Paper MATHS 745 Assignment 1

Please hand in in class by the end of first week after study break

1.[8] Sketch the graph of $F_4^{(n)}$ where $F_4(x) = 4x(1 - x)$ on the unit interval.

Prove that F_4 has at least 2^n periodic points of period n.

2.[4x5] Discuss the bifurcations that occur in the following families of maps for the indicated parameter value. Illustrate with a graph and sample orbits:

(a) $\lambda\, sin(x)$, $\lambda = 1$. (b) $\lambda x^2 + x - \lambda$, $\lambda = 0$. (c) $x^2 - 2x + \lambda$, $\lambda = 9/4$. (d) λe^x, $\lambda = -e$, $1/e$

3.[8+8]

(a) Show that the sequence space Σ_2 is complete as a metric space.

Hint: A complete space is one which contains all of its limit points. A Cauchy sequence $\{x_i\}$ in a metric space X satisfies $\forall \varepsilon > 0\ \exists N \in ¡ : m,n > N \Rightarrow |x_m - x_n| < \varepsilon$. It is sufficient to show every Cauchy sequence (of sequences) in Σ_2 is convergent in Σ_2. A Cauchy sequence (of sequences) in Σ_2 will consist of sequences coming closer and closer to a consensus sequence since for $\varepsilon < \frac{1}{2^n}$. their first n terms must be the same. Define this consensus sequence inductively to find the limit.

(b) Show that Σ_2 is a compact metric space. A subset of R^n is compact if and only if it is closed and bounded, but more generally a metric space is compact if every sequence has a convergent subsequence.

Hint: Consider the first terms s_{i0} of a sequence (of sequences) $\{s_i = s_{i0}s_{i1}\cdots s_{ij}\}$. If all but a finite number of the s_{i0} are 0, we can form a subsequence consisting of only those sequences whose first terms are 1. Conversely with a finte number of 1s. If there are an infinite number of both we can choose either. We can also include a finite number of the sequences we excluded. Proceed inductively to find a convergent subsequence.

4.[8] Show that the mapping $S : \Lambda \to \Sigma_2$ defined by $S(x) = s_0, s_1, ...s_n$ where $s_n = 0$ if $F_r^{\ n}(x) \in I_0$, $s_n = 1$ if $F_r^{\ n}(x) \in I_1$ $F_r\ (x) = r.x.(1 - x)$ has a continuous inverse S^{-1}, when $r > 2 + \sqrt{5}$.

5.[18] A point p is a *non-wandering* point for *f* if for any open interval J containing *p* there exists $x \in$ J and n>0 such that $f^{(n)}(x) \in$ J. Note that *p* does not have to return to J. Let Ω(f) denote the set of non-wandering points for *f.*

(a) Prove that $\Omega(f)$ is a closed set.

(b) If $F_r\ (x) = r.x.(1 - x)$ $r > 2 + \sqrt{5}$, show that $\Omega(f_r) = \Lambda$.

(c) Identify $\Omega(f_r)$ for each $0 < r < 3$.

6.[12] A point p is *recurrent* for *f* if for any open interval J about *p* there exists $n > 0$ such that $f^n(p) \in J$. Clearly all periodic points are recurrent.

(a) Give an example of a non-periodic recurrent point for

$F_r\ (x) = r.x.(1 - x)$ $r > 2 + \sqrt{5}$

(b) Give an example of a non-wandering point for $F_r\ (x) = r.x.(1 - x)$ $r > 2 + \sqrt{5}$ which is not recurrent.

7.[9] Show that any orientation-reversing differentiable homeomorphism of the circle must have two fixed points.

8.[9] Prove that any two lifts of the circle map must differ by an integer. Conversely show that if *F(x)* is a lift of f then so too is *F(x)* + *k* where *k* is any integer.