The entire multifractal spectrum $f(\alpha)$ was calculated from 1024 mode-locked intervals by Cvitanović, Jensen, Kadanoff, and Procaccia [CJKP 85]. The maximum of $f(\alpha)$ equals approximately 0.868 and corresponds to the Hausdorff dimension D_0 of the multifractal underlying the mode-locking staircase.

Mediants, Farey Sequences, and the Farey Tree

In order to calculate the dimensions D_q of the mode-locking fractal and its multifractal spectrum $f(\alpha)$, some order has to be imposed on the rational numbers P/Q representing different frequency ratios. One such ordering is used in the standard proof that the rational numbers (as opposed to the irrational numbers) form a countable set. Here we need a different ordering, one that better reflects the physics of mode locking.

Suppose the parameter Ω in equation 20, the bare winding number, is such that the dressed winding number falls somewhere between $\frac{1}{2}$ and $\frac{2}{3}$ without actually locking into either one. What is the most likely locked-in frequency ratio for a nonlinear coupling strength just below the value that would cause mode locking at $\frac{1}{2}$ or $\frac{2}{3}$. It seems reasonable that it should be a frequency ratio P/Q in the interval $(\frac{1}{2}, \frac{2}{3})$ with Q as small as possible.

Indeed, this is precisely what happens in dynamic systems modeled by the circle map. Adjust the nonlinear coupling strength K and the bare winding number Ω to a point just below the crossing of the two Arnold tongues for the locked frequency ratios $\frac{1}{2}$ and $\frac{2}{3}$. The dressed winding number w for this point in the Ω-K plane must be rational because $K > 1$. In fact, the rational value P/Q that w assumes is given by $\frac{1}{2} < P/Q < \frac{2}{3}$ with Q as small as possible.

This raises an interesting mathematical question with a curious but simple answer: What *is* the ratio following $\frac{1}{2}$ and $\frac{2}{3}$ with the *smallest* denominator? If you ask a kindergartner to add $\frac{1}{2}$ and $\frac{2}{3}$, he or she may well add numerators and denominators separately and write

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$$

and in so doing will have discovered the looked-for "locked-in" fraction with the smallest denominator.[5]

5. This is somewhat reminiscent of S. N. Bose (1894–1974), the celebrated Indian physicist, who, in deriving photon statistics, "forgot" to take account of the photon's (nonexistent) distinguishability. When *Nature* (not nature) turned his paper down, Bose wrote to Einstein, who saw the light and recognized Bose's "mistake" as the long-sought-after answer in the statistical physics of light. Bose's name has become enshrined ever since in the *Bose-Einstein* distribution, *bosons* (integer-spin particles, such as the photon) and *Bose condensation*, which gives us superconductivity and other macroscopic marvels of the microscopic quantum world.

What can such a strange strategy for forming intermediate fractions possibly mean? Physically, the frequency ratio 1/2 of two oscillators can be represented by a pulse (1) followed by a "nonpulse" (0) of the faster oscillator during every period of the slower oscillator. Thus, the frequency ratio 1/2 is represented by the sequence 101010 . . . or simply $\overline{10}$. Similarly, the frequency ratio 2/3 is represented by two 1s repeated with a period of three: $\overline{110}$.

Now, to form an intermediate frequency ratio, we simply *alternate* between the frequency ratios 1/2 (i.e., $\overline{10}$) and 2/3 (i.e., $\overline{110}$), yielding $\overline{10110}$, which represents the frequency ratio 3/5 (3 pulses during 5 clock times). So, in averaging frequency ratios, taking *mediants*, as this operation is called, is not such a strange thing after all.

In general, given two reduced fractions P/Q and P'/Q', the desired intermediate fraction is given by

$$\frac{P''}{Q''} = \frac{P + P'}{Q + Q'}$$

and is called the *mediant* by number theorists. In a penetrating analysis of Diophantine equations, John Horton Conway showed that numerators and denominators can be interpreted as the components of a two-dimensional *vector* and that the intermediate fraction with the lowest denominator is obtained by componentwise vector addition [unpublished, personal communication, 1989]. Thus, for example, the mediant of $\frac{5}{13}$ and $\frac{2}{5}$ equals $\frac{7}{18}$ (the revolutionary frequency ratio that Jupiter and Pallas selected for their gravitationally coupled orbits around the sun). (As it happens, there is not a single fraction between $\frac{5}{13}$ and $\frac{2}{5}$ with a denominator smaller than 18.) For this to be true, the two parent fractions must be sufficiently close. More precisely, they must be *unimodular*. The *modularity* of two reduced fractions P/Q and P'/Q', which measures their closeness for our purposes, is defined as the absolute difference $|QP' - PQ'|$, and unimodular fractions are those for which $|QP' - PQ'|$ equals 1.

The mediant of two fractions has the same modularity with its two parents as the parents have between them: modularity is another hereditary trait. Inheritance is a pivotal property, in self-similarity, including the self-similarities found in mode locking.

Mediants occur naturally in *Farey sequences*. A Farey sequence is defined as the sequence of fractions between 0 and 1 of a given largest denominator (called the *order* of the sequence). Thus, the Farey fractions of order 5 are (in increasing magnitude):

$$\frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}$$

Notice that each fraction is the mediant of its two neighbors. The modularity between all adjacent fractions equals 1, but they are not uniformly spaced.

However, Riemann's famous hypothesis, concerning the zeros of his zeta function, guarantees that the spacings between adjacent fractions are relatively uniform [Schr 90].

While Farey sequences have many useful applications and nice properties, such as classifying the rational numbers according to the magnitudes of their denominators (in fact, there are entire books listing nothing but Farey fractions), they suffer from a great irregularity: the number of additional fractions in going from Farey sequences of order $n - 1$ to those of order n equals the highly fluctuating Euler's function $\phi(n)$, defined as the number of positive integers smaller than and coprime with n. For example, $\phi(5) = 4$, $\phi(6) = 2$, and $\phi(7) = 6$. A much more regular order is infused into the rational numbers by *Farey trees,* in which the number of fractions added with each generation is simply a power of 2.

Starting with two fractions, we can construct a Farey tree by repeatedly taking the mediants of all numerically adjacent fractions. For the interval [0, 1], we start with $\frac{0}{1}$ and $\frac{1}{1}$ as the initial fractions, or "seeds". The first five generations of the Farey tree then look as follows:

$$\frac{0}{1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{1}$$

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad\qquad\qquad\qquad \frac{2}{3}$$

$$\frac{1}{4} \qquad\quad \frac{2}{5} \qquad\quad \frac{3}{5} \qquad\quad \frac{3}{4}$$

$$\frac{1}{5} \quad \frac{2}{7} \quad \frac{3}{8} \quad \frac{3}{7} \quad \frac{4}{7} \quad \frac{5}{8} \quad \frac{5}{7} \quad \frac{4}{5}$$

Each rational number between 0 and 1 occurs exactly once somewhere in the infinite Farey tree. The tree's construction reflects precisely the interpolation of locked frequency intervals in the circle map by means of mediants. The Farey tree is therefore a kind of mathematical skeleton of the Arnold tongues.

The location of each fraction within the tree can be specified by a binary address, in which 0 stands for moving to the left in going from level n to level $n + 1$ and 1 stands for moving to the right. Thus, starting at $\frac{1}{2}$, the rational number $\frac{3}{7}$ has the binary address 011. The complement of $\frac{3}{7}$ with respect to 1 (i.e., $\frac{4}{7}$) has the complementary binary address: 100. This binary code for the rational numbers is useful in describing coupled oscillators.

Note that any two *numerically* adjacent fractions of the tree are unimodular. For example, for $\frac{4}{7}$ and $\frac{1}{2}$, we get $2 \cdot 4 - 1 \cdot 7 = 1$.

Some properties of the Farey tree are particularly easy to comprehend in terms of continued fractions, which for numbers w in the interval [0, 1] look as follows:

$$w = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3} \cdots}}$$

but are more conveniently written as $w = [a_1, a_2, a_3, \ldots]$, where the a_k are positive integers. Irrational w have nonterminating continued fractions. For quadratic irrational numbers the a_k will (eventually) repeat periodically. For example, $1/\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \ldots] = [1, \overline{1, 2}]$ is preperiodic and has a period of length 2; $1/\sqrt{17} = [\overline{8}]$ has period length 1 and $1/\sqrt{61}$ has period length 11. (It is tantalizing that no simple rule is known that predicts period lengths in general.)

Interestingly, for any fraction on level n of the Farey tree, the sum over all its a_k equals n:

$$\sum_k a_k = n \qquad n = 2, 3, 4, \ldots$$

We leave it to the reader to prove this equation (by a simple combinatorial argument, for example).

There is also a direct way of calculating, from each fraction on level $n - 1$, its two neighbors or direct descendants on level n. First write the original fraction as a continued fraction in two different ways, which is always possible by splitting off a 1 from the final a_k. Thus, for example, $\frac{2}{5} = [2, 2] = [2, 1, 1]$. Then add 1 to the last term of each continued fraction; this yields $[2, 3] = \frac{3}{7}$ and $[2, 1, 2] = \frac{3}{8}$, which are indeed the two descendants of $\frac{2}{5}$.

Conversely, the close parent of any fraction (the one on the adjacent level) is found by subtracting 1 from its last term (in the form where the last term exceeds 1, because $a_k = 0$ is an illegal entry in a continued fraction). The other (distant) parent is found by simply *omitting* the last term. Thus, the two parents of $\frac{3}{7} = [2, 3]$ are the close parent $[2, 2] = \frac{2}{5}$ and the distant parent $[2] = \frac{1}{2}$. (But which parent is greater, in general—the close or the distant one? And how are mediants calculated using only continued fractions?)

Interestingly, if we zigzag down the Farey tree from its upper right ($\frac{1}{1} \to \frac{1}{2} \to \frac{2}{3} \to \frac{3}{5} \to \frac{5}{8}$, and so on), we land on fractions whose numerators and denominators are given by the Fibonacci numbers F_n, defined by $F_n = F_{n-1} + F_{n-2}$; $F_0 = 0$, $F_1 = 1$. In fact, on the nth zig or zag, starting at $\frac{1}{1}$, we reach the fraction F_{n+1}/F_{n+2}, which approaches the golden mean $\gamma = (\sqrt{5} - 1)/2 = 0.618\ldots$ as $n \to \infty$ [Schr 90]. (Starting with $\frac{0}{1}$ we land on the fractions F_n/F_{n+2}, which converge on $\gamma^2 = 1 - \gamma$.) The binary address of γ in the Farey tree is 101010 . . . .

The continued fraction expansions of these ratios F_n/F_{n+1} have a particularly simple form. For example,

$$\frac{F_3}{F_4} = \frac{2}{3} = [1, 1, 1]$$

and in general

$$\frac{F_n}{F_{n+1}} = [1, 1, \ldots, 1] \qquad \text{(with } n \text{ 1s)}$$

Obviously, continued fractions with small a_k converge relatively slowly to their final values, and continued fractions with only 1s are the slowest converging of all. Since

$$\gamma = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = [1, 1, 1, \ldots] = [\overline{1}]$$

where the bar over the 1 indicates infinitely many 1s, the golden mean γ has the most slowly converging continued fraction expansion of all irrational numbers. The golden mean γ is therefore sometimes called (by physicists and their ilk) "the most irrational of all irrational numbers"—a property of γ with momentous consequences in a wide selection of problems in nonlinear physics, from the double swing to the three-body problem.

Roughly speaking, if the frequency ratio of two coupled oscillators is a rational number P/Q, then the coupling between the driving force and the "slaved" oscillator is particularly effective because of a kind of a resonance: every Q cycles of the driver, the same physical situation prevails so that energy transfer effects have a chance to build up in resonancelike manner. This resonance effect is strong, of course, particularly if Q is a *small* integer. This is precisely what happened with our moon: resonant energy transfer between the moon and the earth by tidal forces slowed the moon's spinning motion until the spin period around its own axis locked into the 28-day cycle of its revolution around the earth. As a consequence the moon always shows us the same face, although it wiggles ("librates") a little.

Similarly, the frequency of Mercury's spin has locked into its orbital frequency at the rational number $\frac{3}{2}$. As a consequence, one day on Mercury lasts two Mercury *years*. (And one day—in the distant future, one hopes—something strange like that may happen to Mother Earth!)

The rings of Saturn, or rather the gaps between them, are another consequence of this resonance mechanism. The orbital periods of any material (flocks of ice and rocks) in these gaps would be in a rational resonance with some periodic force (such as the gravitational pull from one of Saturn's "shepherding"

moons). As a consequence, even relatively weak forces have a cumulative significant effect over long time intervals, accelerating any material out of th gaps.

For rational frequency ratios with large denominators Q, such a resonan effect would, of course, be relatively weak, and for *irrational* frequency ratio resonance would be weaker still or absent.

For strong enough coupling, however, even irrational frequency ratios migh be affected. But there is always one irrational frequency ratio that would be lea disturbed: the golden mean, because, in a rational approximation to within certain accuracy, it requires the largest denominators Q. This property is als reflected in the Farey tree: on each level n the two fractions with the large denominators are the ones that equal F_{n-1}/F_{n+1} and F_n/F_{n+1}, which for $n \to$ o approach $\gamma^2 = 0.382\ldots$ and $\gamma = 0.618\ldots$, respectively. (Conversely, the frac tions with the smallest Q on a given level of the Farey tree are from the harmoni series $1/Q$ and $1 - 1/Q$.)

Another way to demonstrate the unique position of the golden mean amon all the irrational numbers is based on the theory of rational approximation, a important part of number theory. For a good rational approximation, one expand an irrational number w into a continued fraction and terminates it after n term to yield a rational number $[a_1, a_2, \ldots, a_n] = p_n/q_n$. This rational approximatio to w is in fact the best for a given maximum denominator q_n. For example, fc $w = 1/\pi = [3, 7, 15, 1, 293, \ldots]$ and $n = 2$, we get $p_n/q_n = 7/22$, and there i no closer approximation to $1/\pi$ with a denominator smaller than 22.

Now, even with such an optimal approximation as afforded by continue fractions, the differences for the golden mean γ

$$\left|\gamma - \frac{p_n}{q_n}\right|$$

exceed c/q_n^2 (where c is a constant that is smaller than but arbitrarily close t $1/\sqrt{5}$) for *all* values of n above some n_0. And this is true only for the golde mean γ and the "noble numbers" (defined as irrational numbers whose continue fractions end in all 1s). Thus, in this precise sense, the golden mean (and th noble numbers) keep a greater distance from the rational numbers than does an other irrational number. Small wonder that the golden mean plays such a important role in synchronization problems.

The golden mean is also visible in visual perception (see Figure 3). For a computer-generated image of a "sunflower" using the *golden angle* $\Delta\phi = 360^\circ\,\gamma \approx 225.5^\circ$ as the angular increment in the placement (r_n, ϕ_n) of successive seeds, where

$$(r_n, \phi_n) = (c \cdot r_{n-1}, \phi_{n-1} + \Delta\phi)$$

we get a realistic image of the sun flower's seed pattern, which uses the golder

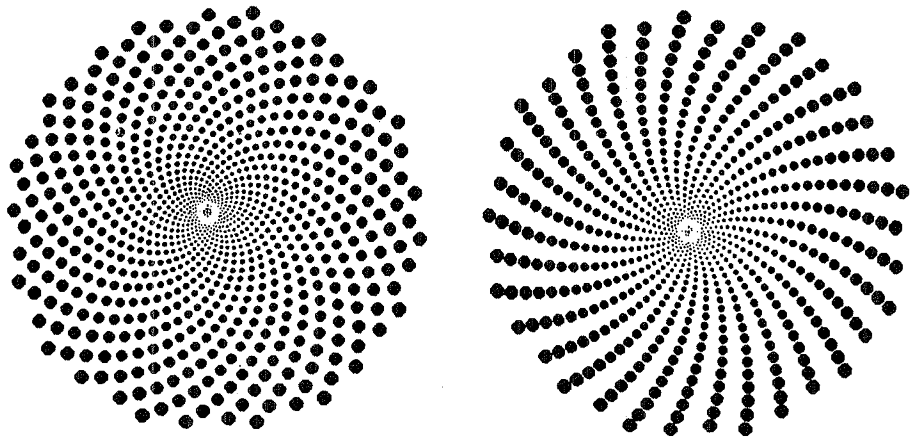

Figure 3 The golden angle in visual perception. (Courtesy T. Gramss, after [RS 87])

angle in its construction (the left part of Figure 3) [RS 87]. But for angular increments $\Delta\phi$ that differ by just 0.04 percent from the golden angle (222.4°), the human eye perceives pronounced spirals (the right part of Figure 3)—a psychovisual mode-locking phenomenon!

The Golden-Mean Route to Chaos

For the critical circle map

$$\theta_{n+1} = \theta_n + \Omega - \frac{1}{2\pi} \sin(2\pi\theta_n) \tag{23}$$

the sequence of the locked-in frequency ratios P/Q equal to the ratio of successive Fibonacci numbers $F_{n-1}/F_n = [1, 1, \ldots, 1]$ is in many respects the most interesting route to aperiodic behavior and deterministic chaos of the variable θ_n. In the transition to chaotic motion, these frequency ratios and equivalent ones, such as $F_{n-2}/F_n = [2, 1, 1, \ldots, 1]$, are usually the last to remain unaffected as the degree of nonlinear coupling is increased. *Chaotic* means, as always, that initially close values of θ will diverge exponentially so that all predictability is lost as the system evolves in time.

In the Farey-tree organization of the rational numbers, introduced in the previous section, the ratios F_{n-1}/F_n or F_{n-2}/F_n lie on a zigzag path approaching

the golden mean γ or its square, $1 - \gamma = \gamma^2$, respectively. Each fraction is the mediant of its two predecessors. For example, the sequence F_{n-2}/F_n, beginning with $\frac{0}{1}$ and $\frac{1}{2}$, equals $\frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \ldots$. The corresponding continued fractions, beginning with $\frac{1}{2}$, keep adding 1s: [2], [2, 1], [2, 1, 1], [2, 1, 1, 1], [2, 1, 1, 1, 1], and so on to $[2, \bar{1}] = \gamma^2$.

The parameter value Ω_n that gives a dressed winding number equal to the frequency ratio F_{n-2}/F_n has to be determined numerically. A simple calculator program that adjusts Ω so that, for $\theta_0 = 0$, $\theta_{F_n} = F_{n-2}$ yields the following approximate parameter values:

$$\begin{aligned} \Omega(\tfrac{1}{2}) &= 0.5 \\ \Omega(\tfrac{1}{3}) &\approx 0.3516697 \\ \Omega(\tfrac{2}{5}) &\approx 0.4074762 \\ \Omega(\tfrac{3}{8}) &\approx 0.3882635 \\ \Omega(\tfrac{5}{13}) &\approx 0.3951174 \\ \Omega(\tfrac{8}{21}) &\approx 0.3927092 \\ \Omega(\tfrac{13}{34}) &\approx 0.3935608 \end{aligned}$$

and so on, converging to $\Omega_\infty \approx 0.3933377$.

These parameter values give rise to superstable orbits because the iterates θ_n include the value $\theta_n = 0$ for which the derivative of the critical circle map vanishes. These Ω values therefore correspond to the superstable values R_n of the quadratic map, and Ω_∞ corresponds to R_∞.

Is there a universal constant, corresponding to the Feigenbaum constant, which describes the rate of convergence of the parameter values $\Omega_n := \Omega(F_{n-2}/F_n)$ to Ω_∞ as n goes to infinity? Numerical evidence suggests that there is, and that the differences between successive values of Ω_n scale with an asymptotic factor:

$$\frac{\Omega_{n-1} - \Omega_n}{\Omega_n - \Omega_{n+1}} \to \delta$$

with $\delta = -2.8336\ldots$, which thus corresponds to the Feigenbaum constant $4.6692\ldots$. (The minus sign signifies that successive differences alternate in sign.)

Other self-similar scaling behaviors can be observed in the iterates of the variable θ_n. For example, for $\Omega = \Omega(F_{n-2}/F_n)$ the differences $\theta_{F_{n-1}} - F_{n-3}$ converge to 0 in an asymptotically geometric progression:

$$\theta_{F_{n-1}} - F_{n-3} \approx \alpha^n$$

with $\alpha = -1.288575\ldots$, which corresponds to the scaling parameter $-2.5029\ldots$ for the iterated variable of the quadratic map.