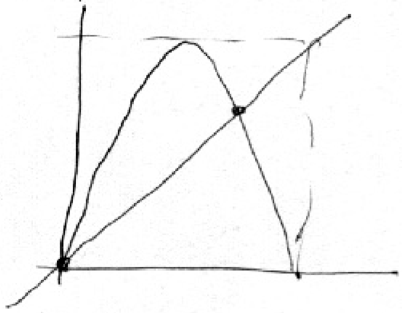
Maths 745.
Assignment 1 Solutions



1. $F_4^{(1)}$ has 2 fixed pts since

$$4x(1-x) = 4x - 4x^2 = x$$
$$\Leftrightarrow 3x = 4x^2 \Leftrightarrow x = 0, 3/4.$$

Furthermore. $F_4[0\ \tfrac{1}{2}] = F_4[\tfrac{1}{2}\ 1] = [0\ 1]$

Consider. F_4^k and assume this has 2^k fixed pts and 2^k alternating continuous transitions between 0 and 1.

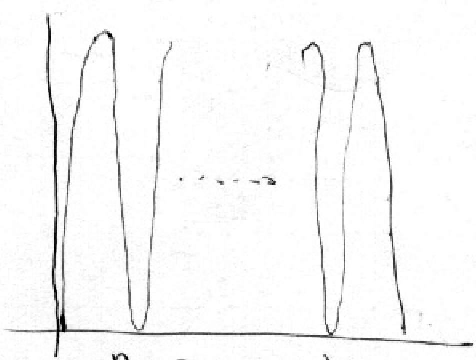
Consider $F_4^{k+1} = F(F_4^k)$. For each transition of F_4^k from 0 to 1 $\exists x_i : F_4^k(x_i) = \tfrac{1}{2}$ (int val thm. There a 2^k of these, dividing each transition between 0 and 1 in two.

$F_4^{k+1}(x_i) = F(F_4^k) = F(\tfrac{1}{2}) = 1.$ and $F_4^{k+1}(x_j) = F_4(1) = 0$ for each $x_j \in [0 1]$; $F_4^k(x_j) = 1$.

Thus F_4^{k+1} has twice as many transitions between 0 and 1 as F_4^k and again by the int val thm. 2^{k+1} fixed pts

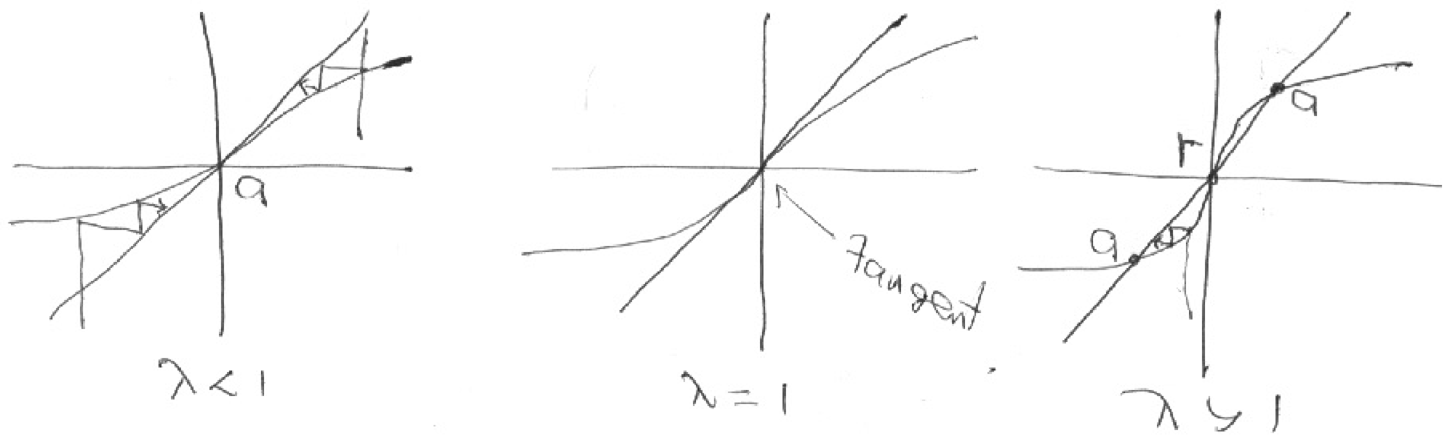
Hence. F_4 has. 2^n period n points.

F_4^n has a graph.



2^n. transitions between 0 and 1.

2 (a) $f_\lambda = \lambda \sin x \quad \lambda = 1$



f_λ has a fixed pt at $x=0$ since
$\lambda \sin x = x = 0$ for $x = 0$

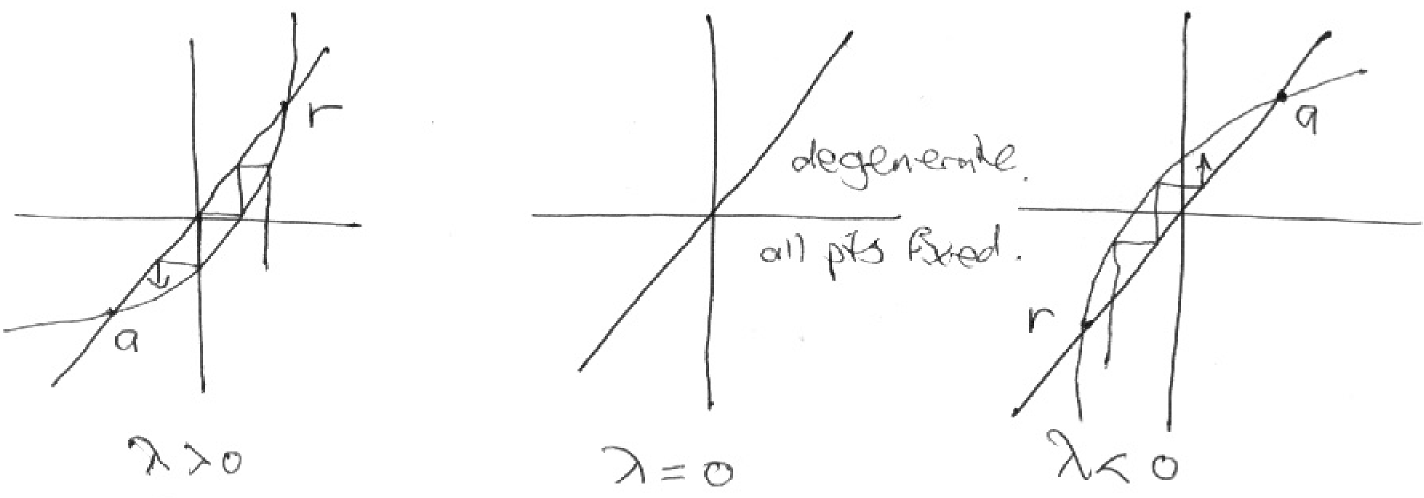
This fixed pt has

$|f'_\lambda(0)| = \lambda \cos x|_{x=0} = \lambda$ — attracting $\lambda < 1$, neutral (tangent) $\lambda = 1$, repelling $\lambda > 1$.

For $\lambda > 1$ there are 3 fixed pts from graphical analysis, since f is bounded by $\pm\lambda$ but $y = x$ is unbounded.

The non-zero fixed pts must have $|f'| < 1$ as the graph is "crossing under" $y = x$.

(b) $f_\lambda = \lambda x^2 + x - \lambda \quad \lambda = 0$



$\lambda x^2 + x - \lambda = x \iff \lambda(x^2-1) = 0 \iff \begin{cases} x = \pm 1 & \lambda \neq 0 \\ x \in \mathbb{R} & \lambda = 0 \end{cases}$

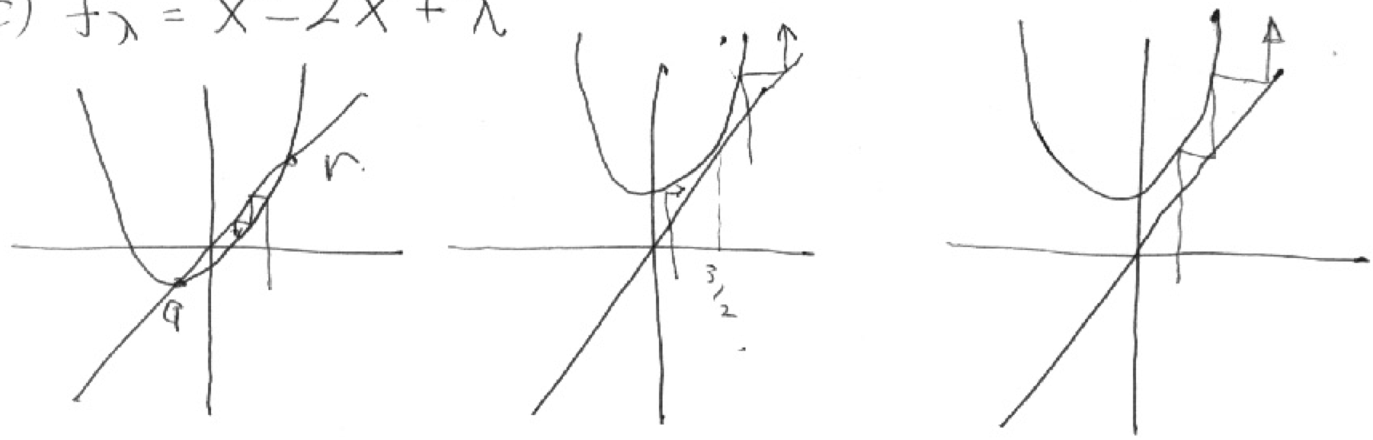
$|f_\lambda'(x)| = |2\lambda x + 1| \qquad f_\lambda' = |1 \pm 2\lambda|.$

$\lambda < 0$ $\quad x=-1$ repelling $\quad x=+1$ attracting

$\lambda > 0$ $\quad x = 1$ " $\quad x = -1$ "

$\lambda = 0$ $\quad$ all points fixed.

(c) $f_\lambda = x^2 - 2x + \lambda$



$x^2 - 2x + \lambda = x \iff x^2 - 3x + \lambda = 0$

$\iff x = \dfrac{3 \pm \sqrt{9-4\lambda}}{2}.$

$\lambda = 9/4$ 1 soln. $\quad \lambda > 9/4$ no solns

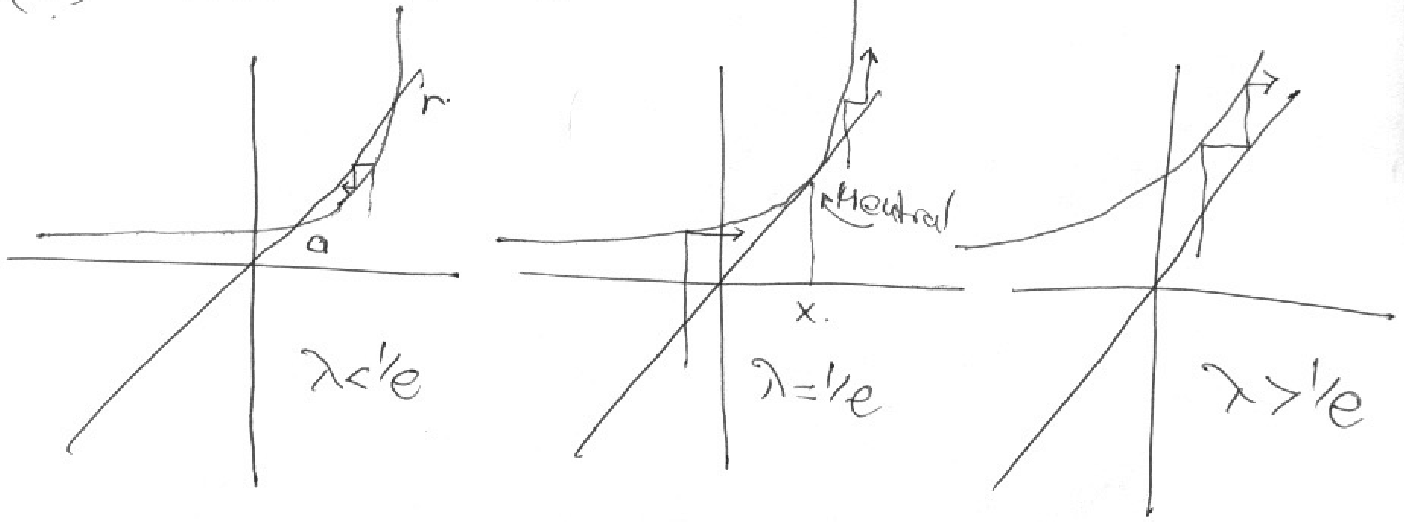
$\lambda < 9/4$ 2 solns

$f_\lambda' = 2x - 2.$

$= \dfrac{2(3 \pm \sqrt{9-4\lambda})}{2} - 2 \qquad \lambda < 9/4$ smaller value attracting, larger value repelling.

$= 1 \pm \dfrac{\sqrt{9-4\lambda}}{2}.$ tangent bifurcation

(d) $\lambda e^x \quad \lambda = 1/e$.

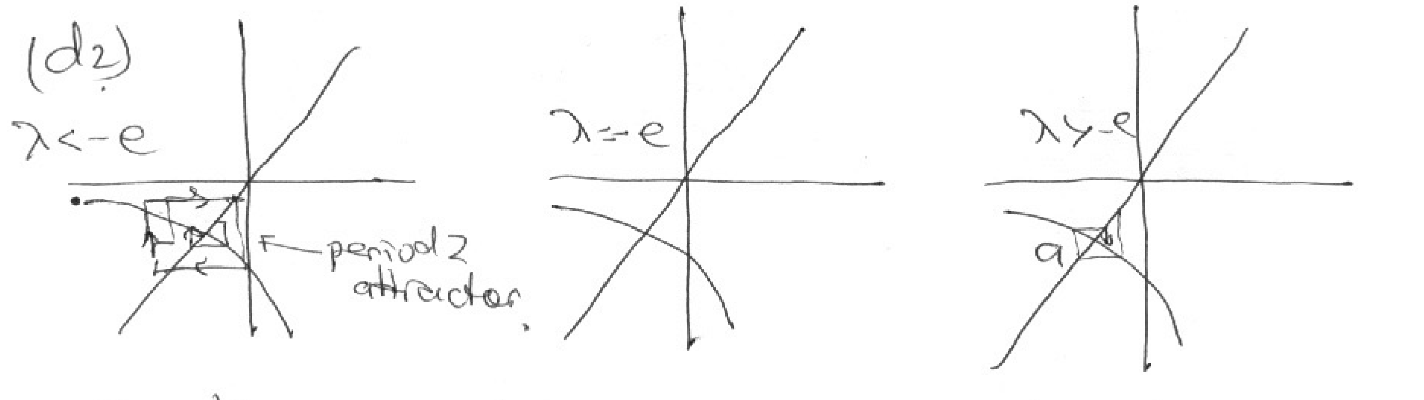


$\lambda e^x = x$. at $1/e$ $\quad \frac{1}{e} e^1 = 1$ $\quad$ soln $x = 1$.

$\lambda < 1/e$. 2 solns. the smaller attracting
the larger repelling

$\lambda > 1/e$ no solns. Tangent bifurcation

(d2)

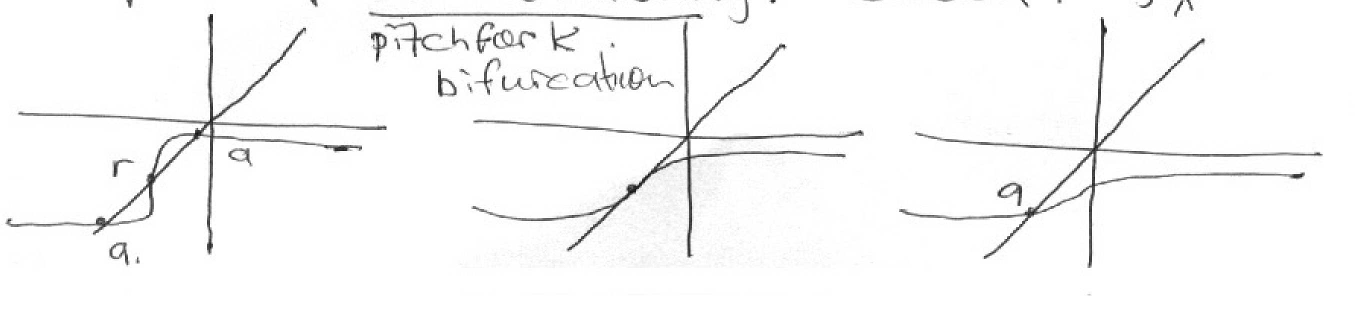


$\lambda e^x = x$ at $-e$. $\quad -e\, e^{-1} = -1$ soln $x = -1$

$|f_\lambda'(x)| = |\lambda e^x| = |\lambda e^{-1}| < 1 \quad \lambda > e^{-1}$.

$> 1 \quad \lambda < e^{-1}$

Suspect period doubling. Check. $f_\lambda^{(2)}$



3(a) Pick a Cauchy sequence $\{\sigma_i\}$

Then if $1/2p < \varepsilon$. $\exists N$: $m,n > N$ $d'(\sigma_m, \sigma_n) < 1/2p$.

So, if $\sigma_m = S_{m0} S_{m1} - - - S_{mi}$
$\sigma_n = S_{n0} S_{n1} - - - S_{ni}$

we have $S_{mi} = S_{ni}$ $i = 1 - p$

For $p = 1, 2, 3 - -$ we can thus define a sequence. $\sigma = S_0 S_1 - - - S_i$ inductively

: $S_p = S_{mp} = S_{np}$.

Then. $\{\sigma_i\} \longrightarrow \sigma$ using the Cauchy criterion above.
so complete.

If you prove (b) first, (a) follows since compact $\Rightarrow$ complete and totally bounded.

(b) Two methods. (i) and (ii)

(i) For an arbitrary sequence $\Sigma = \{\sigma_i\}$ consider.

$$\Sigma = \begin{cases} \sigma_1 = S_{10} S_{11} S_{12} - - - S_{1i} \\ \sigma_2 = S_{20} S_{21} S_{22} - - - S_{2i} \\ \quad\quad\ \text{1st} \ \ \text{2nd} \quad\quad\quad \text{ith}. \end{cases}$$

For the 1st position each S_{j0} is either 0, or 1. Pick S_0 either 0 or 1 so there are an infinite number of sequences with this 1st value and. remove all but a finite number of sequences with $S_{j0} \neq S$ from Σ to form a subset. Σ_0 pick a sequence. from Σ_0 which has this first term S_0. Call it. σ'_0

Now consider. Σ_0 and do the same with the second position choosing either 0, or 1 to be S_1 and choose σ'_1. far enough down the sequence of sequences Σ_1 to be beyond σ'_0

Inductively continue to create
$\Sigma \supseteq \Sigma_1 \supseteq \Sigma_2 - - -$ subsequence of Σ $\{\sigma'_j\}$
and $\sigma = S_0 S_1 - - - S_i - - -$

Now. $\{\sigma_j'\}$ is a subsequence of Σ
and $\{\sigma_j'\} \to \sigma$.

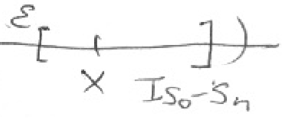
(ii) Σ_2 has an ε-net $\varepsilon = \frac{1}{2^n}$.

since periodic sequences are dense in Σ_2.
Now. from (a) Σ_2 is complete and has an ε-net so is totaly bounded and hence compact by the extended Heine-Borel theorm,

4. $\underset{\delta}{\Sigma_2} \xrightarrow{S^{-1}} \underset{\varepsilon}{\Lambda}$.

Need to show for any ε-nhd in Λ $\exists$ δ-nhd in Σ_n

Pick ε and consider $(x-\varepsilon, x+\varepsilon)$. $x \in \Lambda$.
$\exists n$: $I_{s_0 \cdots s_n} \subseteq (x-\varepsilon, x+\varepsilon)$



as the $I_{s_0-s_n}$ are nested.
their $\cap$ is a point and $x \in \cap$.
Now choose $\delta < \frac{1}{2^n}$ and any $\sigma \in \Sigma_2$ in a δ-nhd has $s_0 - s_n$ the same as $I_{s_0 - s_n}$.

5. (a) $\underset{\text{wandering}}{x \notin \Omega(f)} \iff \exists J \ni p : \forall x \in J, \forall n \in \mathbb{N}.\ f^n(x) \notin J$
so. all $y \in J$ also satisfy this so wandering
Hence $\overline{\Omega(f)}$ open so $\Omega(f)$ closed.

(b) If $x \in \Lambda$ then. $x \leftrightarrow \sigma \in \Sigma_2$
but then. since periodic points are dense in Σ_2
$\exists y : |x-y| < \delta$: $f^n(y) = y$ $n \in \mathbb{N}$, so x non-wandering.
so. $x \in \Lambda \Rightarrow x \in \Omega$.

If $x \notin \Lambda$ then $x \in A_n$ some n, or $x \notin [0,1]$. If $x \in A_n$ $f^{(i)}(x) \in A_{n-i}$ and $A_i \cap A_j = \phi$. If $x \notin [0,1]$ x is wandering to superattractor at $-\infty$.

Hence $x \notin \Lambda \Rightarrow x \notin \Omega$,

(c) $0 < r < 3$. The only non-wandering points are attractors and repellers. $0 < r < 3$ period 1 so we have only fixed points $rx - rx^2 = x \Leftrightarrow x = 0, \frac{r-1}{r}$.

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(a) Let $x \in \Lambda$: $S(x) = \{01\ 00\ 01\ 10\ 11 \cdots\} \in$

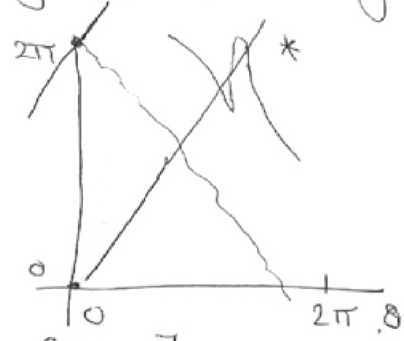
Since $S(x)$ is dense in Σ_2 it is non-periodic but recurrent to all points in Σ_2. Hence x is recurrent but not periodic.

(b) Let $x \in \Lambda$; $S(x) = \sigma = \{0\ 1\ 1\ \bar{1} \cdots\} \in \Sigma_2$.

x non-wandering by 5(b) but x not recurrent as, from σ,

$x \in [0\ \tfrac{1}{2})$ but $f^n(x) \in (\tfrac{1}{2}\ 1]$.

7. If f is an orientation-reversing map, $f(\theta)$ must go through $2\pi \to 0$ at least once as θ goes through $0 \to 2\pi$.



Consider $g(\theta) = f(\theta) - f(0)$

Then $g(0) = 0 = 2\pi \pmod{2\pi}$

so the graph of g must go from 2π at 0 to 0 in $[0\ 2\pi]$.

By the intermediate value theorem $\exists\, x \in [0\ 2\pi]$: $g(x) = x$. since $x \in [0\ 2\pi]$ and g has every value between by continuity

Also $g(0) = 0 = 2\pi$ is fixed, so g has 2 fixed pts

If f is also a diffeomorphism then $f(\theta)$ and hence $g(\theta)$ can only traverse $2\pi \to 0$ once and so there are just 2 fixed pts. since f and g 1-1 prevents multiple fixed pts (see *).

Since f is simply a translate of g vertically and "$y = x$" repeats vertically each 2π. f also has 2 fixed pts by continuity.

8. Two lifts F, G share. $e^{2\pi i x} = e^{2\pi i y}$.
where $x = F(\theta)$, $y = G(\theta)$

hence. $\sin 2\pi x = \sin 2\pi y$ $\quad \cos 2\pi x = \cos 2\pi y$

and. $x = y + k$. $k \in \mathbb{Z}$

Conversely if $x = y + k$ $\quad e^{2\pi i x} = e^{2\pi i y}$.