# THE UNIVERSITY OF AUCKLAND

MOCK TEST FOR MA MSc BSc(Hons) ETC 2008

Mathematics 745

Chaos Fractals and Bifurcation

( Time Allowed : TWO hours )

**PLEASE ATTEMPT 5 QUESTIONS OUT OF 8**

**The use of calculators is permitted.**

1. (a) Consider $(\Sigma_2, d)$ the metric space, where $\Sigma_2$ is the space of sequences $<u> = (u_1, u_2, \ldots u_k, \ldots)$ $u_i = 0, 1$ with metric d defined by

$$d(<u>,<v>) = \sum_{i=0}^{\infty} \frac{|u_i - v_i|}{2^i}.$$

Show that $d(<u>,<v>) < \frac{1}{2^n} \Rightarrow u_i = v_i,\ i = 1, \ldots, n$ and that

$$u_i = v_i,\ i = 1, \ldots, n \Rightarrow d(<u>,<v>) \le \frac{1}{2^n}$$

(b) Show how to define a Cantor subset $\Lambda$ of [0,1] and a map S: $\Lambda \to \Sigma_2$ from $\Lambda$ to the sequence space $\Sigma_2$ which can be used to define a conjugacy between $f : \Lambda \to \Lambda$, $f(x) = r\,x\,(1 - x)$, $r > 2 + \sqrt{5}$ and the shift map $\sigma: \Sigma_2 \to \Sigma_2$.

(c) Prove that $S \circ f = \sigma \circ S$ and explain the significance of the conjugacy.

2. (a) Show that the iteration $f(x) = (r + 1)\,x - r\,x^2$ has two fixed points which are independent of r. Discuss the bifurcation that happens at r = 0 by examining typical orbits on either side of r = 0, and determining the nature of the fixed points.

(b) Discuss the nature of the bifurcation of the iteration $f(x) = x^2 + r$ at $r = 1/4$.

3. (a) (i) For an orientation-preserving differentiable homeomorphism $f : S^1 \to S^1$ define the winding number of f

(ii) Show that the winding number $\rho$ of an orientation-preserving differentiable homeomorphism $f : S^1 \to S^1$ with a periodic point of period m is

$$\rho = \frac{k}{m}, \quad k \in \mathbb{Z}.$$

(b) Define the Hausdorff metric and show that it is a metric on the space $\mathcal{H}(X)$ of compact subsets of a metric space (X,d).

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4. Show how readings from a single time series may be used to determine the correlation dimension of a suitably embedded strange attractor, indicating how to test for the correct embedding dimension and adjust for suitable sampling delays and window times.

5. Write a brief essay ( 1 to 2 pages only) on any topic in 745 you found interesting.

6. (a) Describe the level set method (LSM) for determining the Julia set $J_c$ of a complex quadratic iterative map $z_{n+1} = z_n^2 + c$. Show how LSM can be modified toform the continuous potential method (CPM).

(b) Demonstrate that the quadratic iterative map $z_{n+1} = z_n^2 + (-0.1+0.75i)$ has a period 3 attractor by performing the first three iterations from $z = c$.

7. (a) Describe the Henon attractor, defining the iteration used.

(b) Describe the solenoid attractor of iterations of the map on the solid torus $T = S^1 \times B^2$

$$F(\theta,p) = \left(2\theta, \frac{1}{10}p + \frac{1}{2}e^{2\pi i\theta}\right).$$

Show that periodic points are dense in this attractor.

8. (a) Determine the fractal dimension of the set P illustrated in (i) below, by counting numbers and lengths of edges of a pair of polygons in a sequence converging to P. The other diagrams give the first few sets in such a sequence.

(b) The fractal shown in (ii) below is defined by an iterated function system consisting of three affine mappings $F_i(x,y)$ on $[0,1]\times[0,1]$. With the help of self similarities, determine the coefficients of the three affine mappings

$$F_i(x,y) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}, \quad i = 1,2,3.$$