that if we look at mathematics as a whole rather than as a series of separate subjects we are able to emphasis the connections, the similarities, the differences, and the way that the various topics enrich our concept of the subject. For me such a unified approach that includes both 'pure' and 'applied' topics would help students develop their mathematical understanding in a more rounded way. Indeed, for many students the words 'pure' attached to mathematics suggests it is an abstract art form to be enjoyed rather than a way of making sense of the world by studying relations that exist between things in the world.

Mathematics and Other Subjects?

If one looks at mathematics as a "way of making sense of the world", then it is illogical to divorce it from other subjects; and this separation of subjects is a second form of silo-thinking. With the alternative, a holistic view, then one must acknowledge that mathematics is only one way of making sense of the world, others also exist, and the mathematical viewpoint does not always deserve primacy. The following two incidents have illustrated the inappropriateness of prioritizing a mathematical approach for me.

Incident 1

A science-teacher colleague had a fellowship to do an M.Ed. here at AUT. Her B.Sc. was in chemistry and biology and she used the year to also increase her science background by upgrading her physics knowledge. She somehow managed to get permission to enrol in both 1st year and 2nd year physics. At the end of the year her results were C for 1st year, and A^+ for 2nd year. When asked to explain this rather unusual situation she gave the following example to describe the difference between the two courses:

> In stage 1 we were given a formula and asked to 'calculate the force of attraction between two magnets of strength x and y when the distance between the poles is d.'
>
> In stage 2 we were asked, 'why do magnets attract or repel?'

Incident 2

A 22-year-old colleague of mine wondered about getting married. He approached the task logically. He decided that his wife-to-be needed to satisfy about ten measurable criteria. From memory these included:

- be no taller than 160 cm (he was 170cm),
- be younger than he was,
- have a reasonable IQ, measured in terms of having a Bachelor's degree (he had a M.A.)
- have good job prospects,
- had not previously been married (i.e. n=0 where n is number of previous marriages).

He stayed single until he was about 55, and now seems happily married to a lovely woman who did not satisfy all his original criteria!

More seriously, most students who study mathematics at school and at university are not going to be mathematicians. Some will be teachers, many will use some mathematics in their careers (but will often not think of this as mathematics), and many will enjoy life in spite of not being successful with formal mathematics in school or beyond. This leads me to wonder:

- Is mathematics as important as we like to think it is?
- Should school mathematics be emphasized more than art, music, physical education, or any other subject?
- Would mathematics have a better image if it were only taught through applications?
- Is university mathematics more important than philosophy?

And, at a deeper philosophical level,

- Why is knowledge split into subjects?
- Could mathematics be taught in a fully integrated way?

I believe that the division of knowledge into subjects is counterproductive and it was interesting in August this year to hear Peter James Smith here at The University of Auckland talking about *Truth & Beauty: The visual delivery of mathematical insights.*

His lecture was particularly interesting to me as this year I took up the challenge to co-teach with an artist, a compulsory class for first-year undergraduate students in the 'Bachelor of Creative Technologies' called *"Mathematics and Art"*. In this paper we seek to continually make connections between the two subjects rather than treating them as separate disciplines, although the students find this somewhat odd after their high school experiences.

In considering alternative ways of structuring knowledge and learning (including curriculum and assessment) one is likely to come up against the response "that's the way it's always been". But is that true? It is interesting to think historically how subject disciplines developed. Education emerged from psychology about 100 or so years ago, and psychology emerged from philosophy about 100 years earlier. Mathematics seems to have been around for a considerable time, though sometimes only in one form, for example geometry; nearly all the early mathematicians were also philosophers; and many mathematical topics are very new.

The age of the specialist is comparatively recent, and I would recommend the book *The Specialist* by Charles (Chic) Sale (1929) for a delightful satirizing of specialization.

One way to break away from this traditional subject specialization is to reconsider the teaching/learning approaches that we use; and this forms my third form of silo-thinking.

Teaching Approaches

When we recall primary school we usually think of informal learning activities as well as formal classes. We chanted tables, we sang, we played, we did projects, and all of these, including the more formal approaches that were also used, were purposeful. At high school things got more serious and considerably more formal. Projects were rare (especially in mathematics) and most lessons were teacher directed. Some of you who are younger than I am may recall fewer formal high-school experiences, but now with national standards the informality in high school education and in primary schools is decreasing!

At university teaching and lessons were replaced by lectures and tutorials, and this was a very significant change. I believe that lectures still dominate tertiary 'teaching' in most universities; and I am not belittling lectures as they have their place, but more is needed.

At undergraduate level, and even at masters level projects seem rare. This is particularly odd today as projects involve doing research and our universities keep going on about research-led teaching. Of course research-led teaching rarely occurs because most lecture topics are from the course curriculum and not from the lecturers' research interests, and most lecturers are interested in their research results rather than the way they reached them.

My belief is that it is highly desirable to emphasize research. It helps achieve the aim of university education—to help students become life-long autonomous learners. However, an emphasis on research is more likely to be achieved by enquiry-based learning and not by research-led teaching. Enquiry is research, and we know from our experiences in primary schools that researching projects that interested us provided valuable learning experiences both in terms of the knowledge gained and the ability to find information. Of course enquiry-based learning at university is not something new. The old Oxford model of reading a subject for three years and then sitting one examination seemed to work quite well! Though personally I would hope that such a final examination would be a very open one and allow the student to discuss what they know rather than attempt to discuss what they do not know.

Perhaps with undergraduate mathematics education it would be worthwhile to set up a parallel series of classes to Math 101, 102…, 201… with an enquiry-led focus and find if today's students are capable of taking responsibility for their learning. Of course one would still have a tutor (as in the Oxford model) who would regularly talk to students, encourage them, and occasionally make suggestions regarding something they might read.

Such parallel classes could also include 'General Studies' as some of the topics that might be investigated could be cross-disciplinary, rather than be limited to mathematics which would also address my second silo problem.

Of course if one only used one method of enquiry in the teaching/learning process with mathematics one might be seen as reinforcing the third form of silo. However, my belief is that an enquiry-based classroom automatically encourages a great number of approaches—reading papers and textbooks (real or web-based), hearing lectures (in person, or from the web), having discussions and seminars with peers and mentors, enquiring cooperatively, and numerous other approaches could all be used within an enquiry-based environment.

Thinking Silos

The fourth form of silo that has an adverse effect on education relates to the forms of thinking that are considered appropriate when studying subjects such as mathematics. Actually to say ‘considered appropriate’ might not be quite right; perhaps I should say ‘assumed to be appropriate’.

Lecturers often mention the importance of logic in mathematics, but rarely mention other forms of thinking that are used in both the study of the subject and in solving related problems. In fact what I have heard most in terms of thinking is not that the students use logic, but mainly complaints about how students are not very logical. No one seems to ask who teaches logical thinking; and it often seems that it is learnt by ‘osmosis’.

Traditionally thinking has been conceptualized as being of three forms:

- critical (or logical/rational including identifying assumptions)
- creative,
- meta-cognitive (or self-monitoring or assessing).

More recently two other forms have been added to the mix:

- caring (related to self, others, culture, and the environment),
- contemplative (with particular emphasis on developing awareness)

While listing these five forms may suggest a partitioning into five silos, this is not intended; they overlap and complement and enrich each other—the complementing not being in the Boolean sense).

Of course these five forms of thinking include sub-forms, and one can ask:

- Do we think in words and is telling stories about important ideas important?
- Do we think visually with diagrams, and are we taught to do this?
- Do we think abstractly with symbols or concretely with examples?
- Do we think divergently and imaginatively and ask ‘what-if questions?’
- Do we think aesthetically and consider the elegance and beauty of our results and proofs?
- Do we think communally and stimulate each other’s thinking?
- Do we pose problems as well as solve them?
- Do we use intuition to enrich our thinking?

My concern is that in the mathematics teaching that I have observed there is little evidence of the use of the five forms of thinking or of the above sub-forms. We tend to privilege memory-based thinking over experience-based; and, what we teach and assess usually focuses on the content (the ‘what’) of mathematics rather than the process (the ‘how’).

Critical (or logical) thinking is the most obvious form of thinking in mathematics at university level. Axioms (assumptions, not self-evident truths) and logic have always been emphasized in mathematics, though the axiomatic nature of the subject is not always emphasized and in my experience too many teachers assume that mathematics is a form of absolute rather than relative truth. However, this logical aspect is often not emphasized in schools to the extent that many university teachers might hope for – so it needs to be taught.

Creative thinking involves divergent thinking, thinking differently, using multiple representations, looking for different approaches, and changing ones ways of seeing things to gain fresh insights. A personal incident emphasized this for me.

Incident 3

I remember as a 3rd year student here at The University of Auckland (actually, The University of New Zealand in its final year) doing an analysis paper, I had been at a lecture on:

- reflexive, symmetric and transitive properties of relations,
- how a relation was defined as being an equivalence relation if it had these properties,
- how an equivalence relation partitioned such a relation.

The lecture was abstract and used the usual symbols (for all x there exists y such that…). I understood every line of the argument and could see how each line related to the previous one. My difficulty was simple – I had no idea or overview of what the lecturer was talking about.

About four years later I was teaching Form 4a at a grammar school in Auckland; the brightest class I have ever had the privilege of teaching. They had completed the traditional school certificate prescription in Form 3 and I was given carte blanche to experiment with any topics from what was then called the 'new maths'. At that stage I was fascinated with a new textbook I had just obtained – Moderne Mathematique by Papy (1963). It was written in French (which I could not read) but was full of colourful diagrams of sets and relations that were self-explanatory. The definition for a relation was "a set of ordered pairs". It quite quickly got to the point of illustrating relations on a set diagram and followed this by showing the reflexive, symmetric and transitive properties on the same set diagram. At that stage the equivalence relationship became obvious – and I wondered why my stage 3 lecturer had not used such a visual approach.

After this experience I have made an effort to present mathematics in a variety of ways knowing that the various modes might appeal to different students.

Meta-cognitive thinking is often used without the students being aware of it. Deciding to change tack and try a different approach, or knowing that one has completed a task are two of the obvious uses. Although, sometimes this unconscious meta-cognitive thinking takes control too quickly. For example, one of my favourite mathematical tasks is the following:

Can a square be cut into exactly eleven squares? Generalize!

Most students find one solution quite quickly, and usually assume that is it. They rarely push on to 3 solutions and virtually never persevere to 10 or 15 solutions. (When using this example I introduce ‘jigsaw’ equivalence to avoid ‘too many’ solutions!).

In addition, it is very rare, unless prompted, that students manage to prove that it is also possible to cut a square into any number of squares apart from 2, 3, and 5 (and there is an elegant visual proof of this), or investigate the feasibility of using shapes other than squares.

Caring thinking involves self-respect, concern for others, appreciation of culture and cultural differences, and eco-thinking. But mathematics is usually taught with an emphasis on individual learning (perhaps hoping for the development of self-respect, though often achieving the opposite). Mathematics is hardly ever approached in a cooperative manner with an emphasis on relationships between participants. And the applications of mathematics that may have ethical implications which are often more important than the mathematics itself, seem to be avoided.

Incident 4

I remember in my last year as a student at high school studying applied mathematics. The topic was projectiles, and the task was to determine the angle to point a gun to shoot down an ‘enemy’ plane that was moving in a fixed direction at a constant speed. Most of us were surprised when the equations gave two answers. Finally we figured that one could shoot the bottom of the plane with the bullet going upwards or the top of the plane when the bullet was descending, but our response to this was laughter rather than any consideration of the ethics of shooting a plane.

Contemplative thinking is probably the most controversial form of thinking for mathematics teachers, yet becoming aware of mathematical ideas and gaining mathematical insight often involves more than logic. It requires ‘being’ with an idea. Imagine a sphere (from the inside), imagine a parallelepiped, imagine a three dimensional graph of $z^2 = x^2 + y^3$, visualize a spiral with equation $r = r_0ø$ and extend this for negative values of ø. Such tasks benefit from quiet contemplation of the situations.

Contemplative thinking can also provide intuitive insights.

Incident 5

The famous NZ mathematician Alexander Craig Aitken (1895–1967) on his retirement was purported to have answered the question, how did you contribute so much? With the response, 'The problem was not solving the mathematical problems but finding them'. When asked how he solved them he said something like, 'Read the problem before you go to bed, hold the idea in your head, and in the morning when you wake up write down the proof.'

This suggests both insight and intuition are important; both are part of contemplative thinking, and my belief is that both have a part to play in the study of mathematics.

For me this fourth aspect of silo-thinking that is concerned with the forms of thinking involved in mathematics is a significant area where our approach to the topic needs to be broadened.

Looking Ahead

For me the alternative to silo-thinking is summarized in the words 'making connections' which is a phrase that has sometimes been used in the 'process' (rather than 'content') strand of school; mathematics curriculum documents. Making connections involves cross-silo thinking – connecting mathematical topics, connecting mathematics with other subjects, connecting a variety of teacher/learning approaches, and connecting and using a range of forms of thinking. The lack of such connection was made evident to me while teaching when the "new maths" was being introduced.

Incident 6

A colleague of mine taught both mathematics and physics to the same Year 11 (Form 5) class. One topic he taught in both classes was vectors. In mathematics he taught:

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

In physics he taught vector addition by drawing a diagram, resolving each vector horizontally and vertically, adding the components, and using Pythagoras and the inverse tan to find the resultant (r, q).

At no stage did he connect these two approaches and some of the students that I knew from extra-curriculum activities asked me if there was any connection?

Later I talked with my colleague and he said that he had never thought there was any connection between vectors in mathematics and in physics!

At a more personal level the importance of 'making connections' became evident when I decided to learn about navigation in the days before GPS black boxes were invented.

Incident 7

The class started with inner harbour navigation, then moved to coastal, and finally ocean navigation. My first surprise was learning that the scale on the bottom of a nautical chart was different from that at the top, which of course, is a problem when projecting part of a sphere onto a plane. Next we had to learn spherical geometry and the related celestial navigation, and this all seemed quite meaningful when one thought one's life might depend on it – learning mathematics at school and university had never seemed quite so important!

Looking Cautiously

Making connections is not without problems, as the following three incidents show:

Incident 8

I was talking to some primary school teachers about a Year 6 cross-curriculum unit on 'the butterfly'. It was intended to integrate (or make connections) between language, art, social studies, mathematics, and science by using this theme. When talking to them about the mathematics being covered I found that the cross-curriculum unit was more like an excuse not to teach more than trivial mathematics in spite of the potential that existed.

Incident 9

Another experience related to using contexts from other cultures without care. It seemed to a number of us that 'kowhaiwhai' patterns provided a rich source of examples for transformations geometry. However, we were cautioned by a Māori elder that this may not be appropriate as to consider these patterns as mathematics would detract from the 'mana' of the artists.

Incident 10

In the 1990s Professor Bridges at the University of Waikato shared his concern about his second year calculus class that consisted of engineering and business students. He said that the engineering students could/would not relate with problems set in a business context, the business students could/would not relate with problems set in a engineering context, and neither group wanted calculus taught in a context free (pure) manner. His solution was to devise two sets of problems involving the same concepts (specific differentiation and integration techniques, and differential equations) and allow students to use one context or the other, and this was also included in the end-of-year examination.

I wondered, did this help the students see the generality of what they were studying, or did it reinforce the subject silos that they positioned themselves in?

So, make connections, think outside the silos, be prepared to experiment a bit, and, good luck!

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Author

Andy Begg, Auckland University of Technology, NZ. Email: andy.begg@aut.ac.nz

Linear Algebra with a Didactical Focus

Barbara Jaworski, Stephanie Treffert-Thomas and Thomas Bartsch
Loughborough University, UK

> How might you construct an introductory Linear Algebra course for first year mathematics students? What decisions would you have to make and what issues would you have to address?

The authors of this paper, as a small research team, set out to address these questions and others relating to a first year, first semester module in Linear Algebra. We are all members of the School of Science at Loughborough University; we all teach mathematics and we do research into mathematics or mathematics education. Thomas Bartsch (TB) is a mathematician, working in the Department of Mathematical Sciences (DMS); Barbara Jaworski (BJ) and Stephanie Treffert-Thomas (STT) are mathematics educators working in the Mathematics Education Centre (MEC).

The MEC was opened in 2002 to provide university-wide support for students engaging with mathematics in any disciplinary area of the university. It includes two drop-in Mathematics/Statistics Learning Support Centres, which are staffed by a mathematician and/or statistician for 6 or 7 hours each day. Members of the MEC do research into mathematics learning and teaching, primarily at university level. They contribute to mainstream teaching of mathematics and provide expertise in teaching mathematics to engineering students.

Background to the Study of Teaching of Linear Algebra

An aim in studying the teaching of Linear Algebra was to try to start to characterise mathematics teaching within the university and to gain access to the perspectives of mathematicians on their teaching of mathematics. A seminar series (entitled *How we Teach*) had been started to share aspects of mathematics teaching and initiate a mathematics teaching discourse through which we could learn from each other and develop our teaching. Seminars in the series were video recorded and a selection of them analysed in order to characterise this discourse (Jaworski & Matthews, 2011). Seminars form a part of the New Lecturer's Course for new mathematics lecturers at Loughborough.

The research study was agreed between BJ and TB before the start of the academic year 2008/09. STT joined the team as a PhD student with this research the focus of her PhD. TB was in his second year of teaching this module. BJ had considerable experience of doing research into mathematics teaching at a variety of levels. Together the team formed a small community of inquiry. We had a common purpose in exploring the teaching of mathematics, trying to understand better the teaching process, recognising the issues that arise for teacher and students, and promoting development of teaching. We had differing roles with TB as lecturer, having responsibility for design of the module and module materials, teaching and monitoring students, and BJ and STT as researchers, having responsibility for conducting research of a largely qualitative nature.

Our research methodology was ethnographic in style, that is, producing qualitative data through conversations and interviews. It was important for the two researchers to gain in-depth access to the thinking and actions of the lecturer in order to develop well-grounded understandings of the lecturer's teaching and design of teaching. Thus the two researchers talked extensively with the lecturer before and after lectures, observed all lectures and tutorials, and collected relevant documents. In addition STT sought students' views with two questionnaires handed out in lecture time and by conducting focus group interviews with a small number of students when Semester 1 teaching had ceased. Research meetings of the team and all teaching by the lecturer were audio-recorded. Analysis of this data was qualitative, involving repeated listening, transcribing, coding and categorising. Atlas.ti software was used extensively to support analysis.

The First Semester Linear Algebra Module

Linear Algebra is a mainstream topic for first year mathematics students. It is taught in a two-semester module with 72 hours of teaching and associated assignments and examination. TB is the lecturer for the first semester (S1); there is a different lecturer in the second semester (S2). The two lecturers collaborate on the year-long design of the module and prepare a joint examination at the end of the year. The first semester offers an introduction to Linear Algebra and the second semester a more abstract treatment. In this study we focus on the first semester, which consists of an introduction to Linear Algebra that tries to avoid the more formal aspects of the material. The second semester involves a repetition of the same material, but from a formal perspective. One purpose of such organisation is to recognise that students coming to university from school are not well prepared for mathematical formalism (see, for example, Nardi, 1996) and need some preparation for dealing with abstraction. The module is taught through two lectures and one tutorial each week (the standard allocation of time).

The module that we observed was taught to a cohort of 240 students of which approximately 180 (based on informal, periodic head counts) attended lectures regularly. The lecturer distributed weekly problem sheets on which students were asked to work in their own time. In addition, each student is a member of a Small Group Tutorial (SGT) in which seven or eight students meet once a week with a tutor who is a mathematics lecturer[2] (not a graduate student). In SGTs some of the tutorial problems could be discussed, at the discretion of the tutors and their students. For all problems the lecturer made detailed solutions available after two weeks. SGT tutors are also personal tutors for students in their group. Through the SGTs they have access to student progress and student experiences of learning and teaching.

The lecturer's design of the module included choosing, sequencing and writing the mathematical content, including the examples used in lectures and the examples/exercises used in the weekly tutorial, designing a weekly problem sheet, and preparing assessment tasks which included on-line tests and written coursework. In the first semester, the lecturer prepared notes-with-gaps which were placed on LEARN (a virtual learning environment) for students to access in advance of a lecture.

The lecturer's notes were structured to guide the course and were used for teaching; that is they were presented to students by the lecturer in each lecture. Students were asked to bring printed copies of the notes to the lecture. Tutorials differed from lectures by focusing only on examples with no progression of the material of the notes.

The lecturer used a data projector to project the course notes, including the outline of examples, onto a big screen, and an overhead projector to work out the solutions to examples, which were missing from the printed notes. He would move physically between the two. Often he stood centrally in the lecture theatre to talk to the students offering his own comments about the mathematics and about ways in which students should approach the mathematics.

One purpose of the gaps in the lecture notes was to encourage students to attend lectures and complete the notes *in the lecture*. This involved completing the *solutions* of key examples that were presented. Often, before presenting a solution, the lecturer gave students some minutes to work on the solution by themselves or with their neighbours, walking around the lecture theatre and talking with some students.

The design of the module gave students the option to engage with the content of the module in a variety of ways. They could download the lecture notes from LEARN. They could attend lectures and tutorials, fill in the gaps in the notes and make their own supplementary notes, attend their own SGT each week, and get access to the lecturer either face to face or by email. They

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[2] In the UK, the academic hierarchy is Lecturer, Senior lecturer, Reader, Professor. Most academics are at the levels of Lecturer or Senior Lecturer. The term 'lecturer' is used both as an academic title and as the *role* of the academic teaching a particular module.

could work on problem sheets and complete assignments marked by their SGT tutor. The SGT provides opportunity for discussion with fellow students, and the lecturer encouraged such discussions also outside of the formal teaching sessions. Students could also attend a support centre and get advice from a lecturer who was not otherwise involved in teaching the module.

The content of the first semester was presented in the course notes in four chapters as follows:

1) Linear Equation Systems
2) Matrices
3) Subspaces of $\mathbb{R}^n$
4) Eigenvalues and Eigenvectors

In Chapter 1 the focus was linear equation systems. The lecturer distinguished systems of linear equations that have one, many or no solutions. He introduced the method of Gaussian elimination to determine the solution set of an arbitrary linear equation system. This method uses elementary row operations on a linear equation system, or its coefficient matrix, in order to produce an equivalent, but simpler system. Gaussian elimination is sometimes also referred to as the method of row-reduction of matrices.

Chapter 2 consisted of an introduction to matrices as representing linear equation systems. The content in Chapter 2 included calculating with matrices (namely the addition, subtraction and multiplication of matrices), finding the inverse and the transpose of a given matrix, and the related rules of matrix algebra.

In the lecturer's own words Chapters 1 and 2 contained the more computational aspects of the module. These two chapters provided students with the necessary computational skills to advance to Chapters 3 and 4, which focused more strongly on concepts.

Chapter 3 dealt with the most important concepts in Linear Algebra, which are vector spaces, subspaces, span and spanning sets, range, linear independence, basis and dimension, and the rank-nullity theorem. These concepts were all introduced in the setting of $\mathbb{R}^n$. The lecturer presented examples and deduced general observations from the examples. Theorems were often presented as "Observations" and in general, no abstract proofs were given throughout the first semester. (There were one or two exceptions.) This was a deliberate strategy employed by the lecturer and one that we discuss further below.

The focus in Chapter 4 was Eigenvalues and Eigenvectors. Chapter 4 included the definition of an Eigenvector/value, an introduction to the theory of determinants, the use of the characteristic polynomial in calculating Eigenvalues (and hence for finding Eigenvectors), and a detailed account of the process of diagonalisation.

The Nature of Research Meetings

Research meetings focused on the lecturer's design, planning and intentions for teaching. The meetings provided an opportunity for the lecturer to talk about his design of the module, his current teaching and perceptions of students' learning and issues arising thereof. The two observers asked questions and offered observations or perceptions. Meetings following a lecture or tutorial focused on what had taken place, and involved the lecturer's reflections interspersed with questions from the observers.

Often our discussions in meetings focused on students' responses to the material and the lecturer's perception of students' understanding in relation to the material of the lecture. The nature of these discussions included the lecturer talking about his own conceptions of the material of the lecture, of his didactical thinking with regard to this material, of his perceptions of students' activity and of his decision-making in constructing notes, examples and assessment tasks. The example below, of the lecturer's talk, shows 'expository mode' (talking about his own conceptions of the material) in normal text and 'didactic mode' (talking about his construction of the teaching of the material) in italic text.

> *Thursday is about defining the characteristic polynomial, understanding that its zeroes are the Eigenvalues, and I'll show an example of an Eigenvalue that has algebraic and geometric multiplicity 2.* Algebraic multiplicity, meaning this is the power with which the factor lambda minus Eigenvalue appears in the characteristic polynomial, and geometric multiplicity is the number of linearly independent Eigenvectors. *And these are the important concepts for determining if a matrix is diagonalisable because, for that, we need sufficiently many linearly independent Eigenvectors.* Now if an Eigenvalue has algebraic multiplicity larger than 1, that means there are correspondingly fewer Eigenvalues. So, in principle, we can fail to find as many Eigenvectors as we need in that case. On the other hand, if an Eigenvector has algebraic multiplicity 3, the geometric multiplicity can be anywhere between 1 and 3. If it's 3, we are fine, if it's less than 3, we're missing out at least one linearly independent Eigenvector. And in such a case the matrix would not be diagonalisable. *And that's the big observation that we need to get at next week, that a matrix is diagonalisable if and only if all the geometric multiplicities are equal to the algebraic multiplicities.*

The distinction between expository mode and didactic mode is not clear-cut. The sentence in italics in the middle of the quotation might also be characterised as expository mode. However, it seems here that the lecturer is *meta*-commenting on the material: i.e. expressing his value judgment regarding important concepts that need to be appreciated, rather than just articulating mathematical relationships. This seems to relate to didactic judgments in terms of what needs to be emphasised for students. We observe that such statements in meetings correspond to what we have called meta-comments, or meta-mathematical comments in lectures. Such comments address what students need to attend to, either in terms of their work on the mathematical content (meta-comments -- A) or of their understanding of the mathematical content (meta-mathematical-comments -- B). Examples A and B follow.

> A: First of all, … if I give you an equation system, this gives you a recipe to decide if that equation system is consistent or inconsistent. You transform it to echelon form and you check if there is such a special row that makes the system inconsistent.
>
> B: But it's important that you be able to understand the language that we're using and to use it properly. So please, pay attention to the new terms and the new ideas that we're going to introduce over this chapter.

We are emphasising this difference in modes of talk about the material of the module to contrast thinking about teaching (the didactic mode) with thinking about mathematics (expository mode). In meta-comment A, the lecturer draws students' attention to the nature of the mathematics and how they work with it. In meta-mathematical comment B, he draws their attention to the processes of working with the mathematics and strategies that can lead to understanding. Both of these are "didactical" approaches on the part of the lecturer. In studying the *teaching* of Linear Algebra, we are interested fundamentally in the didactic nature of the lecturer's presentation of the mathematics.

The Lecturer's Approach to Teaching

From analysing the audio-recordings of the meetings between the lecturer and the two researchers, we gained insight into the lecturer's motivations, intentions and strategies for teaching. Based on his experience of teaching undergraduate mathematics for one year prior to this research, the lecturer devised an examples-based approach to the teaching of Linear Algebra for this module. In a research meeting, the lecturer said,

> Yes. …. Generally speaking, I decided that I would focus on doing the development of the argument on examples, and then trying to abstract a general fact from the example, as I have done in most cases so far. And so then, what I am doing is go through the example, and then highlight the important facts on the example, and then condense them into a general observation. And I have several times mentioned to students that this is what

> we're doing, and that it's a good idea to see an example not as an isolated example but rather as a representative of a big class.

In taking this approach the lecturer 'avoided' the introduction of theorems although many of the 'observations' that he made were in fact equivalent to theorems. Few of the observations were proved in a formal sense.

We termed his approach EAG, where EAG stood for 'example-argument-generalisation'. The lecturer's approach could thus be summarised as

- we introduce an **E**xample,
- we make an **A**rgument on the example, and then
- we **G**eneralise to an observation, another example or set of examples.

The term "observation" above agreed with the use of this term in the lecture notes, where the lecturer used the term 'observation' rather than 'theorem'.

This approach could be described as 'bottom-up'. The lecturer demonstrated a mathematical phenomenon on a 'typical' example that served as a representative for a class of similar cases. He explained the example in a manner that was intended to highlight the general features rather than the specific details of the particular example. Where necessary, he introduced definitions to provide relevant terminology. General statements could then be abstracted from the arguments that were applied in the example. Because these statements arose from the study of an example they were called "Observations" rather than "Theorems", as they would be in more formal presentations of Linear Algebra.

The course covered all the standard results of introductory Linear Algebra. Because most of them were presented as Observations that were justified by reasoning about an (typical) example, the first semester included hardly any formal proofs. The proofs were provided in the second semester, in which the results were revisited in the abstract context of vector space theory. By proceeding in this manner, the lecturer hoped to offer his students a gentle introduction to mathematical reasoning about objects and their properties that is required at university level.

An example-based approach as outlined above can be viewed in contrast to the more traditional ('top-down') deductive style of teaching mathematics at university. The latter is often referred to as DTP (definition-theorem-proof) or DLPTPC (definition-lemma-proof-theorem-proof-corollary) style (see, for example, Uhlig, 2002; Dorier *et al.*, 2002). In a traditional approach (DTP), the statement "The range of a matrix is a subspace", for example, is introduced as a theorem. The theorem is then proved by checking that the three properties of a subspace (the set is closed under addition and scalar multiplication and contains the zero vector) are satisfied.

In our study, however, using the EAG approach, the lecturer set up an (concrete) example and asked a series of questions as follows:

Example. 3.14. Consider an unknown 2 x 3 matrix A. We know that A satisfies $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$, where

$$\mathbf{b_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{b_2} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \quad \mathbf{x_1} = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

(a) Is $\mathbf{b}_1$ in the range of A? Is $\mathbf{b}_2$ in the range of A?

(b) Is $\mathbf{b}_1 + \mathbf{b}_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of A?

(c) Take the number $\lambda = 3$. Is $\lambda\mathbf{b}_1 = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of A?

(d) Is the zero vector $\mathbf{0}$ in the range of A?

Figure 1: An example offered to students in the module.

Earlier in the course, the lecturer had introduced the null space of a matrix *A,* i.e., the solution set of the homogeneous equation system $Ax=0$. He had shown that the null space has similar properties to the set of all n-component vectors: It is closed under addition and scalar multiplication and contains the zero vector. This observation had motivated the definition of a subspace. The four questions (a) to (d) in the present example were designed to lead the student to recognise the correspondence between the answers to the questions and the definition of a subspace. As a result the students were to arrive at, and recognise that the range of a matrix is a subspace. This was then summarised in what the lecturer called "Observation 3.15". This "observation" is the theorem "The range of a matrix is a subspace". The lecturer chose the terminology of "Observation" (rather than "Theorem") because he did not give a formal proof at this point in the course.

This example is less abstract than a general proof because specific values are given for the various vectors. On the other hand, because the matrix *A* is unknown, the questions cannot be answered by direct calculation. The solutions make use of numerical values, but they are not essential for the argument. It is this observation that allows the specific example to serve as representative of a wider class: The same arguments that are used in the example could be used for arbitrary matrices and vectors. The lecturer emphasised this fact in lectures, to his students, on several occasions.

Below we include the full solution to Example 3.14. The notes that were available to the students during the lecture contained blank spaces instead of the solutions. Observation 3.5 states that the null space of a matrix has the properties of a subspace.

3.3. THE RANGE OF A MATRIX 6

$$= \{\boldsymbol{b} \ : \ \textit{There is a vector } \boldsymbol{x} \textit{ such that } A\boldsymbol{x} = \boldsymbol{b}.\}$$
$$= \{\boldsymbol{b} \ : \ \textit{There is a vector } \boldsymbol{x} \textit{ such that } \varphi_A(\boldsymbol{x}) = \boldsymbol{b}.\}$$

Remark. The range of a matrix A is the range of the linear transformation (or function) φ_A in the sense used in Calculus: range A is the set of all values $\boldsymbol{b}$ that the function φ_A takes.

Example 3.14. Consider an unknown 2×3 matrix A. We know that A satisfies $A\boldsymbol{x}_1 = \boldsymbol{b}_1$ and $A\boldsymbol{x}_2 = \boldsymbol{b}_2$, where

$$\boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \boldsymbol{b}_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \qquad \boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 3 \\ -7 \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}.$$

(a) Is $\boldsymbol{b}_1$ in the range of A? Is $\boldsymbol{b}_2$ in the range of A?

Solution:
$\boldsymbol{b}_1 \in \operatorname{range} A$ because the equation system $A\boldsymbol{x} = \boldsymbol{b}_1$ is solvable ($\boldsymbol{x}_1$ is a solution).
$\boldsymbol{b}_2 \in \operatorname{range} A$ because the equation system $A\boldsymbol{x} = \boldsymbol{b}_2$ is solvable ($\boldsymbol{x}_2$ is a solution).

(b) Is $\boldsymbol{b}_1 + \boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the range of A?

Solution: Yes. The equation system $A\boldsymbol{x} = \boldsymbol{b}_1 + \boldsymbol{b}_2$ is solvable, and $\boldsymbol{x}_1 + \boldsymbol{x}_2 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$ is a solution because

$$A(\boldsymbol{x}_1 + \boldsymbol{x}_2) = A\boldsymbol{x}_1 + A\boldsymbol{x}_2 = \boldsymbol{b}_1 + \boldsymbol{b}_2.$$

(c) Take the number $\lambda = 3$. Is $\lambda \boldsymbol{b}_1 = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$ in the range of A?

Solution: Yes. The equation system $A\boldsymbol{x} = \lambda \boldsymbol{b}_1$ is solvable, and $\lambda \boldsymbol{x}_1 = \begin{pmatrix} 3 \\ 4 \\ -21 \end{pmatrix}$ is a solution because

$$A(\lambda \boldsymbol{x}_1) = \lambda\, A\boldsymbol{x}_1 = \lambda \boldsymbol{b}_1.$$

(d) Is the zero vector $\mathbf{0}$ in the range of A?

Solution: Yes. The equation system $A\boldsymbol{x} = \mathbf{0}$ solvable, and $\boldsymbol{x} = \mathbf{0}$ is a solution because $A\mathbf{0} = \mathbf{0}$.

In this example, we have verified that the range of a matrix has the three properties of Observation 3.5. We can therefore conclude:

Observation 3.15. *The range of a matrix is a subspace.*

Figure 2: A page from the course notes showing the solution to Example 3.14.

Student Feedback

Students' views were sought with two questionnaires that highlighted students' preferences and work habits. These were followed by focus group interviews in which STT probed students' views further. As a result the research team learned that students (a) liked the notes-with-gaps, (b) found Linear Algebra difficult, and (c) focused on learning computations and algorithms rather than engaging with the conceptual understanding as desired by the lecturer. We explain these responses.

a) Students liked the way that the lecturer had designed the course with the use of notes-with-gaps since they felt it engaged them more. They generally printed the notes and brought them to lectures. One student compared the lecture notes to "a workbook", and the design of the course as providing a "stepping stone" from A-level to university. Despite the positive attitude towards the 'gappy' notes this did not necessarily mean that students worked actively on the solution to the examples in lectures. As one student pointed out, "It depended … whether or not I could do it". Students in the focus groups generally acknowledged that many students waited for the solution to be presented by the lecturer, rather than working on it themselves.

b) Students found Linear Algebra difficult and particularly challenging at the start. They said that they were unprepared for the conceptual nature of the topic. As one student said, she did not realise "that definitions were important", she was revising from the exercises and examples instead, and realised [too late] that understanding definitions was a requirement for the exams.

c) Students frequently referred to computational aspects of Linear Algebra. The Gaussian elimination procedure was taught in the beginning of the module, in chapter 2. One student commented that you always had to use Gaussian elimination somewhere at some time, so if she didn't know what to do, she would always do a Gaussian elimination on the matrix. She expressed the view that this was likely to gain at least some marks (in an exam, say).

Synthesis of the Teaching Approach and its Relation to Students

We have drawn attention to the informal nature of the teaching approach and its EAG structure. We have also talked about the lecturer's observed levels of commenting. It is important to recall that what we have described is the first semester of the module in which the second semester offers a more formal treatment of the same material; so students are then introduced to vector spaces more generally in a more abstract DTP approach. The first semester is the students' introduction to university mathematics. Thus, the teaching seeks to bridge the school-university transition and prepare students to deal with abstraction.

The EAG approach describes the *structure* of the teaching. Examples are chosen carefully to lead to key concepts through the succeeding argument and generalisation, but without formal proof. The lecturer's commenting is central to this process, offering first a mathematical treatment of the topic in consideration, then a commentary on the relationships involved, emphasising key ideas and ways in which these fit into the broader picture, and finally suggesting to students how they should think about and work on these concepts.

Our data showed that students liked the course structure and the course notes. Nevertheless, many students found the transition to argumentation at this level a difficult one, seeking examples that they could follow and taking a more broadly computational approach. Anecdotal evidence from small group tutors suggests that students tackled problem sheets by looking for examples that demonstrated the required approach. Although such responses from students suggest a dependency on the lecturer, a desire for given procedures and a computational approach, towards the end of the year students seemed able to deal with the more abstract treatment, gaining confidence from recognising the material and their earlier struggles with it. They reported that the first semester approach had been valuable in enabling them to address the more abstract formulation in the second semester. A quotation from a focus group shows how two students thought about this.

> S1: I think my understanding of the subject got a bit better and I understand what a lot of the words mean a lot better now [i.e., in Semester 2], so many things like range, basis, then rank, rank-nullity, span, and there are so many of them and try and cram them all in …The way we've used them again and again this term and my small group tutor … we've gone over it so many times that I'd be pretty stupid if I didn't get it by now … and we went through the class test afterwards in my tutorial and I kind of thought that's really silly, I should have done better.
>
> S2: Yeah, it did seem very easy afterwards and once we looked at the solutions for it.

In Conclusion

Given that students find the transition to abstraction and formalism in university mathematics a difficult one, our research documents an approach that offers an alternative to the traditional DTP. We have shown briefly the key elements of this approach, but in the short space of this article have been able to present only little specific detail and almost no treatment of the ways in which the lecturer's thinking and intentions were realised in the teaching practice and in the responses of students. The latter (intentions and their realisation) is the focus of the PhD thesis of the second author, which is forthcoming. In this, STT reports on an *activity theory* analysis of the observational data in order to relate teaching intentions with practical outcomes and link teaching with learning in the mathematical context of Linear Algebra. We welcome interest in these ideas and invite those interested to get in touch with us for discussion and debate.

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Authors

Barbara Jaworski[3], Stephanie Treffert-Thomas and Thomas Bartsch, Loughborough University, England.
Email: b.jaworski@lboro.ac.uk; s.thomas@lboro.ac.uk; t.bartsch@lboro.ac.uk

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[3] Corresponding author

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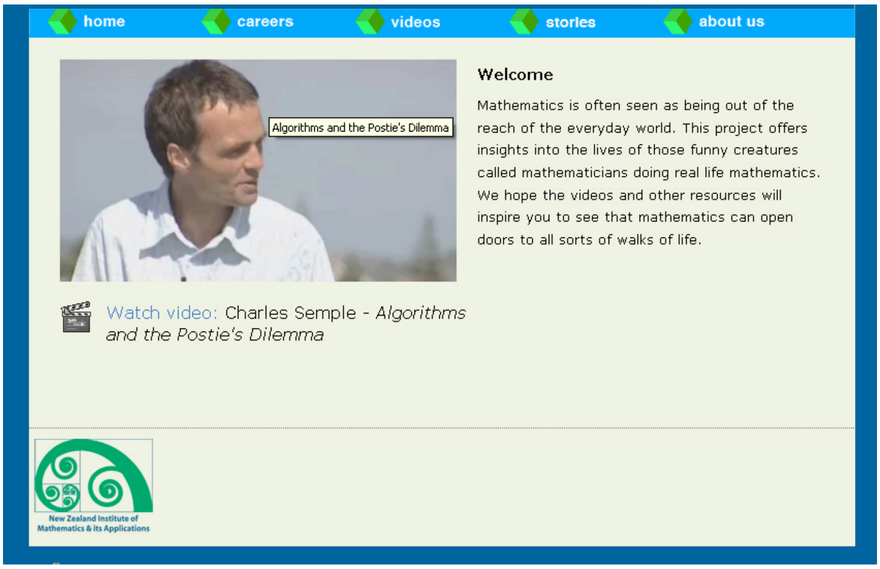

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