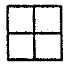
# Fractal Dimension

For the line and the plane, we obviously have $N(\varepsilon)$ the number of neighbourhoods of diameter $\varepsilon$ required to cover the region grows by $\varepsilon^{-D}$ where D is the dimension of the space:

—+— $D=1$. $2^1$ units of diameter 1/2  $D=2$. $2^2$ units of diameter 1/2

If the number $N(\varepsilon)$ of neighbourhoods of diameter $\varepsilon$ required to cover a given space grows with $\varepsilon^{-D}$ then we say the space has fractal (or Hausdorff) dimension D.

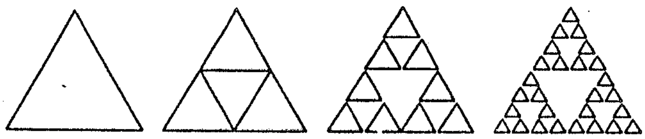
If we have a sequence of spaces $A_n$ converging to a fractal attractor A by a transformation which preserves A then if the transformation replaces a given region with k times as many regions of 1/r the size then we have:

$N(\varepsilon) = p.\varepsilon^{-D}$ and so $k\,N(\varepsilon) = N(\varepsilon/r) = p.(\varepsilon/r)^{-D}$

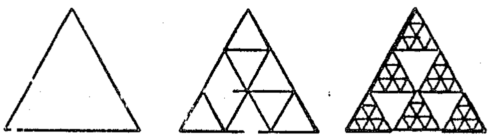
so dividing $k = (1/r)^{-D}$ or $\log k = -D\log(1/r) = D\log r$ and $D = \frac{\log k}{\log r}$.

Example 1: Cantor ternary set. $A_1 = [0,1]$ $A_2 = [0,1/3] \cup [2/3,1]$ etc. $k = 2$ $r = 3$ $D = \frac{\log 2}{\log 3} = 0.631$.

Example 2: Sierpinski gasket. 3 sides of length 1 become 9 sides of length 1/2 $D = \frac{\log 3}{\log 2} = 1.584$.



Example 3: 3 sides of length 1 become 18 sides of length 1/3 $D = \frac{\log 6}{\log 3} = 1.631$.



If we have a non-overlapping or just touching IFS with contractivity factors $s_1, \ldots s_n$

then we have the relation $\sum_{n=1}^{N} s_i^D = 1$.

This generalizes the above result for the Sierpinski gasket to allow for distinct contractivity factors.

For example the Sierpinski gasket with 3 maps of contractivity 1/2 we have $3.(1/2)^D = 1$ i.e. $(1/2)^D = 1/3$

i.e. $D.-\log(2) = -\log(3)$, $D = \frac{\log 3}{\log 2}$.

# SCIENCE

## Mandelbrot set is as complex as it could be

William Bown

IT'S OFFICIAL: the Mandelbrot set is a fractal. A Japanese mathematician has finally proved something that everyone with a picture of chaos on their wall thought they already knew.

In doing so, Mitsuhiro Shishikura of the Tokyo Institute of Technology has proved one of the longest-standing conjectures in fractal mathematics—that the dimension of the boundary of the Mandelbrot set is 2.

This means that the set's boundary is as complicated as it is possible to be. It has the same dimension as a two-dimensional area, despite being a curve.

The key to any fractal is its dimension—in the case of a curve, this is a measure of how "wiggly" it is. The boundaries of circles and triangles have a dimension of 1 because they are not wiggly, whereas the region enclosed by either of these shapes has a dimension of 2.

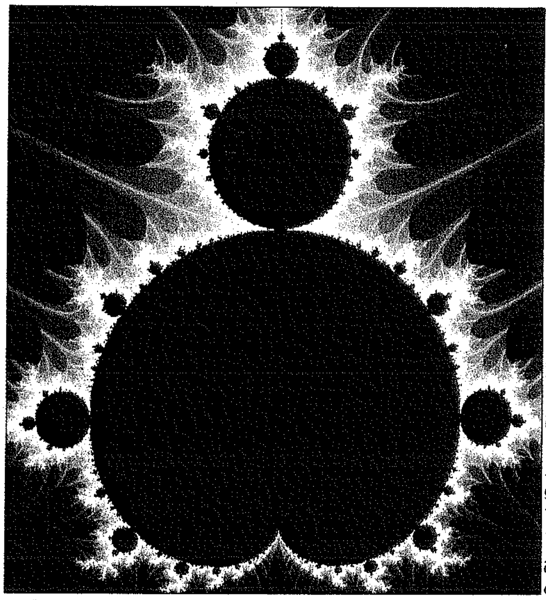
But fractal boundaries have fractional dimensions. Some are so convoluted that they have a dimension between 1 and 2; others are so un-wiggly that their dimension is as little as zero. Mathematicians have long wondered just where the Mandelbrot set fitted in.

The Mandelbrot set is highly complex. It is self-similar—that is, the set contains mini-Mandelbrot sets, each with the same shape as the whole. Indeed, the set is self-similar on all scales: if you examine bits of it, no matter how small, you will always see a complete facsimile of the whole. Another peculiarity of the Mandelbrot set is that even the most far-flung bits of it are connected to the main body through fine tendrils.

Shishikura has calculated the so-called Hausdorff dimension of the Mandelbrot set's boundary for the first time. This quantifies the complexity of the set, and is conclusive evidence of the edge's fractal nature. "The detail of the Mandelbrot set is more complicated than the whole," says Shishikura. "It seemed obvious that the dimension of the boundary must be greater than 1, but we did not know it was 2."

The proof relies on a sequence of carefully constructed fractals called Julia sets. These are closely related to the Mandelbrot set. A Julia set is created by taking a point on a piece of paper, transforming the coordinates of the point to another point by means of a simple formula, and repeating the process over and over again. All the points taken together make the Julia set.

Pictures of the Mandelbrot set are usually generated dot by dot. As with the Julia set, the coordinates of each point are transformed ver and over by a simple formula. But this e the objective is not to trace the sequence ints but to see how big the coordinates some cases the coordinates remain t in others they eventually become infinite. The point you started with is defined as part of the Mandelbrot set if the coordinates remain finite.



S. Shostack/Science Photo Library

*The boundary of the Mandelbrot set has a dimension of 2, making it as wiggly as it is possible to be*

Because the Mandelbrot set and Julia sets are generated in such different ways, they are very different. But they are connected by a deep theorem.

In fact, the Mandelbrot set can be generated by checking every Julia set. All Julia sets which are connected—that is, they are not broken into separate pieces—correspond to points in the Mandelbrot set.

Shishikura's proof relies on this connection. He shows first that the dimension of the Mandelbrot set is at least as big as the dimension of Julia sets in his sequence. Then he shows that the dimension of the sets in the sequence gets larger until eventually it reaches 2. Hence the dimension of the Mandelbrot set is 2.

The difficult bit is figuring out exactly what the sequence of Julia sets is. In fact, Shishikura does not work out the sequence, but just demonstrates that it would be possible to construct it. Nevertheless, the method he used has attracted widespread acclaim.

According to Curt McMullen, a leading researcher into fractals at the University of California, Berkeley, Shishikura's method of attack has enabled mathematicians to get deeper into the detail of the complex dynamics associated with the Mandelbrot set. "The importance of this is not the result itself, but the new techniques introduced," he says.

Shishikura's success leaves two outstanding question marks over the Mandelbrot set.

Firstly, is it possible to draw the Mandelbrot set with perfect accuracy? Mathematicians now doubt this will ever be possible. For this they need a single equation. Although mathematicians generate the set dot by dot, this is not the same as having a single equation which defines every point.

Secondly, is the boundary a space-filling curve? Some other sets with dimension 2 also have a measurable area, while some—like a sponge which is all holes—do not. □

## Stretched bonds fuel chemical controversy

BRITISH chemists claim to have found evidence for a controversial new type of isomerism. Compounds which share the same chemical composition and molecular mass yet differ in a physical or chemical property are known as isomers. The researchers say they have found a compound which can exist in two forms, differing only in the length of a single chemical bond (*Angewandte Chemie*, vol 30, p 980).

Vernon Gibson and his colleagues from the University of Durham and Mary McPartlin and her team at the Polytechnic of North London prepared and analysed two pairs of compounds which they claim exhibit this "bond-stretch" isomerism. Both pairs are compounds of the metal niobium.

Isomerism can exhibit many forms. Structural isomerism, for example, occurs when the same collection of atoms can be joined together in a number of different ways—in a different order round a ring or along a chain, for example.

Stereo-isomerism, on the other hand, arises because of the different ways that atoms joined in the same order can be arranged in space.

Chemists are generally sceptical of claims for bond-stretch isomerism. They tend to believe that for two molecules to be isomeric, there must be a difference in the three-dimensional arrangement of their atoms, not just in their bond lengths.

One of the pairs of niobium compounds which Gibson and his colleagues produced differ only in the length of their niobium-oxygen bond, the researchers say. The molecules of the other pair differ in the length of a niobium-sulphur bond.

Each pair of isomers had different colours—yellow and green for the niobium-oxygen pair, and orange and green for the niobium-sulphur pair. They also displayed different bond stretching frequencies in their infrared spectra, suggesting that their bond lengths are different. **Lionel Milgrom**