CULMS Newsletter

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Community for Undergraduate Learning in the Mathematical Sciences

The CULMS Newsletter

CULMS is the Community for Undergraduate Learning in the Mathematical Sciences.

This newsletter is for mathematical science providers at universities with a focus on teaching and learning.

Each issue will share local and international knowledge and research as well as provide information about resources and conferences.

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Guest Editorial

This issue's Guest Editorial is by Birgit Loch, Senior Lecturer in the Mathematics Discipline and Head of the Mathematics and Statistics Support Centre, Swinburne University of Technology. Birgit has an interest in technology use and adoption in mathematics education, and was co-coordinator of the Special Session on Mathematics Education at the recent AustMS meeting in Brisbane.

In their editorial for the first edition of this newsletter, Bill Barton and Mike Thomas raised a number of important issues in research into the nature of undergraduate teaching and learning in the mathematical sciences. I will touch on the following here: *Is the lecture and tutorial model the best teaching approach, or can we improve on it? If so, how? What should be the role of technology in the undergraduate mathematical sciences? Should it be used, and if so what kind of technology and to what extent? How might course content change when technology is integrated?* These are all valid questions. Researching answers and sharing already existing knowledge is a timely task considering the growing disjunction between the current use of technology in mathematics learning and teaching at undergraduate level, and the adoption, proliferation and ubiquitousness of technologies in the world around us. Let me explain this further.

Many students entering university in the near future, in Australia at least, will have been using laptop or tablet computers for learning for years, thanks to an Australian Government initiative which provided funding to schools to purchase these computers. Many will own an iPad or similar tablet device. Others will have iPhones, or one of an ever-growing range of other smartphones. These devices are always with the students. They are used for communication, making friends, file storage, input of information, sharing, collaboration, searching for answers and definitions, playing music and videos, and gaming – and self-directed learning. They are embedded in students' private and university lives and the students no longer regard them as technology.

How could we benefit from this enthusiasm for technologies? Can we transfer it to the learning of mathematical concepts in Matlab or Maple, two tools that are implemented into undergraduate teaching at many universities? If so, how? Can we slip in the learning of mathematical concepts, undercover, so students don't think of it as learning mathematics? This may sound far fetched now, but could we develop a mathematical game that will teach students how to find an integral by substitution, where they will willingly follow all the steps because they can't advance a level in the game they are addicted to without providing the solution? (You may remember Bill

Barton suggesting that we get students addicted to mathematics in the first edition of this newsletter).

Could we get across the importance of mathematics and its direct relevance to the student's professional life after university by getting them to enter an interactive web-based game that lets them face situations they will encounter in their future jobs, and which require mathematical problem solving skills on the spot? Could this reverse a lack of interest in mathematics by "service course" students? I can think of a nurse calculating drug dosages, and an engineer working out the maximum weight a bridge can take.

Could we redefine assessment and ask students to go on the Web with their always connected devices, in teams, to search not for answers to problems, but for methods for solving them, and then explain to other teams how the methods are used? Would this be better than having the lecturer provide all methods with interaction from only a small number of good students? When the methods have been presented and discussed, then the focus could move towards practicing the actual solving.

Could we give students a screencast to show them how we want an assignment question written, rather than wait until we see their first assignment and mark them down for inadequate explanation of how they derived their answer? Could we provide students with recorded "lectures" that introduce new concepts, expect them to watch them before class, and focus on resolving misunderstandings in face-to-face classes instead?

This is where I see the disjunction. How many of us are doing at least one of the above now?

Many of the suggestions and questions I've raised also apply to other disciplines. So I come to the questions that I would hope research and shared knowledge published in this newsletter could attempt to answer. What is best practice in the use of technologies in learning and teaching in the mathematical sciences? How can we get lecturers involved in the effective use of contemporary technologies, and what type of professional development is needed?

What if a regular contribution to this newsletter was a column on "the three technological interventions that have made the most difference in my students' learning"?

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A Changing Cohort—What Do High School Students Remember and What Does it Mean for University Lecturers?

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Until the late-1990s the first-year engineering cohorts at The University of Queensland (UQ) were very different to recent cohorts. At school, students had to have studied both intermediate *and* advanced mathematics, plus chemistry *and* physics, in order to enter engineering. Since the late-1990s, students have only needed intermediate mathematics plus chemistry *or* physics to enrol. As such, only 60% of recent first-year engineering students have studied both intermediate and advanced mathematics at school (UQ entry data, 2010).

The number of students choosing engineering in the 1990s was also very different. In 1991 there were 400 first-year students and in 2008-2010 there were 970. This increase in numbers has meant an increased diversity of backgrounds, knowledge and abilities, but there has not been a significant change in the way that engineering and mathematics is taught. In order to better understand this new group of students, staff decided in 2007 to reintroduce diagnostic testing.

Diagnostic testing had been conducted on first-year engineering cohorts at UQ from 1972-1994 (Pemberton & Belward, 1996). The test was the same every year with questions covering the Years 11 and 12 intermediate and advanced mathematics syllabi. The 2007 diagnostic test was quite different. Questions came from both the senior intermediate mathematics syllabus and the Queensland Years 1-10 mathematics syllabus (specifically topics which form the basis for the senior secondary topics). A typical course of study in Queensland senior intermediate mathematics can be found in Table 1.

Test questions involved purely mathematical calculations as well as worded real-life problems. The test was given to all first-semester advanced mathematics bridging students (n=457) and Calculus and Linear Algebra 1 students (n=583) in their first lecture. It was a pen-and-paper test with no prior notice. Demographic and enrolment data collected as part of this test revealed that most students studying the advanced mathematics bridging course have either completed intermediate mathematics at a Queensland high school or the equivalent subject interstate or overseas. Calculus and Linear Algebra 1 students have usually studied intermediate and advanced mathematics at secondary school (or completed the advanced mathematics bridging course at UQ). Both first semester cohorts are typically made up of first-year engineering students (17-18 years old) who completed secondary school in Queensland.

Table 1. Typical Queensland Intermediate Mathematics Outline

Date	Topics
Year 11 Term 1	Fundamental concepts, applied statistical analysis, periodic functions and applications
Year 11 Term 2	Functions – limits, composite, inverse, non-linear
Year 11 Term 3	Periodic functions and applications, exponential and log functions
Year 11 Term 4	Rates of change, optimisation using derivatives, applied statistical analysis
Year 12 Term 1	Rates of change, optimisation using derivatives, introduction to integration – numerical
Year 12 Term 2	Periodic functions, exponential and log functions
Year 12 Term 3	Exponential and log functions, optimisation, introduction to integration – area under & between curves, rates of change – log functions, derivatives & graphing
Year 12 Term 4	Applied statistical analysis, optimisation

Students had approximately 20-25 minutes to complete the test and were asked not to use calculators. Students did not need to show working but had three options when answering each question. They could write their answer in the box, or tick one of two boxes: "never seen" or "can't remember". One reason these two options were included was to discover if some students, particularly the non-Queensland students, had not seen some of the topics before. The second reason was to gauge which topics the students felt comfortable in answering and which they did not. An explanation of the test was given to students beforehand, which included students being told that if a question looks familiar but you can't remember how to solve it, then tick the "can't remember" box and move on. A summary of test items is given in Figure 1.

Students' answers to Question 1 were perhaps the most surprising. Only 27% of the advanced mathematics bridging students and 57% of the Calculus and Linear Algebra 1 students answered correctly. The most common wrong answers were $\frac{8}{2x+2}$ (added numerators and added denominators, 8% of students) and $\frac{10}{x+2}$ (added 2 to both the numerator and denominator of the first fraction, 6% of students). Four percent of students had the correct denominator but incorrect numerator (eight different numerators overall).

Percentages of "can't remember" responses were quite high for Questions 7, 12-16. These were questions on topics that students had seen only in Years 11 and 12. An in-depth analysis of the 2007 test results can be found in Jennings (2008, 2009); however, students performed considerably better in topics to which they had more exposure. Questions on calculus, an area only studied in Years 11 and 12, had the lowest success rate. Little feedback was given to the students, other than the test and solutions being posted on course websites.

1. Write as a single fraction $\frac{3}{x}+\frac{5}{x+2}$	9. Given the right-angled triangle below (picture supplied), state the value of $\cos\theta$.
2. Solve $5+\frac{x}{2}=2+x$	10. A surveyor standing at a point B, 40m from the base of the tower, has measured the angle to the top of the tower as 60° (pictured supplied). Write an expression for the height of the tower in terms of the angle.
3. Expand and simplify $(2x-y)^2$	11. Let $f(x) = x^2 - \sqrt{x}$. Determine $f(4)$.
4. Factorise $9x^2 - 64$	12. When is $P(t) = t^2 - 6t + 16$ a maximum?
5. Solve $x^2 + 6x + 8 = 0$	13. Determine the first derivative of $f(x) = xe^x$
6. Simplify $(x^{1/2} \times y)^2/x^2$	14. Determine the first derivative of $f(x) = sin(7x)$
7. Evaluate $\log_3 9 + \log_4 2$	15. Evaluate the integral $\int \sqrt{x}\, dx$
8. You need to make 500mL of a solution that contains 10% (by volume) hydrochloric acid (HCl). What volumes of pure 100% HCl and distilled water do you need to make this solution?	16. Evaluate the definite integral $\int_0^2 (-2x+3)dx$

Figure 1. Summary of test items.

The time taken to mark the 1000-plus tests was lengthy so online tests were considered for 2008. Unfortunately it was not possible to design one in time for the beginning of Semester 1. In late 2008 a team of engineering and science academics received a UQ Teaching and Learning grant to design an electronic diagnostic test to assess knowledge of high school maths, physics, and chemistry, and also the ability of the students to apply this knowledge.

Survey Monkey was used to run the test; however, limitations on entering mathematics symbols meant that the test was multiple-choice, with carefully chosen distracters. Also included were the "never seen" or "can't remember" options.

The test was run just before Semester 1 began 2009, again with no prior notice, with students accessing the test at university or at home. The participation rate was disappointing (n=388), just over a third of the cohort. Given the poor results in the algebraic fractions and calculus questions in 2007, it was decided to add:

a numerical fractions question – Write $\frac{2}{3}+\frac{3}{4}$ as a single fraction,

a simple differentiation question –

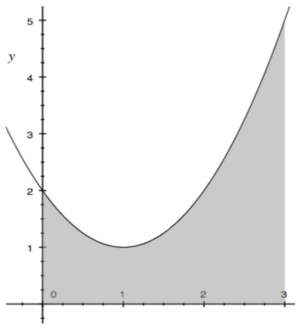
Determine the first derivative of $f(x)=x^3+2x^2-7x+4$,

and a simple integration question –

Find the integral $\int 3x^2+4x-1dx$.

Another integration question was also added.

Find the area of the shaded region that is bounded by the curve, the x-axis, the y-axis and the line x=3, giving the answer to one decimal place. The graph shown here is a graph of $y=x^2-2x+2$.



The remaining questions were the same as the 2007 test. While the raw percentages for most questions were better than the 2007 test, the order of difficulty remained the same. That is, questions on topics that students had first seen in Years 11 and 12 were not done as well as topics that students had had longer exposure to. It was not possible to analyse students' responses in terms of how many mathematics subjects they studied at high school. Feedback to students was limited to the correct answer appearing after each question was answered.

A re-test (*n*=103) was undertaken at the beginning of Semester 2, 2009, to see how students performed in comparison to the beginning of the year. Improvement was seen in all questions, particularly in solving a quadratic, chain and product rules, composition of functions, and definite integral.

Staff now had two years' worth of data so it was a matter of working out what do. Midway through 2009 mathematics staff met with the engineering staff who taught applied mechanics to students studying both their course and the advanced mathematics bridging course at the same time. The applied mechanics lecturers had previously mentioned that students' integration skills were poor, but integration was taught at the end of semester in mathematics. A decision was made to reorder the topics in the advanced mathematics bridging course from 2010, bringing differentiation and integration to the start of semester and moving sequences and series to the end. Integration questions on the end of semester examinations in the advanced mathematics bridging course were done poorly, so the hope was that this reordering of topics would not only benefit students in their applied mechanics study but also in their mathematics study.

In addition to this change in order of topics, a new drop-in centre was opened in 2010 for two hours a week, run by a lecturer, to complement the First Year Learning Centre, staffed by postgraduate students (open for two hours a day Monday to Friday). Online and paper resources were also developed for topics in the advanced mathematics bridging course (Jennings, 2008).

In 2010 the same test was run again (n=623, approximately two-thirds of the cohort). Feedback to students was greatly improved, with each student receiving a personalised report detailing their mark for each question (correct or incorrect), which first-year course(s) each question was important for, and for each question at least one website to visit in order to improve their knowledge and understanding. Many students commented positively on this detailed feedback.

The 2010 cohort was made up of students with higher overall high school results compared with the 2009 cohort, and they performed better in all questions. It was possible to analyse student performance according to how many mathematics subjects they studied at high school, with the students who had studied both intermediate and advanced mathematics performing considerably better than those who had only studied intermediate mathematics. As in 2007 and 2009, students performed considerably better in topics to which they had more exposure. A re-test (n=107) was undertaken at the beginning of Semester 2, 2010; however, of these 107 students only 46 did the original test so comparisons are difficult to make.

The applied mechanics staff reported at the end of Semester 1, 2010, that their students seemed to be better prepared as a result of integration being taught earlier in the advanced mathematics bridging course. However, there appeared to be little change in the advanced mathematics bridging course as students still had considerable difficulty answering integration questions on the final exam.

Where to From Here?

The diagnostic test as it stands gives teaching staff some idea of what the students can do, but it doesn't give staff much of an idea of what students *understand*. In 2011 the plan is to run the test electronically, yet with as few multiple-choice questions as possible. Some of the questions will remain; others will be replaced with questions that allow students to demonstrate their understanding, as opposed to demonstrating which algorithms they remember.

One likely question, and one that was asked on last semester's advanced mathematics bridging course exam, is to give students a graph with a local maximum and minimum (e.g., $y = x^3 + 3x^2 - 9x + 4$) and ask them on which interval(s) the derivative is negative. A very young student can be taught the algorithm to derive a function, but knowing where a derivative is negative is a more complex question. Other possible questions include:

- giving students a graph of a function and asking which of four other graphs is the graph of the derivative (or integral);
- find the integral $\int_{-3}^{3} |x+2|\,dx$ (Eisenberg & Dreyfus, 1991);
- find the maximum slope of $y = -x^3 + 3x^2 + 9x + 27$ (Eisenberg & Dreyfus, 1991).

Another question to show students' understanding of calculus is the one designed by Yoon, Dreyfus and Thomas (2009). Students are given the gradient graph of a tramping track and are asked to find the distance-height graph of the original track. The format of the current diagnostic test does not allow for graphs to be drawn; however, this question could be done in tutorials.

Conclusion

Are the test results surprising? Apart from the algebraic fractions question, not really. Should we complain about how much the students remember from high school? Probably not. The title of this article is *A changing cohort - what do high school students remember and what does it mean for university lecturers?* The answer to the first part of this question is "it depends". Given the nature of the curriculum in Queensland (no external exams, integration only studied for part of Year 12), plus the fact that students would have done no mathematics for the best part of four or five months, it doesn't come as a surprise that students don't remember the product and chain rules or how to find $\int \sqrt{x}\,dx$.

As for what does it mean for university lecturers, it is not possible to squeeze 200 hours of high school mathematics into 39 50-minute lectures. It is therefore important to have a good understanding of what students remember and understand from high school so staff can appropriately tailor their courses. Suggestions for changes to the test and for new questions that allow staff to see what students *understand* have been made in this article.

Students who do the one-semester advanced mathematics bridging course are considered by the university to be at the same level as those students who did both intermediate and advanced mathematics at school. An analysis of students' final exams and overall grades in Calculus and Linear Algebra 1 for the past five years show that students who had studied both intermediate and advanced mathematics at school performed considerably better than those who had only studied intermediate mathematics. This appears consistent with Barry and Chapman's (2007) and Wilson and MacGillivray's (2007) research that showed performance at the tertiary level is dependent on secondary school performance.

Several questions require thinking about:

1. Should the advanced mathematics bridging course run for a year?
2. Should the diagnostic test be run in Week 2 or 3, once students' minds have been working for a few weeks?
3. Should the test be run before semester and then again in Week 2 or 3?
4. In addition to the feedback given to students, should a revision manual like McMaster University's be created? (Kajander & Lovric, 2005).

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Today's Students are Technoliterate: Texting for Communication

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Introduction

The Tertiary Foundation Certificate Programme (TFC) at The University of Auckland is a full year pre-degree programme designed to prepare students for tertiary study. Each year it is offered to two hundred students who do not have the required entry qualifications for university, but who have the desire to succeed academically. Participants may have been out of school for some time, or may be recent school leavers.

The University of Auckland has produced a guide to parents on the transition from school to university. The key message is the shift towards students taking responsibility for and control of their own lives. Parents are advised that the university can offer some support, but only at the student's request. The stylized graph, shown in Figure 1, was produced in the guide, to document the emotional rollercoaster that students experience in their transition into first year degree study. Foundation students are similar to Stage One students in the transition but we would claim 'even more so', because they are often living at the margins, with greater problems in terms of finance, transport, living conditions and parental support.

Students applying for TFC undergo a selection process. Over five hundred applicants attend for entry tests in English and Mathematics, about three hundred of these are interviewed, from which two hundred are chosen for the available places. The excitement phase is a feature of their year as much as it is for any other student. However the anxiety phase often sets in too quickly for comfort and needs addressing as it manifests itself in absence from class, and in worry about the work. As these students may have parents with no experience of the demands of university education, the tutors need to be in a position to step in effectively. Because the TFC mathematics classes are relatively small at fewer than forty students, the tutors can act as a caring parent might. The aim is to stop the anxiety phase tracking down to a point where it is irretrievable.

The TFC student body is diverse, not only in terms of age and ethnicity, but most importantly in background knowledge of mathematics, which is a compulsory subject along with English (or Academic Literacy). The students in the TFC mathematics courses are a step back from first year undergraduates in many ways. Those who have come straight from school have already experienced failure in their attempt to gain university entrance through the usual channels. Those who are mature students returning to study are unsure and lack confidence for a variety of reasons. As a whole, the students are likely to be more fragile, more needy, and more prone to dropping out when compared with the usual first year undergraduate.

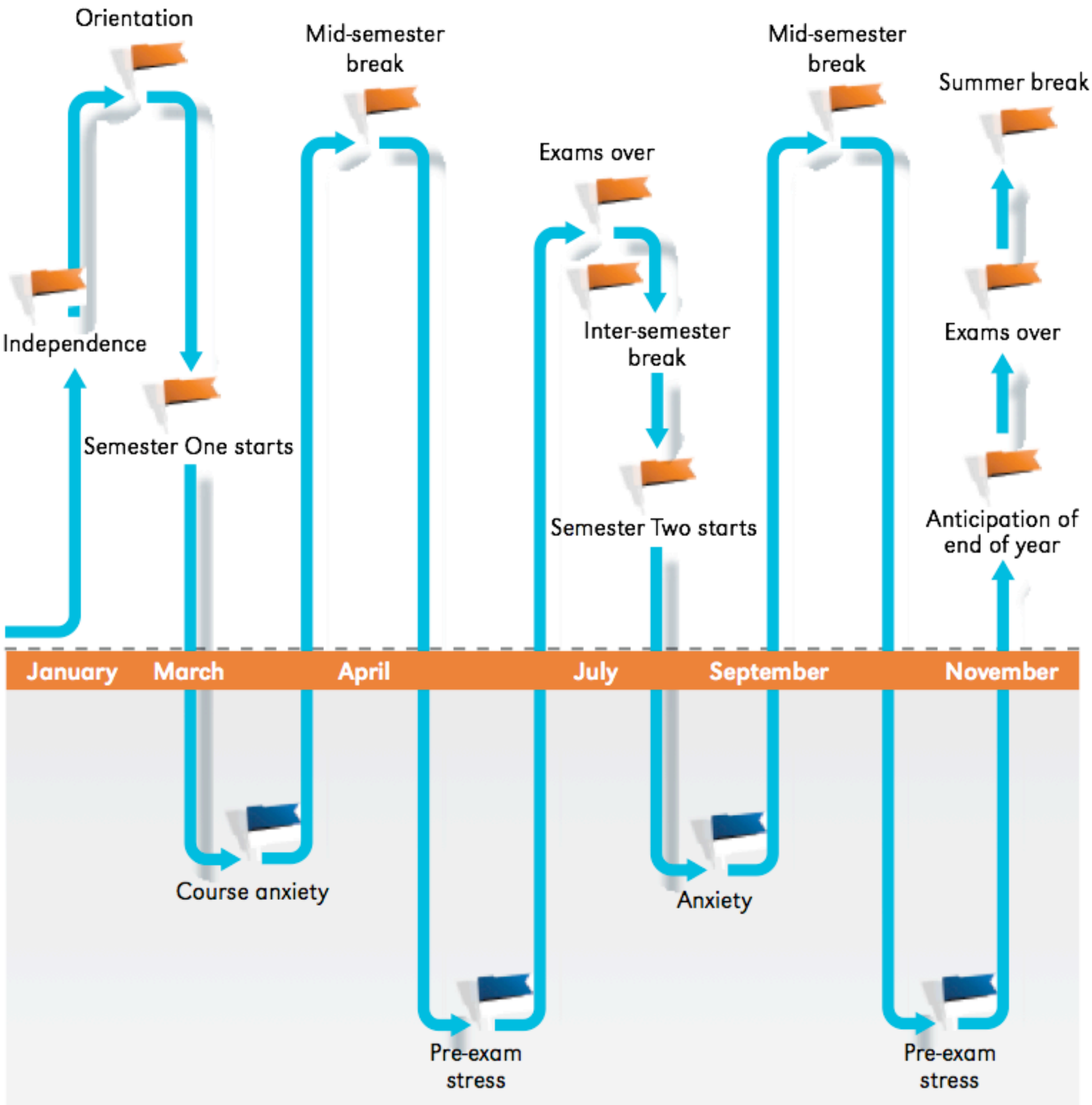

Figure 1. Highs and lows: A guide to a typical first year at university for TFC students.

As a first step towards counteracting some foundation students' traditional negative emotions and low confidence (Anthony, 2000) we need to promote good communication between tutors and students. We are aiming to turn around their failure, to boost their confidence, to stop them dropping out and to make them into successful learners. These are changes in their own disposition, so as adult mentors what can we do to help? Over the last few years there has been a noticeable and rapid change in the nature of students' expectations that is linked to the advent of technology. As tutors in a university–based foundation programme, this made us think about ways that technology could play a part in improving tutor to student communication. One of the most obvious pieces of technology was the cell-phone: they all have cell-phones, they all text incessantly. We decided to use texting to support good attendance as a first step to promoting a work ethic. By demonstrating that attendance is a first priority we hoped to focus their attention on factors for success.

Text Messaging Made Easy: Web to Text Using an Office Computer

Although all students can theoretically be contacted via email, in practice as Carnevale (2006) indicates, this is dependent on them accessing a computer. Secondly the time delay can be significant.

> As some students reduce their use of e-mail in favor of other means of communication, colleges are trying new technologies to reach them. Among the new techniques: Cell-phone Text Messages. Students live and die by their cell-phones. (Carnevale, 2006)

Many schools in NZ are using texting as a way to improve attendance. When students are absent without notification, parents are texted and the responsibility is passed to the parent to explain the absence or to round up their offspring and get them to school. In the TFC programme, responsibility for attendance needs to be clearly shifted to the student.

The TFC Mathematics tutors applied to Vodafone to set up a computer web to text system. Our office computers are used, via a commercial web site, to access students' cell-phones. This involves compiling a database of cell-phone numbers in the appropriate format for all of the students. To contact a student by text, it is only necessary to type their name in the 'To' line of an email. Replies from students come back to our computers and the site provides a permanent record of all texts and all replies. Some success in 2008, even beginning at the halfway point of the programme, encouraged us to keep the system in place for the future. At present we make continual use of the system and it has become part of the everyday means of communication between tutors and students.

The features of this messaging tool include:

- sending messages as an email to and receiving responses from standard mobile phones using a maximum of 160 characters per message;
- message tracking;
- centralised address books;
- standard web browser access;
- reply functionality delivering return messages back to the originating person; and

- one account per organization and messages are free for the students to receive on their mobiles.

The benefits for us were:

- the time saved in communicating with students;
- being able to contact multiple mobiles with one message;
- the fact that there were no changes required to the current computer network setup, just simple browser access;
- it was quick and simple to use.

Absence from class

In the first weeks of a semester, at the first missed class, we text the student. This is the first indication to those students coming straight from school that now they need to take responsibility for themselves, but that we care about them. The message is couched in a friendly, concerned tone with a request for a reply to set up the communication that we are promoting. In most cases we have instant contact with students, and we hope that early intervention in problems with attendance will prevent their escalation into large irreversible issues. Generally the student will respond immediately, by reply text or by appearing in person.

Absence from a test

Facing up to the threat of a test can be a problem for foundation students. One of their avoidance techniques is to be ‘sick’. At the beginning, we chase these up via text, asking sensitively if they are in any trouble, and can they contact us. At this time we will offer encouragement and the opportunity to take the test when they are ‘better’. Again the communication will often forestall further deterioration in the student’s disposition.

Summary

In a practical sense, we need to retain these students in order to give them their only real chance of gaining a university entrance qualification. Foundation programmes like TFC offer the opportunity for institutions to tap into a valuable pool of non-traditional students. The interest in such programmes is growing, diversity in the student body is increasing and the lecturers responsible for teaching the students need to keep pace with the demands of a technologically aware clientele. Maintaining personal contact with students via text messaging has shown to be useful.

> The tenacity shown by students, often in the face of compounding negativity, was positively correlated to connections made with teaching staff. An awareness by teaching staff of personal impinging factors appeared to create definitive turning points for individuals, if staff were actively engaged in pastoral as well as academic assistance. (Morgan, 2004, p. 25)

By providing opportunities for students to communicate more with their tutors, it is perhaps possible to alleviate some of the issues in the affective domain between attitudes and emotions. Conversational support is a non-threatening and technologically possible strategy to enhance the experience of foundation students in a university environment. The 21st Century student has different expectations for communication. Tapping into their technoliteracy makes extra lines of communication to enhance their student experience.

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Report: AustMS Teaching and Learning Workshop

The inaugural Effective Teaching, Effective Learning workshop chaired by Diane Donovan (The University of Queensland (UQ)) and Birgit Loch (Swinburne University of Technology) was held in September 2010 in conjunction with the 54th Annual Conference of the Australian Mathematical Society at UQ. There were 65 participants from around 20 universities. The workshop is part of a professional development program for lecturers and tutors teaching in disciplines in the mathematical sciences funded by the Australian Learning and Teaching Council (ALTC) and offered in conjunction with the Australian Mathematical Society (AustMS). The project team is Leigh Wood (Macquarie University (MQ)), Nalini Joshi (The University of Sydney), Diane Donovan (UQ), Birgit Loch (Swinburne), Walter Bloom (Murdoch University), Matt Bower (MQ), Jane Skalicky and Natalie Brown (University of Tasmania).

Each year in Australia, around 18,000 undergraduate students study a subject in the mathematical sciences. Mathematics teaching staff receive some training in learning and teaching but many of the courses run at university level are not tailored to the mathematical sciences. This workshop and associated professional development activities will offer teaching staff in quantitative disciplines a way to enhance their teaching and the learning of their students in conjunction with their discipline professional body, the AustMS.

At the workshop, PhD students and early career academics came together with ‘old hands’ to discuss and brainstorm strategies for responding to the continually changing landscape of teaching university-level mathematics. The workshop was a chance to showcase the teaching and learning programs, staff and facilities of the host university. Three local UQ staff from a variety of departments facilitated workshop sessions, a UQ mathematics graduate delivered a plenary focused on the transition to industry, and a session was run in UQ’s renowned Advanced Concepts Teaching Space laboratory where participants experimented with using tablet screens. After an inspiring plenary from Bill Barton (The University of Auckland), interactive sessions focused on class and unit planning, assessment, planning your career, evidence-based teaching and service teaching. The sessions on educational technology were very popular and looked at delivery platforms, online tutorials and the use of mathematical typesetting for students. The teaching strategies offered in the workshop are applicable across the quantitative disciplines.

The workshop was designed to benefit all, irrespective of whether they are teaching into mathematics or into service disciplines. The evaluations were positive and the vast majority would attend again and recommend the workshop to their colleagues. After such positive evaluations, we are already planning the next workshop for the 55th Annual meeting of AustMS at the University of Wollongong on 29-30 September 2011. Staging the workshop each year will build capability across the higher education sector as the AustMS meeting moves around Australia and different universities take ownership of the development each year. Regular face-to-face workshops are a key avenue for fostering enthusiasm, inspiration and innovation as linked to focussed knowledge on best-practice teaching. Many PhD students and early career researchers participated: these are our future academics and academic leaders. We hope the impact on the profession and the student experience will be significant.

The professional development works at three levels, teaching classes, coordinating units (papers) and leading programs. The three levels should be aligned as in Table 1.

Table 1. Alignment Overview for the *Planning Classes* Session

Planning	Teaching classes	Coordinating units	Leading programs
Goals/outcomes	• hook • clear outcomes for session	• unit outcomes • communication • links to other ideas and disciplines	• program outcomes • communication • graduate skills
Learning tasks – how to achieve the goals	• audience • moving between representations • student destinations	• variety of learning tasks • appropriate delivery media	• scaffolding learning tasks across a program
Assessment – how to test goals	• questioning • working in groups • short quizzes	• formal assessments • coordination	• policy • quality assurance
Feedback – how to report on achievement	• informal • questioning • group/individual • verbal/written • online	• formal feedback on tasks – group • management of feedback in large classes	• policy on feedback • assurance of learning • providing resources

The sessions at the workshop were interactive and several case studies were presented which generated lively discussion. For example, try Exercise 1 with your new staff.

Exercise 1: Case Study for the *Planning Classes* session.

Your students are really keen and needy. They want you to post solutions to everything – extra examples fully worked through. You are their hero/heroine because you are doing so much for them. What do you do?

Quotes from participants

- You're never too old to learn more.
- Planning and structuring a lecture is important.
- Tasks can and should assess a variety of outcomes.
- Such a variety of ways to engage students online!
- That I need to become familiar with more technology so I'm not left behind!!!

We extend the invitation to teaching staff in the quantitative disciplines to attend the next workshop in Wollongong, Australia on 29-30 September 2011. Why not come for the Australian Mathematical Society conference from 26 September as well?

What is the Probability That a Randomly Chosen Triangle is Obtuse?

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In May 2010 Gilbert Strang (who has spent much of his teaching career at MIT) gave an interesting lecture about the "probability of a randomly chosen triangle being obtuse". The problem seems to be over 100 years old and was published by Charles Dodgson in "Pillow Problems". Gilbert and his colleagues had spent some time on the problem, investigated a number of approaches, and in the lecture gave a solution that involved a 3-D graph.

I wondered after the lecture, how many of the audience took the problem home and worked on it, or are we as mathematicians, like our students, becoming consumers rather than producers of mathematics.

I did keep thinking about it and thought why do we need three dimensions for a problem in two-dimensional geometry. My working went as follows (and I have left it to each reader to do some mathematics by constructing the graph):

To find the probability that a randomly chosen triangle is obtuse.

Consider any triangle with angles X, Y, and Z, and we know that
$Z = 180 - X - Y$

(or instead of 180˚, 2 right angles, or 1 half turn).
and that: $0 < X < 180$
and $0 < Y < 180$
and $0 < 180 - X - Y < 180$ (i.e. $180 > X + Y > 0$)

If a graph is drawn with these three sets of parallel boundaries, then the intersection of the three regions represents the sample space for all randomly chosen triangles.

If the triangle is acute then:
$0 < X < 90$,
and $0 < Y < 90$
and $0 < 180 - X - Y < 90$ (i.e. $180 > X + Y > 90$)
and these three sets of parallel boundaries intersect for the sample space for all acute-angled triangle.

From the graph it is obvious that

$$P(\text{obtuse}) = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

Perhaps this begs the question, is a visual proof a proof?

Reference

Dodgson, C. L. (1894). *Curiosa Mathematica, Part 2: Pillow problems thought out during wakeful hours.* City: Kessinger Publishing.

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Perceptions, Operations and Proof in Undergraduate Mathematics

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Teaching and learning undergraduate mathematics involves the introduction of ways of thinking that at the same time are intended to be more precise and logical, yet which operate in ways that are unlike students' previous experience.

When we think of a vector, in school it is a quantity with magnitude and direction that may be visualized as an arrow, or a symbol with coordinates that can be acted upon by matrices. In university mathematics it is an element in an axiomatic vector space.

As I reflected on this situation I realised that these three entirely different ways of thinking apply in general throughout the whole of mathematics (Tall 2004, 2008). The two ways encountered in school depend on the one hand on our physical perception and action and dynamic thought experiments as we think about relationships, on the other they depend on operations that we learn to perform such as counting and sharing which in turn are symbolised as mathematical concepts such as number and fraction.

At university, all this is turned on its head and reformulated in terms of axiomatic systems and formal deduction. Our previous experiences are now to be refined and properties are only valid if they can be proved from the axioms and definitions using mathematical proof. The formal approach gives a huge bonus. No longer do proofs depend on a particular situation: they will hold good in any future situation we may meet provided only that the new context satisfies the specific axioms and definitions. However, the new experience is also accompanied by mental confusion as links, previously connected in perception and action, now require reorganisation as formal deductions, and subtle implicit links from experience may be at variance with the new formal setting.

Further analysis of the development of mathematical thinking reveals three quite different forms of thinking and development that I term *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*. These operate in such different ways—not only at a given point in time, but also in their long-term development—that I called them three mental *worlds* of mathematics.

Conceptual embodiment and operational symbolism develop in complementary ways in school mathematics in which physical operations relate to algebraic symbolism (Thomas, 1988). The world of conceptual embodiment is based on our operation as biological creatures, with gestures that convey meaning, perceptions of objects that recognise properties and patterns, thought experiments that imagine possibilities, and verbal descriptions and definitions that formulate relationships and deductions as found in Euclidean geometry and other forms of figures and diagrams. The world of operational symbolism involves practising sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number). Gray and Tall (1994) formulated this flexibility by speaking of such symbols as 'procepts' that act dually as *pro*cess and con*cept*. The operational world of symbolism develops in a spectrum of ways from limited procedural learning to flexible proceptual thinking.

The third world of axiomatic formalism builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure. Its major criterion is that relationships must in principle be deducible by formal proof. However, students and mathematicians interpret formalism in a variety of ways, depending on the links with embodiment and symbolism. Some build *naturally* on their previous experience to *give meaning* to definitions. For instance, the idea of a sequence (s_n) tending to a limit may be seen by plotting the successive points (n,s_n) and seeing that, the sequence tends to a limit L if, given a required error $\varepsilon > 0$, then from some value N onwards, (for $n \geq N$) the terms s_n lie between two horizontal lines $L \pm \varepsilon$. Others build *formally* by *extracting meaning* from the definition by learning to reproduce it and practising formal proofs until it becomes a familiar mode of operation. Both approaches are possible and can lead to successful formal thinking, although both can fail, either because the new formal ideas conflict with beliefs built from earlier experience or because the multi-quantified definitions are just too difficult to handle (Pinto, 1998; Pinto & Tall, 1999).

The question arises as to how this framework of three worlds of mathematics can help us as mathematicians to encourage our students to think in successful mathematical ways. The framework is general. Although embodiment starts earlier than operational symbolism, and formalism occurs much later still, when all three possibilities are available at university level, the framework says nothing about the sequence in which teaching should occur. Indeed, in the learning of mathematical analysis some students clearly follow a natural approach based on their thought experiments and concept imagery while others are more comfortable working in a purely formal context. Not only is it possible to use embodied examples to give meaning to a formal theory, it is also possible to use a formal theory to highlight the essential properties in an embodied example.

The framework can be better understood by reflecting on specific cases. Consider, for example the notion of continuity. Embodiment clearly gives powerful insights that can be used to motivate symbolic relationships and formal definitions. For instance, the dynamic idea of natural continuity arises from the physical drawing of a graph with a 'continuous' stroke of the pencil remaining on the paper and leaving a continuous trace. While this is often considered to be an 'intuitive' notion of continuity that lacks a formal definition, it is also possible to envisage the graph as a stroke of a pencil which covers the theoretical graph with a stripe of height $\pm\varepsilon$. If a small portion of the graph is stretched horizontally, while maintaining the vertical height, the graph will 'pull flat' in the sense that, for some $\delta > 0$, then for any x between $x_0 - \delta$ and $x_0 + \delta$, the value of $f(x)$ will lie between $f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon$ (Tall, 2009). In this way it is possible to have a natural transition from embodied continuity to the formal definition in mathematical analysis, which may help a natural learner but may be unnecessary for a formal learner.

Elementary calculus is highly amenable to a natural approach that links together visual insight and symbolic manipulation without introducing formal epsilon-delta definitions. Using computer technology to magnify graphs reveals the property that many continuous graphs visibly approximate to a straight line under high magnification. Such a graph is said to be 'locally straight'. The slope of a locally straight graph can be *seen* by highly magnifying a portion of a graph to visualize it as essentially straight and to measure its slope. This gives a natural distinction between continuity of a graph drawn with a pencil or with pixels on a graphic display (which will 'pull flat') and differentiability (which involve graphs that are 'locally straight'). It enables students to visualize non-differentiability (with 'corners' having different left and right derivatives, or even functions that are so wrinkled that they do not look straight no matter how much they are magnified) and to realise that most continuous functions are *not* differentiable (Tall, 2009). Such an approach, although based on visual and symbolic techniques only, gives far greater insight into the *meaning* of the notions of continuity and differentiability.

Furthermore, for a locally straight function, the Leibniz notation dy/dx may be interpreted as a quotient of the components of the tangent vector, as originally conceived by Leibniz himself. In such an interpretation, *dx* and *dy* can be called differentials, representing the components of the tangent vector up to a scalar multiple. Now a first-order differential equation is just that: it formulates the direction of the tangent in which the differentials are the components *dx* and *dy*.

Software can be programmed to build up the numerical slope of a graph dynamically by shifting along and computing $(f(x+h)-f(x))/h$ for variable *x* and fixed *h*. This can be drawn as a practical slope function that stabilizes on a visible graph on screen for small values of *h*, revealing the stabilized graph as the derivative. The embodied action of looking along a graph, imagining its changing slope operates on a visual *object*, (the graph of *f*) and gives a new *object* (the stabilized graph *Df*). For instance, if $f(x)=\sin x$, then looking at the changing slope along the graph gives $Df(x)=\cos x$. The symbol *D* is here an *embodied* operator that means 'look along the graph and *see* its slope function *Df*).

Focusing on a specific point *x*, gives the equation

$$Df(x)=\frac{dy}{dx}$$

where $Df(x)$ is the value of the function produced by the operation *D* calculated at *x* and *dx* and *dy* are differentials (components of the tangent). This leads to the natural idea of blending of the two meanings by writing

$$\frac{dy}{dx}=\frac{d(f(x))}{dx}=\frac{d}{dx}f(x)$$

and allowing the symbol d/dx to be interchanged with the operation *D*.

This approach is a quite different from that suggested by the APOS theory of Dubinsky (e.g. Asiala et al., 1996), which speaks of focusing on a *process*, here the limit process $\lim_{h\to 0}(f(x+h)-f(x))/h$, and encapsulating it as an *object*. Fundamentally, operating on an *object* to construct a visible object is far more elementary than encapsulating a *process* to give an as yet unknown *object*. Research results speak for themselves: the visual approach is highly successful (Tall, 1986) whereas the APOS view, programming functions symbolically to compute a practical derivative that is to be encapsulated as a symbolic object proves to be far more elusive (Cottrill et al., 1996).

There is a clear distinction between a natural approach to elementary calculus and a formal approach to mathematical analysis. Elementary calculus blends together experiences in embodiment and symbolism without entering the complicated formal world of mathematical analysis that is characterised by the multi-quantified epsilon-delta definition of limit.

Notice that I am not saying that one approach should be privileged over another. It is not a question of whether one should teach the formal definition of limit or not, it is a question of the objective of the particular course and its appropriateness for the current development of the learner.

If the objective is to give insight into the calculus as an operational system in applications in which the Leibniz notation plays its part, then a locally straight approach gives both human meaning and operational symbolism. If the objective is to develop logical mathematical analysis (preferably as a course that follows elementary calculus), then the handling of multi-quantified definitions is part of the toolkit required for rigorous mathematical thinking. The most important aspect is to decide upon the aims of the course and not to inflict formal subtleties on students who are better served by a meaningful blend of embodiment and symbolism.

The three worlds of mathematics each offer their own distinct advantages:

- embodiment gives a basis of human meaning that can be translated into flexible symbolism,
- symbolism offers a powerful tool for suitably accurate computation and precise symbolic solutions,
- formalism offers precise logical deduction that will operate in any context where the axioms and definitions are satisfied.

Consider, for example, the manipulation of multi-quantified statements. Embodiment will allow thought experiments to think about how to negate such a statement, to allow one to realise that to prove that a universal statement is not true, one only needs a single counter-example and that to prove an existence statement is not true requires a universal statement of its falsehood.

Symbolism translates these statements into $\neg\forall \equiv \exists\neg$ and $\neg\exists \equiv \forall\neg$. In this way the definition of continuity of a function *f* at a point *x* on a domain *D* can be written as

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall y \in D\left(\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\right)$$

and its negation can be found by placing the negation symbol in front and passing it successively over each quantifier, swapping one to the other to get

$$\exists\varepsilon > 0\ \forall\delta > 0\ \exists y \in D\left(|x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon\right).$$

This symbolic manipulation is easier to handle than thinking through the full embodiment of the meaning all at once. It enables a more compressed form of thinking that is supportive in building formal proofs.

What is essential in learning is to build on the previous experience of the students to enable them to make personal sense of the new constructs.

In the case of vectors, a vector also has three different meanings: as a geometric quantity with magnitude and direction, as an algebraic entity written as a column vector and as an element in a formal vector space. As a geometric quantity, it can be represented as a physical action, say as a translation of an object such as a triangle on the surface of a flat table. A given point *A* on the object will be shifted by a translation to a point *B* and represented as the shift $\overrightarrow{AB}$ from a starting point to a finishing point in which any two such arrows will all have the same magnitude and direction. The translation can therefore be represented by a *single* arrow of given magnitude and direction that can be placed anywhere to represent the start and end of the shift of a particular point. This gives an embodied arrow of given magnitude and direction that represents the translation. Again we start with an *object* on the table and a process of translating it to represent the translation as an embodied object, the *free vector*. Representing the composite of two translations, one after another, the result is represented by the unique free vector that has the same effect. This conception of a free vector then has a meaning that translates naturally to the triangle law or the parallelogram law.

A scalar multiple of a translation can be imagined as retaining the direction but multiplying the magnitude by the scalar (or reversing the direction if the scalar is negative). This applies to free vectors by multiplying the magnitude of the vector by the scalar in the same way.

The symbolic representation of a vector arises naturally through the solution of a system of linear equations in n variables. For $n = 1, 2, 3$, such equations can be represented in 1, 2, or 3 dimensional space. The symbolic techniques naturally extend to n variables and, even if the ideas are no longer easily visualised in higher dimensions, they can be represented by coordinate vectors with n components with transformations represented by matrices.

The formal representation of a vector is quite different. A vector space is specified as an additive abelian group V with the action of a field of scalars F that satisfies appropriate axioms. Such vectors now no longer have magnitude or direction, but by introducing the notion of linear independence and spanning set, a structure theorem may be proved to show that any finite dimensional vector space over F is isomorphic to a space F^n represented as n-dimensional coordinates. In the case of $n = 2$ or 3 and $F = \mathbb{R}$ gives an embodiment of the vector space almost like $\mathbb{R}^2$ or $\mathbb{R}^3$. I say 'almost' because the vectors in the vector space do not yet have a conception of magnitude or direction. To do this, one needs to add an inner product to enable one to specify lengths and angles.

The problem for the teacher and the student is to be aware what assumptions are being made. Are vector spaces being studied *formally* based only on deductions from axioms or *naturally*, based on experiences of perceptions and actions in two and three-dimensional real space? The choice is up to the teacher, but it needs to be explicit.

A natural approach would involve beginning from conceptions that are familiar: solving linear equations in one, two and three variables and *generalizing* them to n variables, which involves essentially the same symbolic solution technique although no longer visualizable in higher dimensional space. A formal approach would begin by *abstracting* the axioms for a vector space and writing down the list of axioms, and eventually proving a structure theorem from the axioms that vectors in a finite dimensional vector space can be represented by coordinate vectors with n

components. Of course, if a natural learners are presented with a formal approach, then the initial theorems and proofs may make little sense and the course may only come alive for them when the structure theorem for finite dimensional vector spaces has been proved and they are asked to solve linear equations operationally using symbolic vectors.

The same can be said for other topic areas, for instance, groups studied as embodied operations of actions on figures with symmetry, or symbolic operations as permutations of n elements prior to a formal axiomatic approach.

Formally, the various lecture courses, be they in analysis, vector space theory, group theory, or whatever, often begin with a formal axiomatic structure and formal deductions. Part of the way through the course a structure theorem is proven to give the axiomatic system a structure that can be embodied in a manner now based deductively on the axioms with an operational symbolism that can be used solve problems symbolically.

For instance, in analysis, the axioms for a complete ordered field identify it uniquely up to isomorphism, allowing it to be visualised as a real line and symbolised as infinite decimals. In vector space theory, a finite dimensional vector space over F is isomorphic to F^n, allowing it to be symbolised as n-tuples and embodied in $\mathbb{R}^2$ or $\mathbb{R}^3$. In group theory, a finite group is isomorphic to a subgroup of permutations.

The roles played by embodiment, symbolism and formalism are very different and the teacher has to make it explicitly clear what approach is being taken. Is the course to be a formal course that requires formal deduction from axioms? This may be built entirely formally until structure theorems give it forms of embodiment and symbolism based on those axioms. Is it a formal course to be constructed naturally to enable students to give meaning to formal definitions through a range of examples? Or is the course intended to develop the necessary symbolic algorithms to enable the ideas to be used in specific applications, with examples relevant to the area of application?

My own view is that it would help students enormously to gain an insight into the strategy, which many lecturers use implicitly but is rarely made explicit. That is that formal mathematics clarifies issues by specifying explicit axioms that are the ‘rules of the game’ and formal proofs deduced using these rules are proven once and for all in any situation where the rules are satisfied. The initial deductions from the rules are often quite technical and form a barrier for many students. But once a structure theorem has been proved, the techniques developed are now proven to work in all situations, whether known now or to be encountered in the future. This formal foundation is a gift worth having and it can be acquired by the formal thinker who deduces only from the axioms using formal proof, or by the natural thinker who sees the generalities bringing together many experiences that give meanings to the formalities.

An understanding of three different approaches to mathematics would be invaluable, made explicit both to teachers and to students to be aware of the different objectives of mathematical thinking, consisting of:

- ideas based on human perceptions and actions with thought experiments to suggest what might be true,
- operations based on actions that give subtle mathematical processes to express and solve problems symbolically, and
- formal axioms, definitions and proof that give a coherent framework of mathematics, supporting perception and operation with an underlying formal structure that applies in any situation where the axioms and definitions hold.

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Thinking about the Teaching of Linear Algebra

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Is there a better way to teach linear algebra? Over the past three decades, a number of researchers have been concerned with the difficulties related to learning problems in first year linear algebra courses. They believe that teaching and learning this 'high cognitively demanding' course (Dorier & Sierpinska, 2001) is a frustrating experience for both teachers and students, and despite all efforts to improve the curriculum, "linear algebra remains a cognitively and conceptually difficult subject" (Dorier & Sierpinska, 2001, p. 255). One major problem seems to be that students often cope with the procedural aspects of the course, manipulating matrices etc, and hence may perform well enough in examinations, but struggle to understand the crucial conceptual ideas underpinning the procedures. They either don't understand the need for definitions of concepts presented in natural language, or don't see how to incorporate them into their thinking. Mathematicians, of course, consider definitions to be a fundamental starting point for concept formation and deductive reasoning in advanced mathematics. Hence they may believe, like Sierpinska, Nnadozie and Okta (2002, p. 1), that many students are wasting their intellectual talents, since "linear algebra, with its axiomatic definitions of vector space and linear transformation, is a highly theoretical knowledge, and its learning cannot be reduced to practicing and mastering a set of computational procedures".

One of the tensions within the teaching of linear algebra is this balancing of the need for a definitions and theorems approach with procedural aspects such as Gaussian elimination or Jacobi's method. Students often find the former hard because it is their first experience of a systematic construction of mathematical theory, and in this respect linear algebra is very different from calculus (Harel, 1997). Moreover, while the formal definitions of calculus concepts, such as function, limit and continuity, may resonate in an embodied way with students' previous experiences, they may have less intuition for linear algebra constructs, such as basis or eigenvector. This is especially the case in countries, such as New Zealand, where matrices and transformations are no longer taught in the school system.

In considering alternative ways of teaching linear algebra we engaged in construction of a theoretical framework that might suggest ways to advance students' conceptual understanding of the basic linear algebra concepts. Our starting point was to consider two theoretical perspectives that we had previously found useful as research descriptors and predictors. First, the action-process-object-schema (APOS) development in learning proposed by

Dubinsky and others (Dubinsky & McDonald, 2001) suggests an approach different from the definition-theorem-proof that often characterises university courses. Instead mathematical concepts are described in terms of a genetic decomposition, divided into their constituent actions, process and objects in the order the learner should experience them in order to construct understanding.

Second, Tall (2004, 2008; see also the article in this issue) introduced a framework for mathematical thinking based on three worlds: the embodied; symbolic; and formal. The embodied world is enactive and visual. It contains embodied objects (Gray & Tall, 2001); it is where we think about the physical world, using "…not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery" (Tall, 2004, p. 30). The symbolic world is the world of procepts, where actions, processes and their corresponding objects are realized and symbolized, and the formal world comprises defined objects (Tall et al., 2000), presented in terms of their properties, with new properties deduced from objects by formal proof. While these worlds describe a hierarchy of qualitatively different ways of thinking that individuals develop as new conceptions are compressed into more thinkable concepts (Tall & Mejia-Ramos, 2006), all the worlds continue to be available to, and used by, individuals as they engage with mathematical thinking. In particular, "formal mathematics does not arise in isolation" (Tall, 2008, p. 4). Instead, the three worlds of mathematical thinking combine so that "three interrelated sequences of development blend together to build a full range of thinking" (Tall, 2008, p. 3).

Our view was that one could benefit from the synergy of combining these two perspectives into a single framework, even though APOS theory refers to learners' general cognitive thinking, while Tall's worlds are specifically about mathematical thinking. This was possible since, we suggest, they are complementary, in the sense that for any given construct one can consider what the mathematical action and process perspectives, which give rise to properties of the object under consideration, would be like in each of the embodied, symbolic and formal worlds. Thus our framework was constructed by blending the two theories orthogonally into a single construct that could be employed to examine embodied, symbolic and formal action, process, and object thought processes. The medium of presentation was a matrix in which the left-hand column comprised the action, process and object dimensions while the top cells represented the three mathematical worlds: embodied; symbolic (divided into algebraic and matrix); and formal. One novel aspect of this approach is the recognition that actions, processes and objects can be instantiated in visual, geometric, and embodied forms. An example of part of the framework for the concept of basis can be seen in Figure 1.

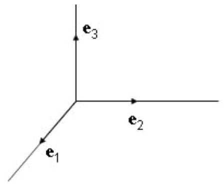
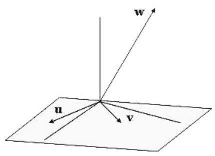
APOS	Embodied World	Symbolic World		Formal World
		Algebra	Matrix	
Action	Can see 3 specific non-coplanar vectors as a basis of R^3  e.g., The standard basis for R^3	Can find a basis for S, where S is a two-dimensional subspace of R^3 satisfying an equation such as $x+2y-z=0$. Can find a specific Nul A, the general solution of a system $Ax=0$.	Can find a basis for the vector space in R^3 spanned by given vectors e.g. $S=\left\{\begin{bmatrix}1\\5\\9\end{bmatrix},\begin{bmatrix}2\\6\\10\end{bmatrix},\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}4\\8\\12\end{bmatrix}\right\}$ Can find the nullspace of a specific matrix A, and a basis for Nul A or Col A.	
Process	Can picture general orth-ogonal (or orthonormal) bases. Can see certain transformations (e.g, rotation, reflection) of a basis as also providing a basis in R^3 	Can describe a basis for any vector space in R^n.	Can generalise the method for finding a basis for Col A or Nul A to describe the resulting bases. Can find a basis for any vector space in R^n	Understands that linear independence ensures that there are not too many vectors in a basis, and spanning ensures that there are not too few.
Object	Can operate on a basis, with certain transform-ations (e.g, rotation, reflection) to provide another basis for R^3.	Can see a set of vectors $\{v_1, v_2,\ldots, v_n\}$ form a basis for all of R^n because they are linearly independent and span R^n.	Can see the columns of an invertible $n\times n$ matrix forming a basis for all of R^n because they are linearly independent and span R^n.	Understanding the formal definition of a basis: A basis for a subspace H of R^n is a linearly independent set in H that spans H.

Figure 1. A section of the framework showing some aspects of the concept of basis.

Our aim was to use the framework to investigate difficulties in understanding linear algebra concepts, and to propose potential paths for preventing them. By constructing such a model for basic linear algebra concepts of scalar and vector, linear combination, linear independence/dependence, span, basis, Eigenvalue and Eigenvector we were able to consider possible teaching trajectories that would enrich students' concept images. Such a framework allows teachers and instructors to cover ideas from a spectrum of representations in the classroom in a manner that can help students build linear algebra knowledge and give them the impression that mathematics is not a long-finished subject. Furthermore, it gives researchers the opportunity to evaluate students' conceptual understanding of linear algebra and provides a foundation for observation of the way students learn.

We assessed the feasibility of this framework as a teaching and research tool for describing the level of students' conceptual and procedural understanding of the linear algebra concepts, their difficulties with these concepts, and the effects of using embodied ideas in the teaching of linear algebra using case studies of stage (year) one and two university students.

Findings from the Research Programme

The findings from the seven case studies we have conducted show that most first and second year university students have difficulties understanding basic linear algebra concepts. Further, the evidence was that the majority of students had little embodied world thinking, instead they were mainly thinking and representing their understanding of the concepts in a manner described by the action-symbolic-matrix and/or process-symbolic-matrix cells of the framework. Lacking embodied aspects of concepts students were trapped in the symbolic world, unable to move to the formal world of mathematical thinking.

However, those exposed to a teaching trajectory based on the framework appeared to construct richer conceptual structures and were better able to make connections between ideas (see Stewart, 2008; Stewart & Thomas, 2007, 2008, 2009, 2010). It seems that having a grasp of the concepts in the embodied world helped them to move more confidently toward the symbolic and the formal worlds and begin to develop more *versatile thinking* (Thomas, 2008). When we examined students' formal understanding of concepts through written definitions we found students in two separate groups. The first group comprised students who struggled to give a clear definition of the required term. Many students could not remember the definition and were confused about the concepts, so they sometimes described an action related to the concept in the symbolic-matrix world. For example, one student said "linear combination, hmm...I can't quite remember the definition, I can just remember those forms something like $b = x_1v_1 + x_2v_2$ and something like that, and x belong to $\mathbb{R}$. I only can remember these things". The second group consisted mainly of students who had experienced embodied teaching; they were happy to give a definition and their definitions usually contained the key elements of the concept. However, in interviews most students said that while solving problems they did not think about definitions. There seems to be a disjunction for many students between thinking about concepts and ideas and solving problems.

It is difficult to evaluate fully the effectiveness of this framework, given the limited data available. Nevertheless, the framework proved to be a valuable tool in analysing students' thinking, providing evidence of the level of thinking based on the specific cells or regions of the framework. For example we could use the framework to trace where students' thinking was, and where the weak points in their understanding and thinking were. Thus the lecturer could see both the areas that need improvement, and how to address them.

A question that we have not yet answered is whether it is possible to track the progression of student understanding of linear algebra over time, either in a traditional course, or using this framework. To our knowledge there has been no study examining the development through the three worlds of student understanding of concepts of advanced mathematics. Thus we do not know whether to construct rich conceptual understanding one has to start from the embodied, travel through the symbolic, and finally arrive at the formal world. Tall proposes that in an *ideal* world this is likely the case, but of course teaching and learning often does not follow this route. Most students need to symbolise the embodiment and embody the symbolism first, and only after fully integrating them they will reach the formal world. However, in the *real* world it is possible to be solely in the symbolic world of thinking, following the steps of the lecturer in class and trying to build the embodiments for oneself. In contrast, a mathematician can comfortably live in the symbolic, embodied and formal worlds, since he is able to reverse and construct embodied views, as well as going forward to the formal world. The possession of a rich schema produces versatile thinking that enables him to tie all the pieces of his knowledge in a way that the student may not be able to. Thus, the hypothesis that still requires justification is that for many students it is the embodied view that gives deep meaning to the concept allowing us move toward the formal world.

In summary we believe that research with this framework has highlighted many areas of difficulties that students have with linear algebra concepts, evidencing major problems they have in understanding concepts that form the foundation of linear algebra courses. However, including embodied ideas and using multiple representations can effect students' understanding positively. Employing a visual, embodied approach to the teaching of linear algebra concepts, rather than simply treating them symbolically or formally, may enrich students' understanding.

As lecturers we may often feel that we would like to show more pictures or describe concepts in more detail, but are constrained by lack of time and so we omit what in many cases are fundamental building blocks of the mathematical concepts. However, we suggest that the investment in time is well worth the effort since investigating concepts from a standpoint such as the theoretical framework in this study can offer a broader view of mathematics, incorporating a variety of representations that may deepen learners' conceptual understanding.

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