e. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

2. Describe the dynamics of the linear maps whose matrix representation is given below. Identify precisely the stable and unstable sets.

a. $\begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

c. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

d. $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

e. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

3. Describe the dynamics of each of the following linear maps, indicating which are non-hyperbolic.

a. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

d. $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

e. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{pmatrix}$

4. Consider the linear map

$$L(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x}.$$

Prove that $L^n\mathbf{x} \to \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^2$. Prove that, if $\mathbf{x}$ does not lie on the y-axis, then the orbit of $\mathbf{x}$ tends to $\mathbf{0}$ tangentially to the x-axis.

5. A function $F: \mathbf{R}^n \to \mathbf{R}$ is called an integral for a linear map L if $F \circ L(\mathbf{x}) = F(\mathbf{x})$, i.e., F is constant along orbits of L. Show that

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$

is an integral for

$$L(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

6. Construct (non-trivial) integrals for each of the following linear maps.

a. $L(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$.

b. $L(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x}$.

§2.3 THE HORSESHOE MAP

Symbolic dynamics, which played such a crucial role in our understanding of the one-dimensional quadratic map, can also be used to study higher dimensional phenomena. In this section, we will study a now-classical example due to Smale, the horseshoe map. This was the first example of a diffeomorphism which had infinitely many periodic points and yet was structurally stable. We will see that this map has much in common with the quadratic map which motivated so much of the material in Chapter One.

To define the map, we first consider a region D consisting of three components: a central square S with side length 1 and two semicircles D_1 and D_2 at either end. See Fig. 3.1. D is shaped like a "stadium."

The horseshoe map F takes D inside itself according to the following prescription. First, linearly contract S in the vertical direction by a factor $\delta < 1/2$ and expand it in the horizontal direction by a factor $1/\delta$ so that S is long and thin. Then put S back inside D in a horseshoe-shaped figure as in Fig. 3.2.

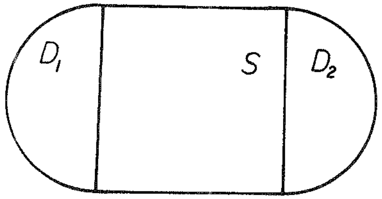

Fig. 3.1. The "stadium" D.

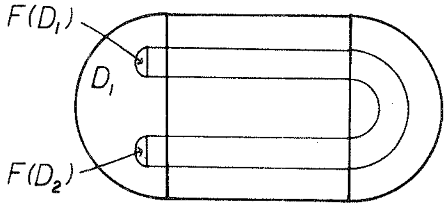

Fig. 3.2. The Smale horseshoe map.

The semicircular regions D_1 and D_2 are contracted and mapped inside D_1 as depicted. We remark that $F(D) \subset D$ and that F is one-to-one. However, since F is not onto, F^{-1} is not globally defined. The remainder of this section is devoted to the study of the dynamics of F in D.

Note first that the preimage of S consists of two vertical rectangles V_0 and V_1 which we may assume are mapped linearly onto the two horizontal components H_0 and H_1 of $F(S) \cap S$. The width of V_0 and V_1 is δ, as is the height of H_0 and H_1. See Fig. 3.3.

By linearity of $F : V_0 \to H_0$ and $F : V_1 \to H_1$, it follows that F preserves horizontal and vertical lines in S. For later use, we note that if h is a horizontal line segment in S whose image also lies in S, then the length of $F(h)$ is $1/\delta$ times the length of h. Similarly, if both v and $F(v)$ are vertical line segments in S, then the length of $F(v)$ is shrunk by a factor of δ.

We claim that the dynamics of F are very similar to those of the quadratic map studied in §1.5. Note first that, since F is a contraction on D_1, F has a unique fixed point p in D_1 and $\lim_{n\to\infty} F^n(q) = p$ for all $q \in D_1$. This follows immediately from the Contraction Mapping Theorem. Since $F(D_2) \subset D_1$, all forward orbits in D_2 behave likewise. Similarly, if $q \in S$ but $F^k(q) \notin S$ for some $k > 0$, then we must have that $F^k(q) \in D_1 \cup D_2$ so that $F^n(q) \to p$

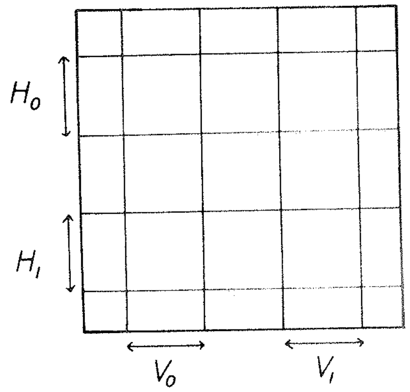

Fig. 3.3.

as $n \to \infty$. Consequently, to understand the forward orbits of F, it suffices to consider the set of points whose forward orbits lie for all time in S. We will do more: we will describe

$$\Lambda = \{q \in S | F^k(q) \in S \text{ for all } k \in \mathbb{Z}\}.$$

Now, if the forward orbit of q lies in S, we must have, first of all, that $q \in V_0$ or $q \in V_1$, for all other points in S are mapped out of S and into $D_1 \cup D_2$. If $F^2(q) \in S$, then, similarly, we must have $F(q) \in V_0 \cup V_1$, i.e., $q \in F^{-1}(V_0) \cup F^{-1}(V_1)$. Here $F^{-1}(V_0)$ means the inverse image of V_0 in S. Clearly, there are substrips in both V_0 and V_1 which map into V_0 as depicted in Fig. 3.4.

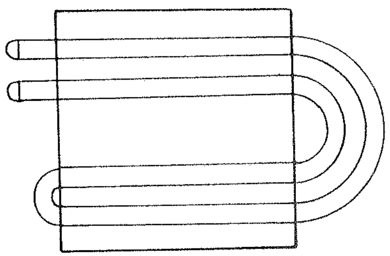

Fig. 3.4. The forward image $F^2(S)$.

This is the inductive step: if V is any vertical rectangle connecting the upper and lower boundaries of S with width w, then $F^{-1}(V)$ is a pair of smaller vertical rectangles of width δw, one in each V_i. Consequently, $F^{-1}(F^{-1}(V_i)) = F^{-2}(V_i)$ consists of four vertical rectangles, each of width δ^2, $F^{-3}(V_i)$ consists of eight vertical rectangles of width δ^3, etc. Hence the same procedure we used in §1.5 shows that

$$\Lambda_+ = \{q | F^k(q) \in S \text{ for } k = 0, 1, 2, \dots\}$$

is the product of a Cantor set with a vertical interval. Arguing entirely analogously, it is easy to check that

$$\Lambda_- = \{q | F^{-k}(q) \in S \text{ for } k = 1, 2, 3, \dots\}$$

consists of a product of a Cantor set with an interval. In this case, the intervals are horizontal. Finally,

$$\Lambda = \Lambda_+ \cap \Lambda_-$$

is the intersection of these two sets.

To introduce symbolic dynamics into the system, we first choose any vertical interval ℓ in Λ_+. Note that $F^k(\ell)$, is a vertical line segment of length δ^k in either V_0 or V_1. Hence we may attach an infinite sequence $s_0s_1s_2\dots$ of 0's or 1's to any point in ℓ according to the rule $s_j = \alpha$ iff $F^j(\ell) \subset V_\alpha$. The number s_0 tells us in which vertical strip the line ℓ is located, s_1 tells where its image is located, etc. We can similarly attach a sequence of integers to any horizontal line segment h. For convenience, we write this sequence $\dots s_{-3}s_{-2}s_{-1}$, where $s_{-j} = \alpha$ iff

$$F^{-j}(h) \subset V_\alpha \quad \text{for } j = 1, 2, 3, \dots.$$

Note again that $F^{-1}(h)$, $F^{-2}(h), \dots$ are horizontal line segments of decreasing lengths.

Consequently, if p is any point in $\Lambda_+ \cap \Lambda_-$, we may associate a pair of sequences of 0's and 1's to p. One sequence gives the itinerary of the forward orbit of p; the other describes the backward orbit. Let us amalgamate both of these sequences into one, doubly-infinite sequence of 0's and 1's. That is, we define the itinerary $S(p)$ by the rule

$$S(p) = (\dots s_{-2}s_{-1} \cdot s_0s_1s_2\dots)$$

where $s_j = k$ if and only if $F^j(p) \in V_k$.

This then gives the symbolic dynamics on Λ. Let Σ_2 denote the set of all doubly-infinite sequences of 0's and 1's:

$$\Sigma_2 = \{(s) = (\dots s_{-2}s_{-1} \cdot s_0s_1s_2\dots) | s_j = 0 \text{ or } 1\}.$$

Impose a metric on Σ_2 by defining $d[(s),(t)] = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$ exactly as before. Define the shift map σ by

$$\sigma(\dots s_{-2}s_{-1} \cdot s_0s_1s_2\dots) = (\dots s_{-2}s_{-1}s_0 \cdot s_1s_2\dots).$$

That is, σ simply shifts each sequence in Σ_2 one unit to the left (equivalently, σ shifts the decimal point one unit to the right). Unlike our previous shift map, this map has an inverse. Clearly, shifting one unit to the right gives this inverse. It is easy to check that σ is a homeomorphism on Σ_2 (see Exercise 2).

The shift map is now the model for the restriction of F to Λ. Indeed, the map S gives a topological conjugacy between F on Λ and σ on Σ_2. We leave the details of this proof to the reader (see Exercise 3).

All of the properties which held for the old one-sided shift hold for σ as well. For example, there are precisely 2^N periodic points of period N for σ. There is a dense orbit for σ as well (see Exercises 4, 5). But there are new phenomena present as well.

Definition 3.1. Two points p_1 and p_2 are forward (respectively backward) asymptotic if $F^n(p_1), F^n(p_2) \in D$ for all $n \geq 0$ (resp. $n \leq 0$) and

$$\lim_{n\to\infty} |F^n(p_1) - F^n(p_2)| = 0$$

(resp. $n \to -\infty$).

Intuitively, two points in D are forward asymptotic if their orbits approach each other as $n \to \infty$. Note that any point which leaves S under forward iteration of F is forward asymptotic to the fixed point $p \in D_1$. Also, if p_1 and p_2 lie on the same vertical line in Λ_+, then p_1 and p_2 are forward asymptotic. If p_1 and p_2 lie on the same horizontal line in Λ_-, then they are backward asymptotic.

As in the linear theory, the notion of forward and backward asymptotic orbits allows us to define the stable and unstable sets of a point.

Definition 3.2. The stable set of p is given by

$$W^s(p) = \{z | |F^n(z) - F^n(p)| \to 0 \text{ as } n \to \infty\}.$$

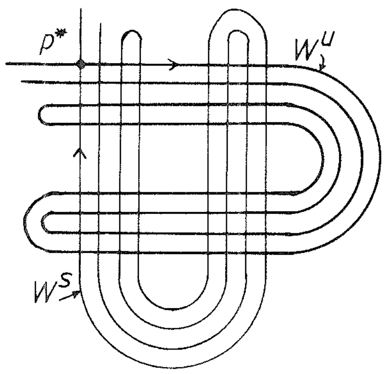

Fig. 3.5. The stable and unstable sets associated to p^*.

The unstable set of p is given by

$$W^u(p) = \{z \mid |F^{-n}(p) - F^{-n}(z)| \to 0 \text{ as } n \to \infty\}.$$

Equivalently, a point z lies in $W^s(p)$ if p and z are forward asymptotic. For example, any point in S which leaves S under forward iteration of the horseshoe map lies in the stable set of the fixed point in D_1.

The stable and unstable sets of points in Λ are more complicated. For example, consider the fixed point p^* which lies in V_0 and therefore has the sequence $(\ldots 00.000\ldots)$ attached. Any point which lies on the vertical segment ℓ_s through p^* lies in $W^s(p^*)$. But there are many other points in this stable set. Suppose the point q eventually maps into ℓ_s. Then there is an integer n such that $|F^n(q) - p^*| < 1$. Hence

$$|F^{n+k}(q) - p^*| < \delta^k$$

and it follows that $q \in W^s(p^*)$. Thus, the union of vertical intervals given by $F^{-k}(\ell_s)$ for $k = 1, 2, 3, \ldots$ all lie in $W^s(p^*)$. The reader may easily check that there are 2^k such intervals. See Fig. 3.5.

Since $F(D) \subset D$, the unstable manifold of p^* assumes a somewhat different form. The horizontal line segment ℓ_u through p^* in D clearly lies in $W^u(p^*)$. As above, all of the forward images of ℓ_u also lie in D. The reader may easily check that $F^k(\ell_u)$ is a "snake-like" curve in D which cuts across S exactly 2^k times in a horizontal segment. See Fig. 3.5.

These stable and unstable sets are easy to describe on the shift level. Let

$$\mathbf{s}^* = (\ldots s^*_{-2}s^*_{-1} \cdot s^*_0 s^*_1 s^*_2 \ldots) \in \Sigma_2.$$

Clearly, if $\mathbf{t}$ is a sequence whose entries agree with those of $\mathbf{s}^*$ to the right of some entry, then $\mathbf{t} \in W^s(\mathbf{s}^*)$. The converse of this is also true, as is shown in Exercise 6.

A natural question that arises is our use of the term "Cantor set" to describe the set $\Lambda = \Lambda_+ \cap \Lambda_-$ for the horseshoe map and the similar set Λ for the quadratic map of Chapter One. Intuitively, it may appear that the Λ for the horseshoe has "twice" as many points. However, both Λ's are actually homeomorphic! This is best seen on the shift level.

Let Σ_2^1 denote the set of one-sided sequences of 0's and 1's and Σ_2 the set of two-sided such sequences. Define a map

$$\Phi: \Sigma_2^1 \to \Sigma_2$$

by $\Phi(s_0 s_1 s_2 \ldots) = (\ldots s_5 s_3 s_1 \cdot s_0 s_2 s_4 \ldots)$. It is easy to check that Φ is a homeomorphism between Σ_2^1 and Σ_2 (see Exercise 11).

Remarks.

1. We have now seen stable and unstable sets in two guises: the stable and unstable subspaces of linear maps and the above collection of horizontal and vertical line segments. This will become a common pattern for higher dimensional systems that are "hyperbolic" in a sense to be made precise later. Each point in a hyperbolic set will come equipped with contracting and expanding directions which will play the role of stable and unstable sets.

2. Unlike the quadratic map, the horseshoe example was defined geometrically rather than algebraically. This is often the case with higher dimensional maps: it is easier to present and work with examples defined geometrically. It is important to realize that it is possible to write down an explicit algebraic expression which gives a map similar to the horseshoe. This map is the Hénon map which we will discuss later in §2.9.

Exercises

1. Prove that $d[(\mathbf{s}),(\mathbf{t})] = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}$ is a metric on Σ_2.
2. Prove that the shift σ is a homeomorphism.
3. Prove that $S: \Lambda \to \Sigma_2$ gives a topological conjugacy between σ and F.
4. Construct a dense orbit for σ.

5. Prove that periodic points are dense for σ.

6. Let $\mathbf{s}^* \in \Sigma_2$. Prove that $W^s(\mathbf{s}^*)$ consists of precisely those sequences whose entries agree with those of $\mathbf{s}^*$ to the right of some entry of $\mathbf{s}^*$.

7. Let $(\mathbf{0}) = (\dots 00.000\dots) \in \Sigma_2$. A sequence $\mathbf{s} \in \Sigma_2$ is called homoclinic to $(\mathbf{0})$ if $\mathbf{s} \in W^s(\mathbf{0}) \cap W^u(\mathbf{0})$. Describe the entries of a sequence which is *homoclinic* to $(\mathbf{0})$. Prove that sequences which are homoclinic to $(\mathbf{0})$ are dense in Σ_2.

8. Let $(\mathbf{1}) = (\dots 11.111\dots) \in \Sigma_2$. A sequence $\mathbf{s}$ is a ***heteroclinic*** sequence if $\mathbf{s} \in W^s(\mathbf{0}) \cap W^u(\mathbf{1})$. Describe the entries of such a heteroclinic sequence. Prove that such sequences are dense in Σ_2.

9. Generalize the definitions of homoclinic and heteroclinic points to arbitrary periodic points for σ and reprove Exercises 7 and 8 in this case.

10. Prove that the set of homoclinic points to a given periodic point is countable.

11. Let Σ_2^1 denote the set of one-sided sequences of 0's and 1's. Define $\Phi\colon \Sigma_2^1 \to \Sigma_2$ by

$$\Phi(s_0 s_1 s_2 \dots) = (\dots s_5 s_3 s_1 \cdot s_0 s_2 s_4 \dots).$$

Prove that Φ is a homeomorphism.

12. Consider the map F on D defined geometrically as in Fig. 3.6. Assume that F linearly contracts vertical lengths and linearly expands horizontal lengths in S exactly as in the case of the Smale horseshoe. Let

$$\Lambda = \{p \in D \mid F^n(p) \in S \text{ for all } n \in \mathbf{Z}\}.$$

Use the techniques of §1.13 to show that F on Λ is topologically conjugate to a two-sided subshift of finite type generated by a 3×3 matrix A. Identify A. Discuss the dynamics of F off Λ.

13. Rework Exercise 12, this time with the map defined geometrically in Fig. 3.7.

14. Let $R\colon \Sigma_2 \to \Sigma_2$ be defined by

$$R(\dots s_{-2} s_{-1} . s_0 s_1 s_2 \dots) = (\dots s_2 s_1 s_0 . s_{-1} s_{-2} \dots).$$

Prove that $R \circ R = id$ and that $\sigma \circ R = R \circ \sigma^{-1}$. Conclude that $\sigma = U \circ R$ where U is a map which satisfies $U \circ U = id$. Maps which are their own inverse are called ***involutions***. They represent very simple types of dynamical systems. Hence the shift may be decomposed into a composition of two such maps.

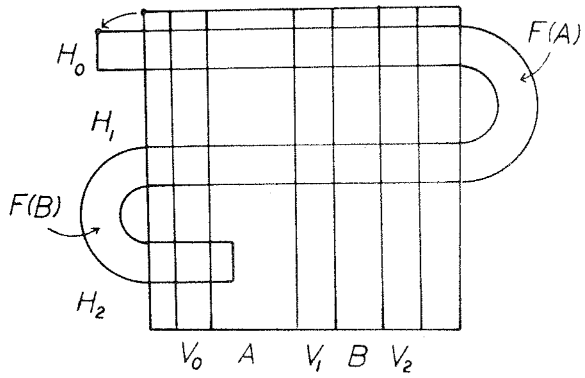

Fig. 3.6.

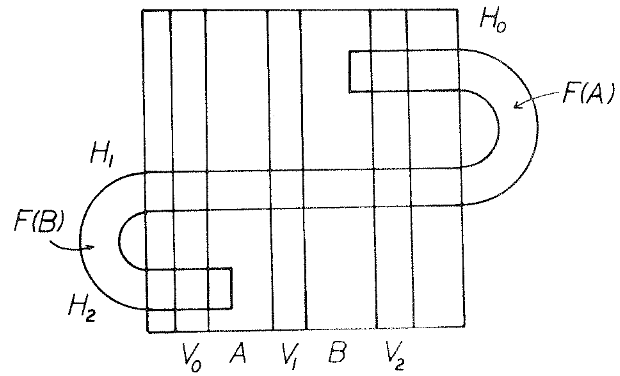

Fig. 3.7.

15. Let $\mathbf{s}$ be a sequence which is fixed by R. Suppose that $\sigma^n(\mathbf{s})$ is also fixed by R. Prove that $\mathbf{s}$ is a periodic point of σ of period $2n$.

16. Rework the previous exercise, assuming that $\sigma^n(\mathbf{s})$ is fixed by U, where U is given as in Exercise 13. What is the period of $\mathbf{s}$?

§2.4 HYPERBOLIC TORAL AUTOMORPHISMS

In this section, we introduce a completely different class of dynamical system, the Anosov systems or hyperbolic toral automorphisms. These maps are important in that they are chaotic everywhere that they are defined. Nevertheless, their dynamics can be described completely. One difference between these maps and those discussed previously is that these maps are naturally defined on a torus or "doughnut" rather than on Euclidean space. Even though the maps are induced by linear maps on Euclidean space (which have extremely simple dynamics), the maps on the tori have extremely rich dynamical structure.

To describe the torus, let us begin with the plane. We will consider as identical all points whose coordinates differ by integers. That is to say, the point (α,β) in the plane is to be regarded as the same as the points $(\alpha+1,\beta)$, $(\alpha+5,\beta+3)$, and, in general, $(\alpha+M,\beta+N)$, where M and N are integers. We let $[\alpha,\beta]$ denote the set of all points equivalent to (α,β) under this relation. To be somewhat more formal, the relation $(x,y)\sim(x',y')$ if and only if $x-x'$ and $y-y'$ are integers gives an equivalence relation on points in the plane. The torus is thus the set of all equivalence classes under this relation.

Geometrically, this procedure can be visualized as follows. Consider the unit square in the plane $0\le x,y\le 1$. Under the above identifications, only points on the boundary of the square need be considered. Indeed, the top boundary $y=1$ should be considered the same as the bottom boundary $y=0$, and similarly the left and right boundaries $x=0$ and $x=1$ should be identified. When this occurs, the square becomes first a cylinder and then a torus, as in Fig. 4.1.

Remarks

1. This procedure is not limited to two dimensions; one may define an n-dimensional torus using the same equivalence relation on $\mathbf{R}^n$. This is shown in Exercise 2.

2. The torus may also be regarded as the Cartesian product of two circles. See Exercise 3.

Let T denote the torus, and let π be the natural projection of $\mathbf{R}^2$ onto

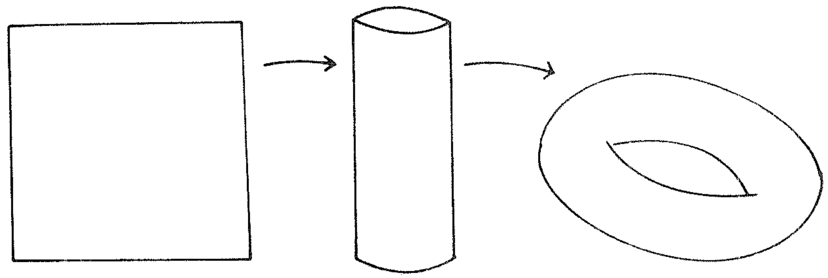

Fig. 4.1. Construction of a torus from a square.

T, i.e.,

$$\pi(x,y)=[x,y]=\pi(x+M,\ y+N).$$

Certain dynamical systems on a torus can be described most efficiently in the plane and then projected onto the torus. For example, suppose $F:\mathbf{R}^2\to\mathbf{R}^2$ has the property that

$$F\begin{pmatrix}x\\y\end{pmatrix}-F\begin{pmatrix}x+M\\y+N\end{pmatrix}$$

belongs to the integer lattice for all points in the plane and all integers M and N. It follows that

$$\pi\circ F\begin{pmatrix}x\\y\end{pmatrix}=\pi\circ F\begin{pmatrix}x+M\\y+N\end{pmatrix}$$

so that F induces a well-defined map $\hat{F}$ on the torus. $\hat{F}$ is defined by the diagram

$$\begin{array}{ccc}\mathbf{R}^2 & \xrightarrow{F} & \mathbf{R}^2\\ \downarrow\pi & & \downarrow\pi\\ T & \xrightarrow{\hat{F}} & T.\end{array}$$

As an example, if L is a linear map whose matrix representation is an integer matrix, then $\hat{L}$ is clearly well-defined on T. $\hat{L}$ is called a *toral automorphism*. For our purposes, we need a few more hypotheses on L.

Definition 4.1. Let $L(\mathbf{x})=A\cdot\mathbf{x}$ where A is a 2×2 matrix satisfying

1. All entries of A are integers.