THE UNIVERSITY OF AUCKLAND

TEST FOR MA MSc BSc(Hons) ETC 2007

Mathematics 745

Chaos Fractals and Bifurcation

(Time Allowed : TWO hours)

PLEASE ATTEMPT 5 QUESTIONS OUT OF 8

The use of calculators is permitted.

1. (a) **[6]** State the three axioms of chaos for an iterative map $x_{n+1} = f(x_n)$.

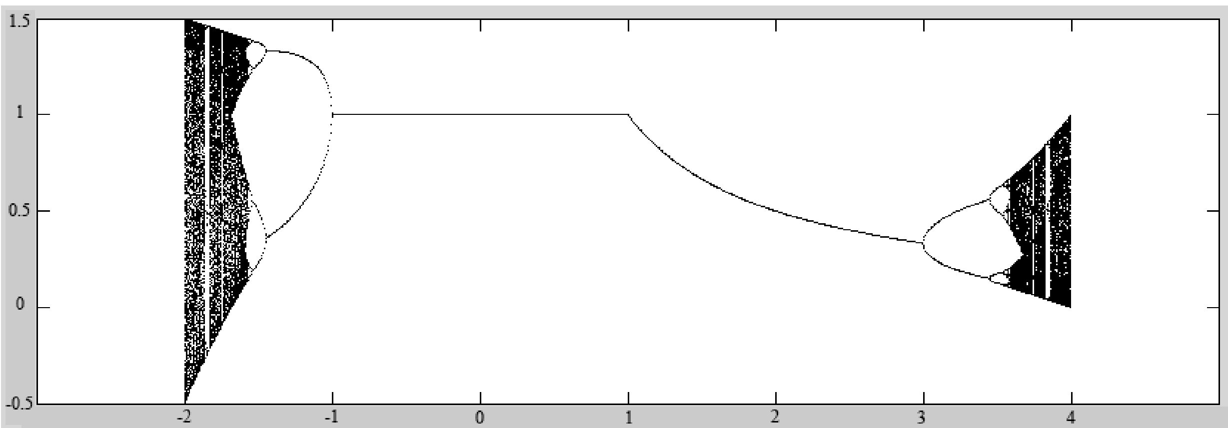
   (b) **[8]** Define the sequence space $\sum_2$ on two symbols and show that the shift map on $\sum_2$ is continuous.

   (c) **[6]** Show that the shift map obeys the three axioms in part (a).

2. Consider the function $f(x) = \lambda x^2 - \lambda x + 1$.

   (a) **[6]** Find the fixed points of the iterated function and hence show that there is a bifurcation at $\lambda = 1$.

   (b) **[8]** Describe the bifurcation using trajectories and determine whether each of the fixed points are attracting or repelling.



   (c) Above is a Feigenbaum diagram for the above function:

   (i) **[4]** Find two values at which period doubling begins, and confirm using $|f'(x)|$ that one of your fixed points above is in transition from attraction to repulsion in each case.

   (ii) **[2]** Find two values at which the iteration is chaotic on an interval.

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3. **[20]** Prove the special case of Sarkovski's theorem :

Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ with principal period 3,

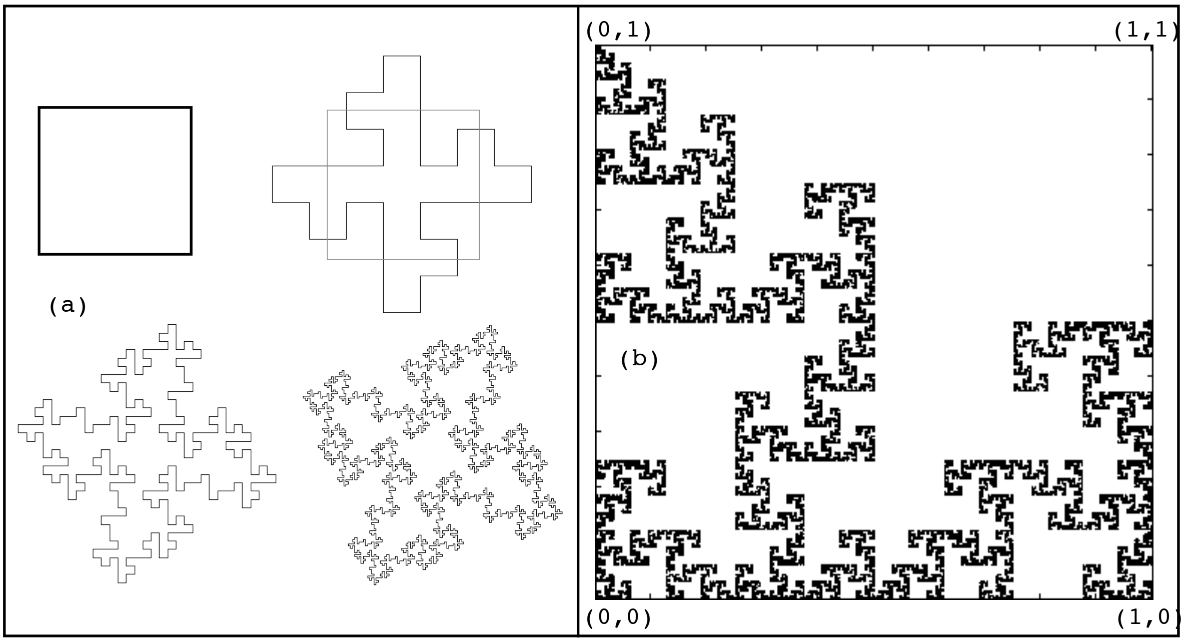
f has periodic points of all periods.

NOTE: You may assume :

(i) If I, J are closed intervals such that $I \subseteq J$ but $f(I) \supseteq J$ then f has a fixed point in I.

(ii) If I, J are closed intervals such that $f(I) \supseteq J$

then there is a sub-interval H of I such that $f(H) = J$.



4 (a) **[10]** Determine the fractal dimension of the set P illustrated in (a) above, by counting numbers and lengths of edges of a pair of polygons in a sequence converging to P, as shown in the upper two figures.
The four diagrams give the first four sets in such a sequence.

(b) **[10]** The fractal attractor shown in (b) above is defined by an iterated function system consisting of three affine mappings $F_i(x,y)$ on [0,1] x [0,1]. With the help of the internal symmetries in the attractor, determine the coefficients of the three affine mappings $F_i(x,y) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}$, $i = 1,2,3$.

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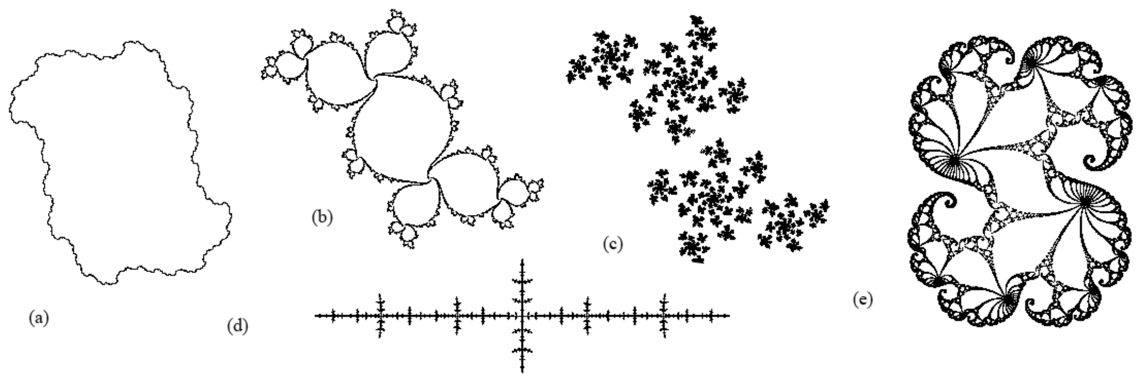

5. (a) **[6]** For each of the above Julia sets of $f(z) = z^2 + c$, state whether it is in the Mandelbrot set and indicate which you think are in the interior of the Mandelbrot set.

   (b) **[14]** Define the Mandelbrot set of a parametric quadratic function in terms of the topological properties of the function's Julia sets. Explain how and why its critical point can be used to provide a computational method to define the points in the Mandelbrot set.

6. (a) **[10]** Give a brief description of multifractality, using a recursive probability distribution on the unit interval.

   (b) **[10]** Explain using continued fractional representation of Fibonacci numbers, why the golden mean $\gamma = \dfrac{-1+\sqrt{5}}{2}$ is worst approximated by fractions of all numbers in the unit interval. What does this mean for mode-locking?

7. (a) **[6]** Define the terms 'trapping region' and 'transitive attractor' for a multidimensional iteration.

   (b) **[8]** Describe the solenoid attractor of iterations of the map $F(\theta,p) = \left(2\theta, \frac{1}{10}p + \frac{1}{2}e^{2\pi i\theta}\right)$ on the solid torus $T = S^1 \times B^2$

   (c) **[6]** Show that this iteration is sensitive to initital conditions.

8. **[20]** Write a short essay on any topic in 745 you found interesting that is not repeated in your other answers.