Detailed calculations in groups of order p^7

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1 Summaries

For $p > 5$ the number of groups of order p^7 is

$$\begin{aligned}&3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455\\&+(4p^2 + 44p + 291)\gcd(p-1,3) + (p^2 + 19p + 135)\gcd(p-1,4)\\&+(3p+31)\gcd(p-1,5) + 4\gcd(p-1,7) + 5\gcd(p-1,8) + \gcd(p-1,9)\end{aligned}$$

The groups are organized as follows. For odd prime p there are 42 groups of order at most p^5 which have immediate descendants of order p^6. Presentations for the corresponding Lie rings are given in Table 1. Of these groups of order at most p^5, 17 have immediate descendants of order p^7, and these groups and the number of their immediate descendants of order p^7 are given in Table 2. The next two tables list the 42 groups of order at most p^5 which have immediate descendants of order p^6, and give the number of these groups of order p^6 which are capable, and the number of descendants of order p^7 of these capable groups of order p^6.

To get the complete total of groups of order p^7 we must add in the 9 six generator groups of p-class 2, and the 1 seven generator group of p-class 1.

Note that the numbers of groups given are only valid when the prime p is greater than the p-class of the groups in question. All the numbers given are valid for $p > 5$.

The following table lists the nilpotent Lie rings of order at most p^5 which have immediate descendants of order p^6. The numbering and the presentations correspond to the list of groups of order at most p^5 which have immediate descendants of order p^6.

1	-	$\langle a \mid \text{ class } 5\rangle$
2	4.6	$\langle a, b \mid ba, \text{ class } 2\rangle$
3	4.7	$\langle a, b \mid pb, \text{ class } 2\rangle$
4	4.8	$\langle a, b \mid pb - ba, \text{ class } 2\rangle$
5	5.37	$\langle a, b \mid \text{ class } 2\rangle$
6	5.38	$\langle a, b \mid pa, pb, \text{ class } 3\rangle$
7	5.41	$\langle a, b \mid pa - baa, pb, \text{ class } 3\rangle$
8	5.39	$\langle a, b \mid pa - bab, pb, \text{ class } 3\rangle$
9	5.40	$\langle a, b \mid pa - \omega bab, pb, \text{ class } 3\rangle$
10	5.42	$\langle a, b \mid pa - baa, pb + bab, \text{ class } 3\rangle$
11	5.45	$\langle a, b \mid pa + bab, pb + \omega baa, \text{ class } 3\rangle$
12	5.47	$\langle a, b \mid ba, p^2b, \text{ class } 3\rangle$
13	5.48	$\langle a, b \mid ba - p^2a, p^2b, \text{ class } 3\rangle$
14	5.49	$\langle a, b \mid baa, bab, pb, \text{ class } 3\rangle$
15	5.50	$\langle a, b \mid bab, p^2a, pb, \text{ class } 3\rangle$
16	5.51	$\langle a, b \mid bab, p^2a, pb - baa, \text{ class } 3\rangle$
17	5.52	$\langle a, b \mid bab, p^2a, pb - \omega baa, \text{ class } 3\rangle$
18	5.54	$\langle a, b \mid baa, p^2a, pb, \text{ class } 3\rangle$
19	5.58	$\langle a, b \mid baa, pb - ba, \text{ class } 3\rangle$
20	5.73	$\langle a, b \mid ba, pb, \text{ class } 4\rangle$
21	5.60	$\langle a, b \mid bab, pa, pb, \text{ class } 4\rangle$
22	5.65	$\langle a, b \mid bab - baaa, pa, pb, \text{ class } 4\rangle$
23	3.1	$\langle a, b, c \mid \text{ class } 1\rangle$
24	4.3	$\langle a, b, c \mid ca, cb, pa, pb, pc, \text{ class } 2\rangle$
25	5.8	$\langle a, b, c \mid ba, ca, cb, pc, \text{ class } 2\rangle$
26	5.9	$\langle a, b, c \mid ca, cb, pb, pc, \text{ class } 2\rangle$
27	5.10	$\langle a, b, c \mid ca, cb, pb - ba, pc, \text{ class } 2\rangle$
28	5.11	$\langle a, b, c \mid ca, cb, pb, pc - ba, \text{ class } 2\rangle$
29	5.12	$\langle a, b, c \mid ca, cb, pa, pb, \text{ class } 2\rangle$
30	5.13	$\langle a, b, c \mid ca, cb, pa - ba, pb, \text{ class } 2\rangle$
31	5.14	$\langle a, b, c \mid cb, pa, pb, pc, \text{ class } 2\rangle$
32	5.15	$\langle a, b, c \mid cb, pa - ba, pb, pc, \text{ class } 2\rangle$
33	5.16	$\langle a, b, c \mid cb, pa, pb - ba, pc, \text{ class } 2\rangle$
34	5.18	$\langle a, b, c \mid cb, pa, pb - ca, pc, \text{ class } 2\rangle$
35	5.19	$\langle a, b, c \mid cb, pa - ba, pb - ca, pc, \text{ class } 2\rangle$
36	5.24	$\langle a, b, c \mid ba, ca, cb, pb, pc, \text{ class } 3\rangle$
37	5.27	$\langle a, b, c \mid bab, ca, cb, pa, pb, pc, \text{ class } 3\rangle$
38	5.32	$\langle a, b, c \mid bab, ca, cb - baa, pa, pb, pc, \text{ class } 3\rangle$
39	4.1	$\langle a, b, c, d \mid \text{ class } 1\rangle$
40	5.2	$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pb, pc, pd, \text{ class } 2\rangle$
41	5.3	$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb, pc, pd, \text{ class } 2\rangle$
42	5.1	$\langle a, b, c, d, e \mid \text{ class } 1\rangle$

The following table lists the groups of order at most p^5 which have immediate descendants of order p^7, and gives the number of descendants.

3	4.7	4
5	5.37	$p^2 + 8p + 25$
6	5.38	$p + 6 + (p^2 + 3p + 10)\gcd(p-1,3)$
21	5.60	$2p^2 + p + 3 + 2(p+1)\gcd(p-1,3) + (2p+4)\gcd(p-1,4) + \gcd(p-1,8)$
22	5.65	$p^3 + p^2 + p - 2 + 2\gcd(p-1,3) + \gcd(p-1,4) + (p+1)\gcd(p-1,5)$
23	3.1	$p + 14$
25	5.8	$p + 8$
26	5.9	$4p^2 + 26p + 107 + 5\gcd(p-1,3) + (p+4)\gcd(p-1,4)$
27	5.10	$2p + 7$
29	5.12	$3p^2 + 17p + 53 + \gcd(p-1,3) + \gcd(p-1,4)$
31	5.14	$2p^5 + 7p^4 + 19p^3 + 49p^2 + 128p + 256 + (p^2 + 7p + 29)\gcd(p-1,3)$
		$+(p^2 + 7p + 24)\gcd(p-1,4) + (p+3)\gcd(p-1,5)$
32	5.15	$3p^2 + 12p + 14 + (p+2)\gcd(p-1,4)$
33	5.16	$p^4 + 2p^3 + 5p^2 + 14p$
34	5.18	$3p^3 + 6p^2 + 6p + 11 + (p+7)\gcd(p-1,3) + (p+1)\gcd(p-1,4) + \gcd(p-1,5)$
39	4.1	1361 if $p = 3$, otherwise $p^5 + 2p^4 + 7p^3 + 25p^2 + 88p + 270 + (p+4)\gcd(p-1,3) + \gcd(p-1,4)$
41	5.3	$p^4 + 5p^3 + 19p^2 + 64p + 140 + (p+6)\gcd(p-1,3) + (p+7)\gcd(p-1,4) + \gcd(p-1,5)$
42	5.1	178 if $p = 3$, otherwise $p^2 + 15p + 125$

In the two tables below we give the list of 42 groups of order at most p^5 which have immediate descendants of order p^6. For each of these groups we give the number of immediate descendants of order p^6 which are capable, and we give the number of ıgrand-childrenȷ of order p^7. Note that by ıgrandchildrenȷ we mean groups whose immediate ancestors have order p^6.

		capable descendants of order p^6
1	-	1
2	4.6	2
3	4.7	16
4	4.8	1
5	5.37	2
6	5.38	3
7	5.41	$p+1+\gcd(p-1,3)$
8	5.39	$1+\gcd(p-1,3)+\gcd(p-1,4)/2$
9	5.40	$1+\gcd(p-1,3)+\gcd(p-1,4)/2$
10	5.42	$p+1$
11	5.45	p
12	5.47	2
13	5.48	0
14	5.49	1
15	5.50	$2+2\gcd(p-1,3)$
16	5.51	$(p+1)/2$
17	5.52	$(p+1)/2$
18	5.54	$3+\gcd(p-1,4)$
19	5.58	1
20	5.73	1
21	5.60	2
22	5.65	1
23	3.1	$3p+27$
24	4.3	$5p+37+\gcd(p-1,4)$
25	5.8	2
26	5.9	9
27	5.10	1
28	5.11	2
29	5.12	3
30	5.13	1
31	5.14	3
32	5.15	$2p+9$
33	5.16	4
34	5.18	11
35	5.19	0
36	5.24	1
37	5.27	4
38	5.32	1
39	4.1	24
40	5.2	1
41	5.3	2
42	5.1	2

		grandchildren of order p^7
1	-	1
2	4.6	4
3	4.7	$15p + 41 + 16\gcd(p-1,3) + 4\gcd(p-1,4)$
4	4.8	2
5	5.37	$5p + 10 + 2\gcd(p-1,3) + \gcd(p-1,4)$
6	5.38	$p^3 + 3p^2 + 8p + 18 + 5\gcd(p-1,3) + (p+5)\gcd(p-1,4)$
		$+3\gcd(p-1,5) + 2\gcd(p-1,8) + \gcd(p-1,9)$
7	5.41	$3p^2 + 4p + (p+1)\gcd(p-1,3) + \gcd(p-1,4)$
8	5.39	$\frac{1}{2}(p^2 + 2p + 1 + (p+5)\gcd(p-1,3) + (p+3)\gcd(p-1,4))$
9	5.40	$\frac{1}{2}(p^2 + 2p + 1 + (p+5)\gcd(p-1,3) + (p+3)\gcd(p-1,4))$
10	5.42	$p+3$
11	5.45	$p+1$
12	5.47	3
13	5.48	0
14	5.49	4
15	5.50	$4p + 5 + (p+7)\gcd(p-1,3) + 3\gcd(p-1,4) + 2\gcd(p-1,5)$
16	5.51	$\frac{1}{2}(p+1)$
17	5.52	$\frac{1}{2}(p+1)$
18	5.54	$7p + 9 + 4\gcd(p-1,3) + 6\gcd(p-1,4) + 2\gcd(p-1,5)$
19	5.58	2
20	5.73	2
21	5.60	$4p + 3 + 2\gcd(p-1,3) + 4\gcd(p-1,5) + \gcd(p-1,7) + \gcd(p-1,8)$
22	5.65	$2p^2 + p + 2p\gcd(p-1,3) + p\gcd(p-1,5)$
23	3.1	$2p^2 + 63p + 362 + (p+19)\gcd(p-1,3) + 5\gcd(p-1,4) + \gcd(p-1,5)$
24	4.3	$p^4 + 4p^3 + 17p^2 + 39p + 72 + (p^2 + 9p + 47)\gcd(p-1,3)$
		$+(2p+8)\gcd(p-1,4) + 2\gcd(p-1,5) + \gcd(p-1,7)$
25	5.8	6
26	5.9	$5p + 49 + 11\gcd(p-1,3) + 4\gcd(p-1,4)$
27	5.10	5
28	5.11	7
29	5.12	$2p + 20 + 7\gcd(p-1,3) + 3\gcd(p-1,4)$
30	5.13	$p+1$
31	5.14	$p^2 + 9p + 36 + (p^2 + 5p + 29)\gcd(p-1,3) + (p+7)\gcd(p-1,4)$
		$+\gcd(p-1,7) + \gcd(p-1,8)$
32	5.15	$10p + 16 + (2p+7)\gcd(p-1,3) + 2\gcd(p-1,4) + 2\gcd(p-1,5)$
33	5.16	$p^3 + 5p^2 + 13p + 6 + 3\gcd(p-1,3)$
34	5.18	$2p^2 + 14p + 10 + (2p+8)\gcd(p-1,3) + 7\gcd(p-1,4) + \gcd(p-1,5)$
35	5.19	0
36	5.24	3
37	5.27	$p^2 + 10p + 34 + (p+14)\gcd(p-1,3) + 13\gcd(p-1,4) + 6\gcd(p-1,5) + \gcd(p-1,7)$
38	5.32	$p^2 + 7p + 3 + 2\gcd(p-1,3) + 3\gcd(p-1,4) + \gcd(p-1,5)$
39	4.1	$p^3 + 13p^2 + 96p + 595 + (3p+21)\gcd(p-1,3) + (p+11)\gcd(p-1,4) + \gcd(p-1,5)$
40	5.2	4
41	5.3	$35 + (p+15)\gcd(p-1,3) + 4\gcd(p-1,4)$
42	5.1	30

The following table lists the groups of order at most p^5 which have immediate descendants of order p^7, and gives the øles containing those descendants.

3	4.7	4.7
5	5.37	5.37
6	5.38	5.38
21	5.60	5.60
22	5.65	5.65
23	3.1	3.1
25	5.8	5.8
26	5.9	5.9
27	5.10	5.10
29	5.12	5.12
31	5.14	5.14
32	5.15	5.15
33	5.16	5.16
34	5.18	5.18
39	4.1	$4.1, 4.1a, 4.1b, 4.1c, 4.1d, 4.11 - 4.16$
41	5.3	$5.3a, b, c, d, e, f, g, h, i$
42	5.1	5.1

And the table below gives the øles containing presentations for the grandchildren of order p^7 of groups of order at most p^5.

		øles for grandchildren of order p^7
1	-	6.0
2	4.6	6.366
3	4.7	$6.368 - 6.383$
4	4.8	6.384
5	5.37	6.386, 6.388
6	5.38	6.394, 6, 399, 6.404
7	5.41	6.420, 6.421, 6.424
8	5.39	6.408, 6.411, 6.412
9	5.40	6.414, 6.417, 6.418
10	5.42	6.425, 6.426
11	5.45	6.427
12	5.47	6.428
14	5.49	6.431
15	5.50	6.435, 6.436, 6.442, 6.445
16	5.51	6.448
17	5.52	6.451
18	5.54	6.454, 6.455, 6.459, 6.460
19	5.58	6.467
20	5.73	6.518
21	5.60	6.469, 6.475
22	5.65	6.507
23	3.1	$6.85 - 6.106, 6.108 - 6.117$
24	4.3	$6.118 - 6.122, 6.125, 6.127, 6.131 - 6.135, 6.138 - 6.140, 6.142 - 6.144, 6.146, 6.148,$
		$6.150 - 6.163, 6.168, 6.172 - 6.176, 6.178, 6.179, 6.16A$
25	5.8	6.182, 6.183
26	5.9	$6.184, 6.187 - 6.192, 6.197, 6.198$
27	5.10	6.207
28	5.11	6.212, 6.215
29	5.12	6.216, 6.218, 6.222
30	5.13	6.228
31	5.14	6.231, 6.256, 6.261
32	5.15	$6.267, 6.269, 6.271, 6.273 - 6.280$
33	5.16	6.281, 6.282, 6.289, 6.290
34	5.18	$6.294 - 6.299, 6.303 - 6.305, 6.312, 6.313$
36	5.24	6.322
37	5.27	$6.325 - 6.328$
38	5.32	6.362
39	4.1	$6.9 - 6.21, 6.23, 6.24, 6.29, 6.33 - 6.36, 6.48, 6.51, 6.52, 6.60$
40	5.2	6.63
41	5.3	6.67, 6.72
42	5.1	6.2, 6.3

The 9 six generator groups of order p^7 and p-class 2 are in group6.1, and the 1 seven generator group of order p^7 and p-class 1 is in group7.1.

The notes that follow are not intended as ıpublishable mathematicsȷ. They are compiled from contemporaneous records of the calculations as I carried them out. In fact, to a very large extent, they are the calculations, as they were written in Scientiøc Workplace, which enables one to perform a lot of basic linear algebra on screen. The only editing that has been done to the original notes is to reorder them, and to insert section headings, etc.

2 Immediate descendants of algebra 3 (4.7)

Let L be an immediate descendant of 4.7 of order p^7. Then L is generated by a, b, L_2 is generated by ba and pa modulo L_3, and L_3 is generated by baa, bab and p^2a. We have $pb \in L_3$ and adding a suitable scalar multiple of pa to b we can assume that $pb = \alpha baa + \beta bab$ for some α, β. If a', b' generate L and if $pb' \in L_3$ then

$$\begin{aligned} a' &= \lambda a + \mu b, \\ b' &= \nu b \end{aligned}$$

modulo L_2 for some λ, μ, ν, and

$$\begin{aligned} b'a'a' &= \lambda^2 \nu baa + \lambda\mu\nu bab, \\ b'a'b' &= \lambda\nu^2 bab. \end{aligned}$$

So we may assume that $pb = 0$, baa, ωbaa or bab, giving 4 algebras. (Checked for $p = 3, 5, 7, 11, 13$.)

3 Immediate descendants of algebra 5 (5.37)

Algebra 5.37 has $p^2 + 8p + 25$ class 3 descendants of order p^7. I have checked that the recipes below give this number of groups for $p = 3, 5, 7, 11, 13$, and I have also checked that this is the right number of groups for $p = 3, 5, 7, 11$. (Out of memory on $p = 13$.) Algebra 5.37 is the free two generator class 2 Lie ring, and so if L is an immediate descendant of 5.37 of order p^7 then L is generated by a, b, L_2 is generated by ba, pa, pb modulo L_3, and L_3 is generated by baa, bab, p^2a, p^2b, pba. The subalgebra generated by baa and bab can have order 1, p or p^2, and we consider these three cases separately. If this subablgebra has order p then we can assume that $bab = 0$ and that $baa \neq 0$.

3.1 $baa = bab = 0$

If $baa = bab = 0$ then L_3 is generated by pba, p^2a and p^2b. If $pba = 0$ then p^2a and p^2b generate L_3. If $pba \neq 0$ we can assume that pba and p^2a generate L_3 and that $p^2b = \alpha pba + \beta p^2a$ for some α, β. Replacing b by $b - \beta a$ we can assume that $\beta = 0$, and we can take $\alpha = 0$ or 1. So we have three algebras here.

3.2 $baa \neq 0$, $bab = 0$

If $baa \neq 0$ and $bab = 0$, then at least one of p^2a, p^2b and pba must lie outside the subalgebra generated by baa.

First consider the case when L_3 is generated by baa and pba. Then

$$\begin{pmatrix} p^2a \\ p^2b \end{pmatrix} = A \begin{pmatrix} baa \\ pba \end{pmatrix}$$

for some 2×2 matrix A. If a' and b' generate L and satisfy the same relations as those already speciøed for a and b, then

$$\begin{aligned} a' &= \lambda a + \mu b, \\ b' &= \nu b \end{aligned}$$

modulo L_2 for some λ, μ, ν and

$$\begin{pmatrix} p^2a' \\ p^2b' \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix} A \begin{pmatrix} baa \\ pba \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix} A \begin{pmatrix} \lambda^2\nu & 0 \\ 0 & \lambda\nu \end{pmatrix}^{-1} \begin{pmatrix} b'a'a' \\ pb'a' \end{pmatrix}.$$

Now

$$\begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \lambda^2\nu & 0 \\ 0 & \lambda\nu \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\lambda u + \mu w}{\lambda^2\nu} & \frac{\lambda v + \mu x}{\lambda\nu} \\ \frac{w}{\lambda^2} & \frac{x}{\lambda} \end{pmatrix}$$

and so if $x \neq 0$ we can take $x = 1$, $v = 0$, $u = 0$ or 1 and $0 \leq w < p$ ($2p$ algebras). If $x = 0$, $w \neq 0$ we can take $u = 0$, $v = 0$ or 1 and $w = 1$ or ω (4 algebras). If $w = x = 0$ then we can take u and v independently equal to 0 or 1 (4 algebras).

Next, consider the case when pba is a scalar multiple of baa. Then we can assume that $pba = 0$ or baa, and we can assume that either L_3 is generated by baa and p^2b and $p^2a = \alpha baa$, or L_3 is generated by baa and p^2a and $p^2b = \beta baa$. In the ørst of these two case we can take $\alpha = 0$ or 1 and in the second we can take $\beta = 0, 1$ or ω if $pba = 0$, but we have to take $0 \leq \beta < p$ if $pba = baa$ ($p + 7$ algebras).

3.3 baa and bab generate L_3

We write

$$\begin{pmatrix} p^2a \\ p^2b \\ pba \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some 3×2 matrix A. If we let

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

modulo L_2 for some non-singular 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then

$$\begin{pmatrix} p^2a' \\ p^2b' \\ pb'a' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \alpha\delta - \beta\gamma \end{pmatrix} A \begin{pmatrix} baa \\ bab \end{pmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \alpha\delta - \beta\gamma \end{pmatrix} A \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}.$$

So we can assume that $pba = 0$ or baa.

If $pba = 0$ then by Theorem 5 we may take

$$\begin{pmatrix} p^2a \\ p^2b \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

where A is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or where A is a matrix of the form

$$\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices give isomorphic algebras, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ give isomorphic algebras. ($p + 7$ algebras.)

If $pba = baa$ then we need $\alpha = 1$ and $\beta = 0$ above, and so we have

$$\begin{pmatrix} p^2 a \\ p^2 b \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some 2×2 matrix A, with A and

$$\delta^{-1} \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix}^{-1}$$

giving isomorphic algebras. There are $p^2 + 4p$ orbits under this equivalence relation (and hence $p^2 + 4p$ algebras).

$$\delta^{-1} \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{-u\delta+\gamma v}{\delta^2} & \frac{1}{\delta^2} v \\ -\frac{-\delta\gamma u - w\delta^2 + \gamma^2 v + \gamma x \delta}{\delta^2} & \frac{\gamma v + x\delta}{\delta^2} \end{pmatrix}$$

If $v \neq 0$ we can take $u = 0$, which requires $\gamma = 0$ and gives

$$\delta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\delta^2} v \\ w & \frac{x}{\delta} \end{pmatrix}.$$

If $v = 0$ we have

$$\delta^{-1} \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u & 0 \\ w & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\delta} u & 0 \\ \frac{\gamma u + w\delta - x\gamma}{\delta} & \frac{x}{\delta} \end{pmatrix}$$

so we can take $u = x = 0$ or 1 with any w, or $u \neq x$ and $w = 0$ with $(u, x) = (0, 1)$ or $(1, 0)$ or $(1, x)$ with $1 < x < p$.

4 Immediate descendants of algebra 6 (5.38)

Algebra 5.38 has $\gcd(p - 1, 3)(p^2 + 3p + 10) + p + 6$ immediate descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7, 11, 13$, and have also checked that this is the right number of descendants for $p = 5, 7, 11, 13$.

Algebra 5.38 has presentation

$$\langle a, b \mid pa, pb, \text{ class } 3 \rangle,$$

and so if L is an immediate descendant of 5.38 of order p^7 then the commutator structure of L is the same as one of the three algebras 7.144 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we may assume that L_4 is generated by $baaa$, $baab$ and that $babb = \lambda baaa$ where $\lambda = 0, 1, \omega$. We also have

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baaa \\ baab \end{pmatrix}$$

for some 2×2 matrix A.

4.1 $babb = 0$

If a', b' generate L and satisfy the relation $b'a'b'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \gamma b \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} A \begin{pmatrix} baaa \\ baab \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha^3\gamma & 2\alpha^2\beta\gamma \\ 0 & \alpha^2\gamma^2 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a' \\ b'a'a'b' \end{pmatrix}.$$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^3\gamma & 2\alpha^2\beta\gamma \\ 0 & \alpha^2\gamma^2 \end{pmatrix}^{-1} = \alpha^{-3}\gamma^{-2} \begin{pmatrix} \gamma(\alpha u + \beta w) & -2\beta\alpha u - 2\beta^2 w + \alpha^2 v + \alpha\beta x \\ \gamma^2 w & -2\gamma w\beta + \alpha\gamma x \end{pmatrix}.$$

So we can take $w = 0$, 1 or (if $p = 1 \bmod 3$) ω or ω^2.

First consider the case when $w = 0$. We then have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u & v \\ 0 & x \end{pmatrix} \begin{pmatrix} \alpha^3\gamma & 2\alpha^2\beta\gamma \\ 0 & \alpha^2\gamma^2 \end{pmatrix}^{-1} = \alpha^{-2}\gamma^{-2} \begin{pmatrix} \gamma u & -2\beta u + \alpha v + \beta x \\ 0 & \gamma x \end{pmatrix}.$$

So we can take $u = 0$ or 1 and if $u = 0$ we can take $x = 0$ or 1. (We cannot alter the ratio of x to u.) If $x \neq 2u$ we can take $v = 0$. If $u = x = 0$ then we can take $v = 0$ or 1. And if $u = 1$, $x = 2$ then we need $\gamma = \alpha^{-2}$ so that $v \sim \alpha^3 v$ and we can take $v = 0$, 1 or (if $p = 1 \bmod 3$) ω or ω^2. ($p + 3 + \gcd(p-1, 3)$ algebras.)

Next consider the case when $w \neq 0$. Then we need $\alpha^3 = 1$ and we can take $u = 0$, though we then need $\beta = 0$. If $\alpha^3 = 1$ then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^3\gamma & 0 \\ 0 & \alpha^2\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{v}{\alpha\gamma^2} \\ w & \frac{x}{\alpha^2\gamma} \end{pmatrix}.$$

So we can take $x = 0$ or 1. Note that α is a square, so that if $x = 0$ we can take $v = 0$, 1 or ω. But if $x = 1$ then we need $\gamma = \alpha^{-2}$, and so we have to take $0 \leq v < p$. ($(p+3)\gcd(p-1, 3)$ algebras.)

So there are a total of $p + 3 + (p+4)\gcd(p-1, 3)$ algebras when $babb = 0$.

4.2 $babb = baaa$

If a', b' generate L and satisfy the relation $b'a'b'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \pm\beta a \pm \alpha a \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{pmatrix} A \begin{pmatrix} baaa \\ baab \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm(\alpha^4 - \beta^4) & \pm 2\alpha\beta(\alpha^2 - \beta^2) \\ 2\alpha\beta(\alpha^2 - \beta^2) & \alpha^4 - \beta^4 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a' \\ b'a'a'b' \end{pmatrix}.$$

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} (\alpha^4 - \beta^4) & 2\alpha\beta(\alpha^2 - \beta^2) \\ 2\alpha\beta(\alpha^2 - \beta^2) & \alpha^4 - \beta^4 \end{pmatrix}^{-1}$$

$$= (\alpha^2 - \beta^2)^{-3} \begin{pmatrix} \alpha u\beta^2 + \alpha^3 u + \beta^3 w + \beta w\alpha^2 - 2\beta\alpha^2 v - 2\beta^2\alpha x & -2\beta\alpha^2 u - 2\beta^2\alpha w + \alpha v\beta^2 + \alpha^3 v + \beta^3 x + \beta x\alpha^2 \\ \beta^3 u + \beta\alpha^2 u + \beta^2\alpha w + \alpha^3 w - 2\alpha v\beta^2 - 2\beta x\alpha^2 & -2\alpha u\beta^2 - 2\beta w\alpha^2 + \beta^3 v + \beta\alpha^2 v + \beta^2\alpha x + x\alpha^3 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} -(\alpha^4-\beta^4) & -2\alpha\beta(\alpha^2-\beta^2) \\ 2\alpha\beta(\alpha^2-\beta^2) & \alpha^4-\beta^4 \end{pmatrix}^{-1}$$

$$= (\alpha^2-\beta^2)^{-3} \begin{pmatrix} -\beta^3 w - \alpha u\beta^2 + 2\beta^2\alpha x + 2\beta\alpha^2 v - \beta w\alpha^2 - \alpha^3 u & -2\beta\alpha^2 u - 2\beta^2\alpha w + \alpha v\beta^2 + \alpha^3 v + \beta^3 x + \beta x\alpha^2 \\ \beta^3 u + \beta\alpha^2 u + \beta^2\alpha w + \alpha^3 w - 2\alpha v\beta^2 - 2\beta x\alpha^2 & 2\alpha u\beta^2 + 2\beta w\alpha^2 - \beta^3 v - \beta\alpha^2 v - \beta^2\alpha x - x\alpha^3 \end{pmatrix}$$

From orbit stabilizer calculations we see that there are

$$\frac{(p^2+3p+11)\gcd(p-1,3)+1}{2}$$

algebras here.

4.3 $babb = \omega baaa$

If a', b' generate L and satisfy the relation $b'a'b'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \pm\omega\beta a \pm \alpha a \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix} A \begin{pmatrix} baaa \\ baab \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm(\alpha^4-\omega^2\beta^4) & \pm 2\alpha\beta(\alpha^2-\omega\beta^2) \\ 2\omega\alpha\beta(\alpha^2-\omega\beta^2) & \alpha^4-\omega^2\beta^4 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a' \\ b'a'a'b' \end{pmatrix}$$

From orbit stabilizer calculations we see that there are

$$\frac{\gcd(p-1,3)(p^2+p+1)+5}{2}$$

algebras here.

5 Immediate descendants of algebra 21 (5.60)

Algebra 5.60 has $2p^2+p+3+2(p+1)\gcd(p-1,3)+(2p+4)\gcd(p-1,4)+\gcd(p-1,8)$ descendants of order p^7 and class 5. I have checked that the recipes below give this number of non-isomorphic groups for $p = 7, 11, 13, 17$, and have also checked that this is the right number of descendants for $p = 7$ (out of space at $p = 11$).

Algebra 5.60 has presentation

$$\langle a, b \mid bab, pa, pb, \text{ class } 4\rangle.$$

Let L be an immediate descendant of 5.60 of order p^7. Then L_5 has order p^2 and is generated by $baaaa$, $baaab$, and we have

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baaaa \\ baaab \end{pmatrix}$$

for some 2×2 matrix A. We also have $bab \in L^4$, but by adding a suitable scalar multiple of baa to b we can assume that $bab = 0$ or $baaaa$.

5.1 $bab = 0$

If a', b' generate L and satisfy the relation $b'a'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \gamma b \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} A \begin{pmatrix} baaaa \\ baaab \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha^4\gamma & \alpha^3\beta\gamma \\ 0 & \alpha^3\gamma^2 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a'a' \\ b'a'a'a'b' \end{pmatrix}.$$

Now

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^4\gamma & \alpha^3\beta\gamma \\ 0 & \alpha^3\gamma^2 \end{pmatrix}^{-1} = \alpha^{-4}\gamma^{-2} \begin{pmatrix} \gamma(\alpha u + \beta w) & -\beta\alpha u - \beta^2 w + \alpha^2 v + \alpha\beta x \\ \gamma^2 w & -\gamma w\beta + \alpha\gamma x \end{pmatrix}.$$

So we can take $w = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3.

First consider the case when $w = 0$. Then we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u & v \\ 0 & x \end{pmatrix} \begin{pmatrix} \alpha^4\gamma & \alpha^3\beta\gamma \\ 0 & \alpha^3\gamma^2 \end{pmatrix}^{-1} = \alpha^{-4}\gamma^{-2} \begin{pmatrix} \gamma\alpha u & -\beta\alpha u + \alpha^2 v + \alpha\beta x \\ 0 & \alpha\gamma x \end{pmatrix}.$$

So we can take (u, x) eqal to $(0,0)$, $(0,1)$, $(1,x)$ with $0 \le x < p$. (We cannot alter the ratio $u : x$.) If $u \ne x$ we can take $v = 0$. If $u = x = 0$ we can take $v = 0, 1$ or ω, and if $u = x = 1$ we can take $v = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. ($p + 4 + \gcd(p-1, 4)$ algebras.)

On the other hand, if $w \ne 0$ then we can take $u = 0$, though we then need $\beta = 0$ and $\alpha^4 = 1$. So we have

$$\alpha^4\gamma^2 \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^4\gamma & 0 \\ 0 & \alpha^3\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \alpha^2\gamma^{-2}v \\ w & \alpha\gamma^{-1}x \end{pmatrix}.$$

We can take $x = 0$ or 1. If $x = 0$ we can take $v = 0, 1$ or ω, but if $x = 1$ we have to take $0 \le v < p$. ($\gcd(p-1, 4)(p+3)$ algebras.

So we have a total of $(p+4)(1 + \gcd(p-1, 4))$ algebras when $bab = 0$.

5.2 $bab = baaaa$

If a', b' generate L and satisfy the relation $b'a'b' = b'a'a'a'a'$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \alpha^3 b \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^3 \end{pmatrix} A \begin{pmatrix} baaaa \\ baaab \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^3 \end{pmatrix} A \begin{pmatrix} \alpha^7 & \alpha^6\beta \\ 0 & \alpha^9 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a'a' \\ b'a'a'a'b' \end{pmatrix}.$$

Now

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^7 & \alpha^6\beta \\ 0 & \alpha^9 \end{pmatrix}^{-1} = \alpha^{-10} \begin{pmatrix} \alpha^3(\alpha u + \beta w) & -\beta\alpha u - \beta^2 w + \alpha^2 v + \alpha\beta x \\ \alpha^6 w & -\alpha^3 w\beta + x\alpha^4 \end{pmatrix}.$$

So we can take $w = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3.

First consider the case when $w = 0$. Then we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} u & v \\ 0 & x \end{pmatrix} \begin{pmatrix} \alpha^7 & \alpha^6\beta \\ 0 & \alpha^9 \end{pmatrix}^{-1} = \alpha^{-10} \begin{pmatrix} \alpha^4 u & -\beta\alpha u + \alpha^2 v + \alpha\beta x \\ 0 & x\alpha^4 \end{pmatrix},$$

and so we can take (u, x) equal to $(0, x)$ where $x = 0, 1, \omega$ or (if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5 or (u, x) where $u = 1, \omega$ or (if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5 and $0 \leq x < p$. (We cannot alter the ratio $u : x$.) If $u \neq x$ we can take $v = 0$. If $u = x = 0$ we can take $v = 0, 1, \omega$ if $p = 3 \bmod 8$ or $7 \bmod 8$, and $v = 0, 1, \omega, \omega^2, \omega^3$ if $p = 5 \bmod 8$, and $v = 0, 1, \omega, \ldots, \omega^7$ if $p = 1 \bmod 8$. And if $u = x \neq 0$ then we need $\alpha^6 = 1$ so we can take $v = 0$ or v in a transversal for the cube roots of unity in the multiplicative group of non-zero elements in $\mathbb{Z}_p$. ($2p - 1 + 2(p + 1)\gcd(p - 1, 3) + \gcd(p - 1, 8)$ algebras.)

Next consider the case when $w \neq 0$. Then we can take $u = 0$, though we need $\beta = 0$ and $\alpha^4 = 1$.

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^7 & 0 \\ 0 & \alpha^9 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & v \\ w & x\alpha^{-2} \end{pmatrix}.$$

If $p = 1 \bmod 4$ we can take $\alpha^2 = \pm 1$ so we have $2p(p + 1)$ algebras, and if $p = 3 \bmod 4$ then we need $\alpha^2 = 1$ and so we have $2p^2$ algebras. So we have $2p^2 + p(\gcd(p - 1, 4) - 2)$ algebras.

So when $bab = baaaa$ we have $2p^2 - 1 + 2(p+1)\gcd(p-1, 3) + p\gcd(p-1, 4) + \gcd(p-1, 8)$ algebras.

6 Immediate descendants of algebra 22 (5.65)

Algebra 5.65 has presentation

$$\langle a, b \mid bab - baaa, pa, pb, \text{ class } 4 \rangle.$$

Let L be an immediate descendant of 5.60 of order p^7. Then L_5 has order p^2 and is generated by $baaaa$, $baaab$, and we have

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baaaa \\ baaab \end{pmatrix}$$

for some 2×2 matrix A. We also have $bab - baaa \in L^4$, but by adding a suitable scalar multiple of b to a, and adding a suitable scalar multiple of baa to b we can assume that $bab = baaa$.

If a', b' generate L and satisfy the relation $b'a'b' = b'a'a'a'$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^2 b \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} baaaa \\ baaab \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha^6 & 0 \\ 0 & \alpha^7 \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a'a' \\ b'a'a'a'b' \end{pmatrix}.$$

Now

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^6 & 0 \\ 0 & \alpha^7 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-5}u & \alpha^{-6}v \\ \alpha^{-4}w & \alpha^{-5}x \end{pmatrix}.$$

First consider the case when $w \neq 0$. Then we can assume that $w = 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, though we then need $\alpha^4 = 1$. If $u \neq 0$ then we can take u in a transversal for the 4th roots of unity, and take v, x arbitrary ($p^2(p-1)$ algebras). If $u = 0$ and $x \neq 0$ we can take x in a transveral for the fourth roots of unity and v arbitrary ($p(p-1)$ algebras). And if $u = x = 0$ and $v \neq 0$ then we can take v in a transversal for squares of fourth roots of unity ($2(p-1)$ algebras. If $u = v = x = 0$ we have $\gcd(p-1, 4)$ algebras. So there are a total of $(p^2 + p + 2)(p-1) + \gcd(p-1, 4)$ algebras when $w \neq 0$.

Next consider the case when $w = 0$, $u \neq 0$. Then we can take $u = 1$ or (if $p = 1 \bmod 5$) $\omega, \omega^2, \omega^3$ or ω^4. We then have x arbitrary and $v = 0$, or v in a transversal for the 5th roots of unity. ($p(p-1) + p\gcd(p-1, 5)$ algebras.)

If $w = u = 0$ and $x \neq 0$ then we can take $x = 1$ or (if $p = 1 \bmod 5$) $\omega, \omega^2, \omega^3$ or ω^4, and $v = 0$ or v in a transversal for the 5th roots of unity. ($p - 1 + \gcd(p-1, 5)$ algebras.)

And ønally, if $w = u = x = 0$ we can take $v = 0, 1, \omega$ or (if $p = 1 \bmod 3$) $\omega^2, \omega^3, \omega^4$ or ω^5. ($1 + 2\gcd(p-1, 3)$ algebras.

So we have a total of $p^3 + p^2 + p - 2 + 2\gcd(p-1, 3) + \gcd(p-1, 4) + (p+1)\gcd(p-1, 5)$ algebras. I have checked that the recipes above give this number of non-isomorphic groups for $p = 7, 11, 13, 17$, and have also checked that this is the right number of descendants for $p = 7$ (out of space at $p = 11$).

7 Immediate descendants of algebra 23 (3.1)

The total number of descendants of 3.1 of order p^7 is $p + 14$. This has been checked for lots of smallish primes. Let L be an immediate descendant of 3.1 of order p^7. Then L is generated by a, b, c, L has class 2, and L_2 has order p^4. There are 3 possible commutator structures on L, corresponding to L^2 having order p, p^2 or p^3.

7.1 L^2 has order p

$$\langle a, b, c \mid ca, cb, \text{ class } 2\rangle$$

7.2 L^2 has order p^2

We can assume that L^2 is generated by ba, ca and that $cb = 0$. Note that $\langle b, c\rangle + L_2$ is a characteristic subalgebra.

We ørst consider the case when pb, pc are linearly independent modulo L^2. Then we can assume that $pa \in L^2$, and hence that $pa = 0$ or ba. So we have

$$\langle a, b, c \mid cb, pa, \text{ class } 2\rangle,$$

$$\langle a, b, c \mid cb, pa - ba, \text{ class } 2\rangle.$$

Next we consider the case when some linear combination of pb, pc lies in L^2. We may then assume that $pb \in L^2$. We can also assume that pa, pc are linearly independent modulo L^2. If a', b', c' generate L, and if $c'b' = 0$ and $pb' \in L^2$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \mu b + \nu c \end{aligned}$$

modulo L^2. So

$$\begin{aligned} b'a' &= \alpha\delta ba, \\ c'a' &= \alpha\mu ba + \alpha\nu ca. \end{aligned}$$

Hence we can assume that $pb = 0, ba$ or ca, giving

$$\langle a, b, c \,|\, cb, pb, \text{ class } 2\rangle,$$

$$\langle a, b, c \,|\, cb, pb - ba, \text{ class } 2\rangle,$$

$$\langle a, b, c \,|\, cb, pb - ca, \text{ class } 2\rangle.$$

7.3 L^2 has order p^3

If L^2 has order p^3 then L^2 is spanned by ba, ca, cb, and we may assume that $pa \notin L^2$, $pb, pc \in L^2$. If a', b', c' generate L, and if $pb', pc' \in L^2$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \mu b + \nu c \end{aligned}$$

modulo L^2. So

$$\begin{aligned} b'a' &= \alpha\delta ba + \alpha\varepsilon ca + (\beta\varepsilon - \gamma\delta)cb, \\ c'a' &= \alpha\mu ba + \alpha\nu ca + (\beta\nu - \gamma\mu)cb, \\ c'b' &= (\delta\nu - \varepsilon\mu)cb. \end{aligned}$$

One possibility is that $pb, pc = 0$, giving

$$\langle a, b, c \,|\, pb, pc, \text{ class } 2\rangle,$$

And if pb, pc are linearly dependent then we may assume that $pc = 0$ and that $pb = ba, ca$ or cb, giving

$$\langle a, b, c \,|\, pb - ba, pc, \text{ class } 2\rangle,$$

$$\langle a, b, c \,|\, pb - ca, pc, \text{ class } 2\rangle,$$

$$\langle a, b, c \,|\, pb - cb, pc, \text{ class } 2\rangle.$$

So suppose that pb, pc are linearly independent. If there is some linear combination of pb, pc in $\langle cb\rangle$ then we may assume that $pc = cb$ and that $pb = ba$ or ca, giving

$$\langle a, b, c \,|\, pb - ba, pc - cb, \text{ class } 2\rangle,$$

$$\langle a, b, c \,|\, pb - ca, pc - cb, \text{ class } 2\rangle.$$

If no linear combination of pb, pc lies in $\langle cb\rangle$ then we may assume that

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some non-singular matrix A. Then if we take a', b', c', d' as above we see that

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = PA \begin{pmatrix} ba \\ ca \end{pmatrix} = \alpha^{-1}PAP^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix}$$

where $P = \begin{pmatrix} \delta & \varepsilon \\ \mu & \nu \end{pmatrix}$. So by Theorem 6 we can take A to be one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices are equivalent to each other, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. An alternative formulation of this (using the rational canonical form in every case) is that we can take A to be one of the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 1 & 1 \end{pmatrix} (\lambda \neq 0).$$

So we have

$$\langle a, b, c \,|\, pb - ba, pc - ca, \text{ class } 2\rangle,$$
$$\langle a, b, c \,|\, pb - ca, pc - ba, \text{ class } 2\rangle,$$
$$\langle a, b, c \,|\, pb - \omega ca, pc - ba, \text{ class } 2\rangle,$$
$$\langle a, b, c \,|\, pb - \lambda ca, pc - ba - ca, \text{ class } 2\rangle \; (0 < \lambda < p).$$

8 Immediate descendants of algebra 25 (5.8)

Algebra 5.8 has $p + 8$ descendants of order p^7 and p-class 3.

Algebra 5.8 has presentation

$$\langle a, b, c \,|\, ba,\ ca,\ cb,\ pc, \text{ class } 2\rangle,$$

so if L is an immediate descendant of 5.8 of order p^7 then L_2 is generated by pa, pb modulo L_3 and L_3 has order p^2 and is generated by p^2a, p^2b. We have $ba, ca, cb, pc \in L_3$, but by adding suitable scalar multiples of pa and pb to c we can assume that $pc = 0$. If a', b', c' generate L, and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \lambda a + \mu b + \nu c, \\ c' &= \xi c \end{aligned}$$

modulo L_2, and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \alpha\mu - \beta\lambda & \alpha\nu - \gamma\lambda & \beta\nu - \gamma\mu \\ 0 & \alpha\xi & \beta\xi \\ 0 & \lambda\xi & \mu\xi \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

So if

$$\begin{pmatrix} ba \\ ca \\ cb \end{pmatrix} = A \begin{pmatrix} p^2a \\ p^2b \end{pmatrix}$$

for some 3×2 matrix a over $\mathbb{Z}_p$ then

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \alpha\mu - \beta\lambda & \alpha\nu - \gamma\lambda & \beta\nu - \gamma\mu \\ 0 & \alpha\xi & \beta\xi \\ 0 & \lambda\xi & \mu\xi \end{pmatrix} A \begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix}^{-1} \begin{pmatrix} p^2a' \\ p^2b' \end{pmatrix}.$$

First consider the case when ca, cb span L_3. We can then choose γ, ν so that $ba = 0$, and by Theorem 6 we can take

$$\begin{pmatrix} ca \\ cb \end{pmatrix} = A \begin{pmatrix} p^2a \\ p^2b \end{pmatrix}$$

where A is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix} \; (x \neq 0).$$

This gives

$$\langle a, b, c \,|\, ba,\; ca - p^2a,\; cb - p^2b,\; pc,\; \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, ba,\; ca - p^2b,\; cb - p^2a,\; pc,\; \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, ba,\; ca - \omega p^2b,\; cb - p^2a,\; pc,\; \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, ba,\; ca - xp^2b,\; cb - p^2a - p^2b,\; pc,\; \text{class } 3\rangle \; (0 < x < p).$$

Next, consider the case when ca, cb span a one dimensional space. Then the centralizer of c in the linear span of a, b has order p, and so we can assume that $ca = 0$, though we then need $\beta = 0$. So we have

$$\begin{aligned} p^2a' &= \alpha p^2a, \\ p^2b' &= \lambda p^2a + \mu p^2b. \end{aligned}$$

So we can assume that $cb = p^2a$ or p^2b.

First consider the case when $ca = 0$, $cb = p^2a$. Then we need $\alpha = \mu\xi$, $\beta = 0$, so if $ba = \rho p^2a + \sigma p^2b$ then

$$b'a' = \mu^2\xi ba - \gamma\mu p^2a = (\mu^2\xi\rho - \gamma\mu)p^2a + \mu^2\xi\sigma p^2b.$$

So we can assume that $ba = 0$ or p^2b:

$$\langle a, b, c \,|\, ba,\; ca,\; cb - p^2a,\; pc,\; \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, ba - p^2b,\; ca,\; cb - p^2a,\; pc,\; \text{class } 3\rangle.$$

Next consider the case when $ca = 0$, $cb = p^2b$. Then we need $\beta = \lambda = 0$ and $\xi = 1$, and o if $ba = \rho p^2a + \sigma p^2b$ then

$$b'a' = \alpha\mu ba - \gamma\mu p^2b = \alpha\mu\rho p^2a + (\alpha\mu\sigma - \gamma\mu)p^2b.$$

So we can assume that $ba = 0$ or p^2a:

$$\langle a, b, c \,|\, ba,\; ca,\; cb - p^2b,\; pc,\; \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, ba - p^2a,\; ca,\; cb - p^2b,\; pc,\; \text{class } 3\rangle.$$

Finally, if $ca = cb = 0$ we can assume that $ba = 0$ or p^2a, giving

$$\langle a, b, c \,|\, ba,\; ca,\; cb,\; pc,\; \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, ba - p^2a,\; ca,\; cb,\; pc,\; \text{class } 3\rangle.$$

9 Immediate descendants of algebra 26 (5.9)

Algebra 5.9 has $4p^2 + 26p + 107 + 5\gcd(p-1,3) + (p+4)\gcd(p-1,4)$ descendants of order p^7 and p-class 3.

Algebra 5.9 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pb,\ pc,\ \text{class } 2\rangle,$$

and so if L is a descendant of 5.9 of order p^7 then L_2 is generated by ba, pa modulo L_3 and L has order p^2 and is generated by baa, bab, and p^2a. We also have $ca, cb, pb, pc \in L_3$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \mu b + \nu c, \\ c' &= \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa + \alpha\beta\mu bab, \\ b'a'b' &= \alpha\mu^2 bab, \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

So we can assume that $baa = 0$ or $bab = 0$, or that $p^2a = 0$, baa, bab or ωbab.

9.1 $baa = 0$

If $baa = 0$ then L_3 is generated by bab and p^2a, though we need $\beta = 0$. Adding a suitable scalar multiple of ba to c we may assume that $cb = \rho p^2 a$, and adding suitable scalar multiple of pa to b and c we can assume that $pb = \sigma bab$, $pc = \tau bab$. We then have

$$\begin{aligned} b'a'b' &= \alpha\mu^2 bab, \\ p^2a' &= \alpha p^2 a, \\ c'a' &= \alpha\xi ca, \\ c'b' &= \mu\xi\rho p^2 a, \\ pb' &= (\mu\sigma + \nu\tau) bab, \\ pc' &= \xi\tau bab. \end{aligned}$$

So we can assume that $pb = pc = 0$, or that $pb = 0$, $pc = bab$, or that $pb = bab$, $pc = 0$, and for every one of these possibilities we can assume that $cb = 0$ or p^2a, except that if $pb = 0$, $pc = bab$ and if $p = 1 \bmod 3$ we also have the possibility that $cb = \omega p^2 a$ or $\omega^2 p^2 a$. Let $ca = xbab + yp^2a$, so that

$$c'a' = \mu^{-2}\xi x b'a'b' + \xi y p^2 a'.$$

First suppose that $cb = pb = pc = 0$. Then we can take $y = 0$ or 1. If $y = 0$ we can take $x = 0$ or 1, and if $y = 1$ we can take $x = 0, 1, \omega$.

Next, suppose that $cb = p^2a$ and that $pb = pc = 0$. Then we need $\alpha = \mu\xi$, but as above we can take $y = 0$ or 1, and if $y = 0$ we can take $x = 0$ or 1, and if $y = 1$ we can take $x = 0, 1, \omega$.

Now consider the case when $cb = pb = 0$, $pc = bab$. We need $\xi = \alpha\mu^2$, but as above we can take $y = 0$ or 1, and if $y = 0$ we can take $x = 0$ or 1, and if $y = 1$ we can take $x = 0, 1, \omega$.

Next suppose that $cb = kp^2a$, $pb = 0$, $pc = bab$, where $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. We need $\alpha = \mu\xi$, $\xi = \alpha\mu^2$ which implies that $\mu^3 = 1$. So we can take $y = 0, 1$ and if $y = 0$ we can take $x = 0, 1$. But if $y = 1$ then we need to take $x = 0$ or x in a transversal for the cube roots of unity.

Now consider the case when $cb = 0$, $pb = bab$, $pc = 0$. We need $\alpha = \mu^{-1}$, but as above we can take $y = 0$ or 1, and if $y = 0$ we can take $x = 0$ or 1, and if $y = 1$ we can take $x = 0, 1, \omega$.

Finally, consider the case when $cb = p^2a$, $pb = bab$, $pc = 0$. Then we need $\alpha = \mu\xi$ and $\alpha = \mu^{-1}$, so that $\xi = \mu^{-2}$. So we can take $y = 0, 1, \omega$. If $y = 0$ we can take $x = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, and if $y = 1, \omega$ we can take $0 \leq x < p$.

So there are $3p + 20 + 3\gcd(p-1, 3) + \gcd(p-1, 4)$ algebras here.

9.2 $bab = 0$

If $bab = 0$ then L_3 is generated by baa and p^2a. Adding suitable scalar multiple of pa to b and c we can assume that $pb = \sigma baa$, $pc = \tau baa$. If we then let

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \mu b + \nu c + \delta ba, \\ c' &= \xi c \end{aligned}$$

modulo L_3, We have

$$\begin{aligned} b'a'a' &= \alpha^2 \mu baa, \\ p^2 a' &= \alpha p^2 a, \\ c'a' &= \alpha\xi ca + \beta\xi cb + \alpha\delta baa, \\ c'b' &= \mu\xi cb, \\ pb' &= (\mu\sigma + \nu\tau) baa, \\ pc' &= \xi\tau baa. \end{aligned}$$

So we can assumne that $pb = pc = 0$ or that $pb = 0$, $pc = baa$ or that $pb = baa$ or ωbaa, $pc = 0$. If $cb = \rho baa + \sigma p^2a$ then

$$c'b' = \alpha^{-2}\xi\rho b'a'a' + \alpha^{-1}\mu\xi\sigma p^2a'.$$

So if $pb = pc = 0$ we can take $\rho, \sigma = 0, 1$. If $pb = 0$, $pc = baa$ then we need $\xi = \alpha^2\mu$ and again we can take $\rho, \sigma = 0, 1$. And if $pb = baa$ or ωbaa, $pc = 0$ then we need $\alpha = \pm 1$ and so once again we can take $\rho, \sigma = 0, 1$.

If $\sigma \neq 0$ we can take $ca = 0$. And if $\sigma = 0$ then we can take $ca = \tau p^2a$, which taking $\beta = \delta = 0$ gives

$$c'a' = \xi\tau p^2a'.$$

So if $pb = pc = \rho = \sigma = 0$ we can take $ca = 0$ or p^2a. If $pb = pc = \sigma = 0$, $\rho = 1$ then we need $\xi = \alpha^2$ so we can take $ca = 0$, p^2a or ωp^2a. If $pb = 0$, $pc = baa$, $\sigma = 0$ then we need $\xi = \alpha^2\mu$ so if $\rho = 0$ we can take $ca = 0$ or p^2a and if $\rho = 1$ we can take $ca = 0$, p^2a or ωp^2a. Finally if $pb = baa$ or ωbaa, $pc = 0$, $\sigma = 0$ then we need $\alpha = \pm 1$ so that if $\rho = 0$ we can take $ca = 0$ or p^2a and if $\rho = 1$ we can take $ca = xp^2a$ with $0 \leq x < p$.

So there are $2p + 22$ algebras here.

9.3 $p^2a = 0$

If $p^2a = 0$ then L_3 is generated by baa, bab. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \mu b + \nu c \text{ modulo } L_2, \\ c' &= \xi c + \eta ba \text{ modulo } L_3, \end{aligned}$$

and

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa + \alpha\beta\mu bab, \\ b'a'b' &= \alpha\mu^2 bab, \\ c'a' &= \alpha\xi ca + \beta\xi cb + \alpha\eta baa + \beta\eta bab, \\ c'b' &= \mu\xi cb + \mu\eta bab, \\ pb' &= \mu pb + \nu pc, \\ pc' &= \xi pc. \end{aligned}$$

Taking $\beta = 0$ we can choose η so that $c'a'$ is a scalar multiple of $b'a'b'$, and it follows that we can take $ca = 0$ or bab.

First consider the case when $ca = 0$. Let $cb = xbaa + ybab$. Then to ensure that $c'a' = 0$ we need $\beta = \eta = 0$ or $\eta = -\xi y$ and $\beta x = \alpha y$. First consider a', b', c' as above, with $\beta = \eta = 0$. Then

$$c'b' = \mu\xi xbaa + \mu\xi ybab = \alpha^{-2}\xi xb'a'a' + \alpha^{-1}\mu^{-1}\xi yb'a'b',$$

and so we can assume $cb = 0$, baa, bab or $baa + bab$. We now consider non-zero possibilities for β, η with these values of cb.

If $ca = cb = 0$ then we need $\eta = 0$ to ensure that $c'a' = 0$, though β can take any value.

If $ca = 0$, $cb = baa$ then we need $\beta = \eta = 0$ to ensure that $c'a' = 0$. This gives $c'b' = \mu\xi baa$, and so we also need $\xi = \alpha^2$ to ensure that $c'b' = b'a'a'$.

If $ca = 0$, $cb = bab$ then we again need $\beta = \eta = 0$ to ensure that $c'a' = 0$. This gives $c'b' = \mu\xi bab$, and so we need $\xi = \alpha\mu$ to ensure that $c'b' = b'a'a'$.

And if $ca = 0$, $cb = baa + bab$ then we need $\beta = \eta = 0$ or $\eta = -\xi$, $\beta = \alpha$ to ensure that $c'a' = 0$. If $\beta = \eta = 0$ then we have $c'b' = \mu\xi baa + \mu\xi bab$ and so we need $\alpha = \mu$, $\xi = \alpha^2$ to ensure that $c'b' = b'a'a' + b'a'b'$. And if $\eta = -\xi$, $\beta = \alpha$ then we have

$$c'b' = \mu\xi baa = \alpha^{-2}\xi b'a'a' - \alpha^{-1}\mu^{-1}\xi b'a'b'$$

and so we need $\xi = \alpha^2$, $\mu = -\alpha$ to ensure that $c'b' = b'a'a' + b'a'b'$.

Now consider the case when $ca = bab$. As above, let $cb = xbaa + ybab$ and consider a', b', c', with $\beta = \eta = 0$. Then

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa, \\ b'a'b' &= \alpha\mu^2 bab, \\ c'a' &= \alpha\xi bab, \\ c'b' &= \mu\xi xbaa + \mu\xi ybab. \end{aligned}$$

To ensure that $c'a' = b'a'b'$ we need $\xi = \mu^2$, and we then have

$$c'b' = \mu^3 xbaa + \mu^3 ybab = \alpha^{-2}\mu^2 xb'a'a' + \alpha^{-1}\mu yb'a'b'.$$

So we can take $y = 0$ or 1, and if $y = 0$ we can take $x = 0, 1, \omega$.

First consider the case when $ca = bab$, $cb = xbaa$. Then taking arbitrary values for β, μ we have

$$c'a' = \alpha\xi bab + \beta\xi xbaa + \alpha\eta baa + \beta\eta bab.$$

So if $x = 0$ we need $\eta = 0$, and we cannot ønd a', b', c' so that $c'a' = 0$, and to ensure that $c'a' = b'a'b'$ we need $\xi = \mu^2$. If $x = 1$ then taking $\alpha = \beta = \xi = 1$, and taking $\eta = -1$, we have $c'a' = 0$, and we are back in the case dealt with above. And if $x = \omega$ then we cannot ønd a', b', c' so that $c'a' = 0$, though we need $\eta = -\alpha^{-1}\beta\xi\omega$, $\xi = \frac{\mu^2}{1-\alpha^{-2}\beta^{-2}\omega}$ to ensure that $c'a' = b'a'b'$. We then have

$$c'b' = \mu\xi\omega baa - \mu\alpha^{-1}\beta\xi\omega bab$$

which can only equal $\omega b'a'a'$ if $\beta = \eta = 0$ and $\mu = \pm\alpha$.

Next consider the case when $ca = bab$ and $cb = xbaa + bab$. As above, taking arbitrary values for β, μ we have

$$c'a' = \alpha\xi bab + \beta\xi xbaa + \beta\xi bab + \alpha\eta baa + \beta\eta bab.$$

If $1 + 4x$ is a square we can arrange for $c'a' = 0$, so that we are back in an earlier case. In particular, if $x = 0$ we are back in an earlier case. If $x \neq 0$, then we take $\alpha = \mu = \xi = 1$, $\beta = \frac{1}{2}x^{-1}$ and $\eta = -\frac{1}{2}$. This gives

$$\begin{aligned}
b'a'a' &= baa + \frac{1}{2}x^{-1}bab, \\
b'a'b' &= bab, \\
c'a' &= bab + \frac{1}{2}baa + \frac{1}{2}x^{-1}bab - \frac{1}{2}baa - \frac{1}{4}x^{-1}bab = (1 + \frac{1}{4}x^{-1})b'a'b', \\
c'b' &= xbaa + bab - \frac{1}{2}bab = xb'a'a',
\end{aligned}$$

So when $ca = bab$ we can assume that $ba = 0$ or ωbaa.

So we can assume that one of the following holds:

- $ca = cb = 0$, with $\eta = 0$,
- $ca = 0$, $cb = bab$, with $\beta = \eta = 0$ and $\xi = \alpha\mu$,
- $ca = 0$, $cb = baa$, with $\beta = \eta = 0$ and $\xi = \alpha^2$,
- $ca = 0$, $cb = baa + bab$, with $\beta = \eta = 0$ and $\alpha = \mu$, $\xi = \alpha^2$ or $\eta = -\xi$, $\beta = \alpha$ and $\xi = \alpha^2$, $\mu = -\alpha$
- $ca = bab$, $cb = 0$, with $\eta = 0$ and $\xi = \mu^2$,
- $ca = bab$, $cb = \omega baa$, with $\beta = \eta = 0$ and $\xi = \mu^2 = \alpha^2$.

Consider the case when $ca = cb = 0$ and $\eta = 0$. Then

$$\begin{aligned}
b'a'a' &= \alpha^2\mu baa + \alpha\beta\mu bab, \\
b'a'b' &= \alpha\mu^2 bab, \\
pb' &= \mu pb + \nu pc, \\
pc' &= \xi pc.
\end{aligned}$$

So we can assume that $pc = 0$, baa or bab. If $pc = 0$ we can assume that $pb = 0$, baa, ωbaa or bab. If $pc = baa$ then we can assume that $pb = 0$ or bab. And if $pc = bab$ then we can assume that $pb = 0$, baa or ωbaa. So we have 9 algebras here.

Next, consider the case when $ca = 0$, $cb = bab$, with $\beta = \eta = 0$ and $\xi = \alpha\mu$. We have

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa, \\ b'a'b' &= \alpha\mu^2 bab, \\ pb' &= \mu pb + \nu pc, \\ pc' &= \alpha\mu pc. \end{aligned}$$

We can assume that $pc = 0$, bab, baa or $baa + bab$. If $pc = 0$ then we can assume that $pb = xbaa + ybab$ where $x = 0, 1, \omega$ and $y = 0, 1$. If $pc = bab$ then we can assume that $pb = 0$, baa or ωbaa. If $pc = baa$ then we can assume that $pb = 0$ or bab. And if $pc = baa + bab$ then we can assume that $pb = xbab$ with $0 \le x < p$. So there are $p + 11$ algebras here.

And now consider the case when $ca = 0$, $cb = baa$, with $\beta = \eta = 0$ and $\xi = \alpha^2$. Then

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa, \\ b'a'b' &= \alpha\mu^2 bab, \\ pb' &= \mu pb + \nu pc, \\ pc' &= \alpha^2 pc. \end{aligned}$$

We can assume that $pc = 0$, bab, baa or $baa + bab$. If $pc = 0$ then we can assume that $pb = xbaa + ybab$ where $x = 0, 1, \omega$ and $y = 0, 1$. If $pc = bab$ then we can assume that $pb = 0$, baa or ωbaa or (if $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. If $pc = baa$ then we can assume that $pb = 0$ or bab. And if $pc = baa + bab$ then we can assume that $pb = xbab$ with $0 \le x < p$. So there are $p + 9 + \gcd(p-1, 4)$ algebras here.

Next consider the case when $ca = 0$, $cb = baa + bab$, with $\beta = \eta = 0$ and $\alpha = \mu$, $\xi = \alpha^2$ or $\eta = -\xi$, $\beta = \alpha$ and $\xi = \alpha^2$, $\mu = -\alpha$. We have

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ b'a'b' &= \alpha^3 bab, \\ pb' &= \alpha pb + \nu pc, \\ pc' &= \alpha^2 pc. \end{aligned}$$

or

$$\begin{aligned} b'a'a' &= -\alpha^3 baa - \alpha^3 bab, \\ b'a'b' &= \alpha^3 bab, \\ pb' &= -\alpha pb + \nu pc, \\ pc' &= \alpha^2 pc. \end{aligned}$$

Let $pc = xbaa + ybab$. Then under the ørst transformation

$$pc' = \alpha^{-1}xbaa + \alpha^{-1}ybab$$

and under the second transformation

$$pc' = -\alpha^{-1}xb'a'a' + \alpha^{-1}(y - x)b'a'b'.$$

So if $x = 0$ we can take $y = 0, 1$, and if $x \neq 0$ we can take $x = 1$ and $0 \le y < p$ with $y \sim 1 - y$.

If $pc = 0$ and $pb = xbaa + ybab$ then under the ørst transformation

$$pb' = \alpha^{-2}xb'a'a' + \alpha^{-2}yb'a'b'$$

and under the second transformation

$$pb' = \alpha^{-2}xb'a'a' - \alpha^{-2}(y-x)b'a'b'.$$

If $x = 0$ then we can take $y = 0$ or 1 or (if $p = 1 \bmod 4$) ω. If $x \neq 0$ then we can take $x = 1$ or ω with $0 \leq y < p$, where $y \sim x - y$. ($p + 2 + \gcd(p-1,4)/2$ algebras.)

If $pc = bab$ then we need $\alpha = 1$ so we can take $pb = xbaa$ with $0 \leq x < p$. (p algebras.)

if $pc = baa + ybab$ and $y \neq \frac{1}{2}$ then we are restricted to the ørst transformation with $\alpha = 1$, and so we can take $pb = xbab$ with $0 \leq x < p$. But if $pc = baa + \frac{1}{2}bab$ then we can have the ørst transformation with $\alpha = 1$, or the second transformation with $\alpha = -1$, and so we can take $pb = xbab$ with $0 \leq x \leq (p-1)/2$. ($(p^2+1)/2$ algebras.)

And next consider the case when $ca = bab$, $cb = 0$, with $\eta = 0$ and $\xi = \mu^2$. We have

$$\begin{aligned}
b'a'a' &= \alpha^2\mu baa + \alpha\beta\mu bab, \\
b'a'b' &= \alpha\mu^2 bab, \\
pb' &= \mu pb + \nu pc, \\
pc' &= \mu^2 pc.
\end{aligned}$$

We can assume that $pc = 0$, baa or bab. If $pc = 0$ we can assume that $pb = 0$, baa, ωbaa or bab. If $pc = baa$ then we need $\mu = \alpha^2$ and $\beta = 0$, and we can assume that $pb = 0$, bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. And if $pc = bab$, then we need $\alpha = 1$ and we can assume that $pb = xbaa$ with $0 \leq x < p$. So we have $p + 5 + \gcd(p-1,3)$ algebras.

And ønally consider the case when $ca = bab$, $cb = \omega baa$, with $\beta = \eta = 0$ and $\xi = \mu^2 = \alpha^2$. We have

$$\begin{aligned}
b'a'a' &= \pm\alpha^3 baa, \\
b'a'b' &= \alpha^3 bab, \\
pb' &= \pm\alpha pb + \nu pc, \\
pc' &= \alpha^2 pc.
\end{aligned}$$

So we can assume that $pc = 0$, bab, baa or $xbaa + bab$ with $1 \leq x \leq (p-1)/2$. If $pc = 0$ then we can assume that $pb = xbaa + ybab$ where $x = 0, 1, \omega$, and where if $x = 0$ then $y = 0, 1$ or (if $p = 1 \bmod 4$) ω, and if $x \neq 0$ then $0 \leq y \leq (p-1)/2$. If $pc = bab$ then $pb = xbaa$ with $0 \leq x < p$. If $pc = baa$ then $pb = xbab$ with $0 \leq x \leq (p-1)/2$. And if $pc = xbaa + bab$ with $1 \leq x \leq (p-1)/2$ then $pb = ybab$ with $0 \leq y < p$. So we have

$$(p^2 + 4p + 5 + \gcd(p-1,4))/2$$

algebras here.

So the total number of algebras when $p^2 a = 0$ is $p^2 + 7p + 39 + 2\gcd(p-1,4) + \gcd(p-1,3)$.

9.4 $p^2 a = baa$

If $p^2 a = baa$ then L_3 is generated by baa and bab. Adding suitable scalar multiples of pa to b and c we can assume that pb and pc are scalar multiples of bab. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned}
a' &= \alpha a + \gamma c \text{ modulo } L_2, \\
b' &= \alpha^{-1} b + \nu c \text{ modulo } L_2, \\
c' &= \xi c + \eta ba \text{ modulo } L_3,
\end{aligned}$$

and

$$\begin{aligned} b'a'a' &= \alpha baa, \\ b'a'b' &= \alpha^{-1}bab, \\ c'a' &= \alpha\xi ca + \alpha\eta baa, \\ c'b' &= \alpha^{-1}\xi cb + \alpha^{-1}\eta bab, \\ pb' &= \alpha^{-1}pb + \nu pc, \\ pc' &= \xi pc. \end{aligned}$$

So we can assume that $ca = 0$ or bab, though we then need $\eta = 0$.

First, consider the case when $ca = 0$, and let $cb = xbaa + ybab$. Then

$$c'b' = \alpha^{-1}\xi xbaa + \alpha^{-1}\xi ybab = \alpha^{-2}\xi xb'a'a' + \xi yb'a'b'$$

and so we can assume that $y = 0$ or 1. If $y = 0$ we can assume that $x = 0$ or 1, and if $y = 1$ we can assume that $x = 0, 1$ or ω. If $ca = cb = 0$ then we can assume that $pb = 0$, $pc = bab$ or that $pb = xbab$ $(0 \le x < p)$, $pc = 0$. If $ca = 0$, $cb = baa$ then we need $\xi = \alpha^2$ and so we can assume that $pb = 0$ and $pc = bab$ or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$ or that $pb = xbab$ $(0 \le x < p)$, $pc = 0$. If $ca = 0$ and $cb = bab$ then we need $\xi = 1$ and so we can assume that $pb = 0$, $pc = bab$ or that $pb = xbab$ with $0 \le x < p$ and $pc = 0$. If $ca = 0$ and $cb = xbaa + bab$ where $x = 1$ or ω then we need $\xi = 1$, $\alpha = \pm 1$, and so we can assume that $pb = 0$, $pc = ybab$ with $0 < y \le (p-1)/2$ or that $pb = ybab$ with $0 \le y < p$ and $pc = 0$. ($6p + 1 + \gcd(p-1, 3)$ algebras.)

Next consider the case when $ca = bab$, and let $cb = xbaa + ybab$. We need $\xi = \alpha^{-2}$ giving

$$c'b' = \alpha^{-4}xb'a'a' + \alpha^{-2}yb'a'b',$$

and so we we can assume that $y = 0$, 1 or ω. If $y = 0$ we can take $x = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, and if $y \neq 0$ then we can take $0 \le x < p$.

If $ca = bab$, $cb = 0$ then we have

$$\begin{aligned} b'a'b' &= \alpha^{-1}bab, \\ pb' &= \alpha^{-1}pb + \nu pc, \\ pc' &= \alpha^{-2}pc. \end{aligned}$$

So we can assume that $pb = 0$, $pc = bab$ or that $pb = xbab$ $(0 \le x < p)$, $pc = 0$.

If $ca = bab$, $cb = kbaa$ with $k = 1, \omega, \omega^2, \omega^3$ then $\alpha^4 = 1$ so we can assume that $pb = 0$, $pc = xbab$ where x lies in a transversal for the fourth roots of unity, or that $pb = xbab$ $(0 \le x < p)$, $pc = 0$.

And if $ca = bab$, $cb = xbaa + kbab$ where $k = 1, \omega$ then we need $\alpha^2 = 1$ and so we can assume that $pb = 0$, $pc = ybab$ where $0 < y \le (p-1)/2$ or that $pb = ybab$ $(0 \le y < p)$, $pc = 0$.

So we have $3p^2 + p + p\gcd(p-1, 4)$ algebras here, giving a total of $3p^2 + 7p + 1 + \gcd(p-1, 3) + p\gcd(p-1, 4)$ algebras with $p^2a = baa$.

9.5 $p^2a = bab$ or ωbab

If $p^2a = bab$ or ωbab then L_3 is generated by baa and bab, and adding suitable scalar multiples of pa to b and c we can assume that pb, pc are scalar multiples of baa. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \pm b + \nu c \text{ modulo } L_2, \\ c' &= \xi c + \eta ba \text{ modulo } L_3, \end{aligned}$$

and

$$\begin{aligned}
b'a'a' &= \pm\alpha^2 baa \pm \alpha\beta bab, \\
b'a'b' &= \alpha bab, \\
c'a' &= \alpha\xi ca + \beta\xi cb + \alpha\eta baa + \beta\eta bab, \\
c'b' &= \pm\xi cb \pm \eta bab, \\
pb' &= \pm pb + \nu pc, \\
pc' &= \xi pc.
\end{aligned}$$

So we can take $cb = 0$ or baa. If $cb = 0$ then we need $\eta = 0$ and we can take $ca = 0$, baa or bab. If $cb = baa$ then we need $\xi = \alpha^2$, $\eta = \alpha\beta$ and we have

$$c'a' = \alpha^3 ca + 2\alpha^2\beta baa + \alpha\beta^2 bab$$

so we can assume that $ca = 0$, bab or ωbab.

First consider the case when $ca = cb = 0$. Then we can assume that $pb = 0$, baa or ωbaa and $pc = 0$, or we can assume that $pb = 0$, $pc = baa$. (4 algebras.)

Next let $cb = 0$ and let $ca = baa$. Then we need $\beta = \eta = 0$ and $\xi = \pm\alpha$. So, as above, we can assume that $pb = 0$, baa or ωbaa and $pc = 0$, or we can assume that $pb = 0$, $pc = baa$. (4 algebras.)

And now let $cb = 0$ and let $ca = bab$. We now need $\eta = 0$, $\xi = 1$ and so we can assume that $pb = 0$, baa or ωbaa and $pc = 0$, or we can assume that $pb = 0$, $pc = baa$ or (if $p = 1 \bmod 4$) ωbaa. ($3 + \gcd(p-1, 4)/2$ algebras.)

Next let $cb = baa$, $ca = 0$. Then we can assume that $pb = 0$, baa or ωbaa and $pc = 0$, or we can assume that $pb = 0$, $pc = xbaa$ with $0 < x < (p-1)/2$. ($(p+5)/2$ algebras.)

And finally let $cb = baa$, $ca = bab$ or ωbab. Then we need $\alpha = \pm 1$, $\xi = 1$, $\eta = \beta = 0$, so we can assume that $pb = xbaa$ ($0 \le x < p$) and $pc = 0$, or we can assume that $pb = 0$, $pc = xbaa$ with $0 < x < (p-1)/2$. ($3p - 1$ algebras.)

So we have $(7p + 25 + \gcd(p-1, 4))/2$ algebras for each of $p^2a = bab$ and $p^2a = \omega bab$.

10 Immediate descendants of algebra 27 (5.10)

Algebra 5.10 has $2p + 7$ descendants of order p^7 and p-class 3.

Algebra 5.10 has presentation

$$\langle a, b, c \,|\, ca,\ cb,\ pb - ba,\ pc,\ \text{class } 2\rangle,$$

and so if L is an immediate descendant of 5.10 of order p^7 then L_2 is generated by ba, pa modulo L_3, and L_3 has order p^2 and is generated by baa, p^2a. We have ca, cb, $pb - ba$, and $pc \in L_3$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned}
a' &= a + \beta b + \gamma c + \delta ba + \varepsilon pa, \\
b' &= \lambda b + \mu c + \nu ba + \xi pa, \\
c' &= \rho c + \sigma ba + \tau pa
\end{aligned}$$

modulo L_3 and

$$\begin{aligned}
b'a'a' &= \lambda baa, \\
p^2a' &= \beta baa + p^2a, \\
c'a' &= \rho ca + \beta\rho cb + (\sigma - \beta\tau)baa, \\
c'b' &= \lambda\rho cb - \lambda\tau baa, \\
pb' - b'a' &= \lambda(pb - ba) + \mu pc + (\beta\xi - \varepsilon\lambda)baa + \xi p^2a - \mu ca - (\beta\mu - \gamma\lambda)cb, \\
pc' &= \rho pc + \sigma baa + \tau p^2 a.
\end{aligned}$$

So we can assume that $pc = 0$, though we then need $\sigma = \tau = 0$. This gives

$$\begin{aligned}
b'a'a' &= \lambda baa, \\
p^2a' &= \beta baa + p^2a, \\
c'a' &= \rho ca + \beta\rho cb, \\
c'b' &= \lambda\rho cb, \\
pb' - b'a' &= \lambda(pb - ba) + (\beta\xi - \varepsilon\lambda)baa + \xi p^2a - \mu ca - (\beta\mu - \gamma\lambda)cb,
\end{aligned}$$

so that whatever the values of ca, cb (or β) we can choose ε, ξ so that $pb' - b'a' = 0$. Let $cb = xbaa + yp^2a$. Then

$$c'b' = \lambda\rho xbaa + \lambda\rho yp^2a = (\rho x - \beta\rho y)b'a'a' + \lambda\rho yp^2a',$$

and so we can assume that $cb = 0$, baa or p^2a.

First consider the case when $cb = 0$. Then if $ca = xbaa + yp^2a$ we have

$$c'a' = \rho xbaa + \rho yp^2a = \lambda^{-1}(\rho x - \beta\rho y)b'a'a' + \rho yp^2a'$$

and so we can assume that $ca = 0$, baa or p^2a.

Next consider the case when $cb = baa$. Note that we need $\rho = 1$. If $ca = xbaa + yp^2a$ we have

$$c'a' = xbaa + yp^2a + \beta baa = \lambda^{-1}(x + \beta - \beta y)b'a'a' + yp^2a'$$

so if $y \neq 1$ we can assume that $x = 0$, but if $y = 1$ then we can assume that $x = 0$ or 1.

Finally consider the case when $cb = p^2a$. Note that we need $\beta = 0$, $\rho = \lambda^{-1}$. If $ca = xbaa + yp^2a$ we have

$$c'a' = \lambda^{-1}xbaa + \lambda^{-1}yp^2a = \lambda^{-2}xb'a'a' + \lambda^{-1}yp^2a'$$

so we can take $y = 1$ and $0 \leq x < p$ or $y = 0$ and $x = 0, 1, \omega$.

So we have

$$\langle a, b, c \mid ca,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - p^2a,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - xp^2a,\ cb - baa,\ pb - ba,\ pc,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa - p^2a,\ cb - baa,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - xbaa - p^2a,\ cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid ca,\ cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - \omega baa,\ cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle.$$

11 Immediate descendants of algebra 29 (5.12)

Algebra 5.12 has $3p^2+17p+53+\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7 and p-class 3.

Algebra 5.12 has presentation

$$\langle a,b,c\,|\,ca,\,cb,\,pa,\,pb,\,\text{class }2\rangle,$$

and so if L is an immediate descendant of 5.12 of order p^7 then L_2 is generated by ba, pc modulo L_3 and L_3 has order p^2 and is generated by baa, bab, and p^2c. If a', b', c' generate L and satisfy the same relations as a,b,c modulo L_3 then

$$\begin{aligned} a' &= \alpha a+\beta b,\\ b' &= \gamma a+\delta b,\\ c' &= \varepsilon c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha(\alpha\delta-\beta\gamma)baa+\beta(\alpha\delta-\beta\gamma)bab,\\ b'a'b' &= \gamma(\alpha\delta-\beta\gamma)baa+\delta(\alpha\delta-\beta\gamma)bab,\\ p^2c' &= \varepsilon p^2c. \end{aligned}$$

So we can assume that $bab=0$ and that L_3 is generated by baa and p^2c, or that L_3 is generated by baa and bab and that either $p^2c=0$ or $p^2c=baa$.

11.1 $bab=0$

There are $4p+16$ algebras when $bab=0$.

If $bab=0$ then L_3 is generated by baa and p^2a, and we need $\gamma=0$. Adding suitable scalar multiples of pc to a and b we may assume that pa and pb are scalar multiples of baa. If we let

$$\begin{aligned} a' &= \alpha a+\beta b \text{ modulo } L_2,\\ b' &= \delta b \text{ modulo } L_2\\ c' &= \varepsilon c+\xi ba+\rho pc \text{ modulo } L_3 \end{aligned}$$

then

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa,\\ p^2c' &= \varepsilon p^2c\\ c'a' &= \alpha\varepsilon ca+\beta\varepsilon cb+\alpha\xi baa,\\ c'b' &= \delta\varepsilon cb,\\ pa' &= \alpha pa+\beta pb,\\ pb' &= \delta pb. \end{aligned}$$

So we can assume that $cb=xbaa+yp^2c$ where $x,y=0,1$. If $y=1$ we can assume that $ca=0$ though we then need $\beta=0$. If $cb=0$ we can assume that $ca=0$ or p^2c, and if $cb=baa$ we can again assume that $ca=0$ or p^2c.

If $ca=cb=0$ we can assume that $pb=0$, baa or ωbaa. If $pb\neq 0$ we can assume that $pa=0$ and if $pb=0$ then we can assume that $pa=0$ or baa. (4 algebras.)

If $ca = p^2c$, $cb = 0$ then we need $\alpha = 1$. We can assume that $pb = xbaa$ with $0 \leq x < p$, and if $x \neq 0$ we can assume that $pa = 0$. If $pb = 0$ then we can assume that $pa = 0$ or baa. ($p+1$ algebras.)

If $ca = 0$, $cb = baa$ then we need $\varepsilon = \alpha^2$. We can assume that $pb = 0$, baa or ωbaa. If $pb \neq 0$ we can assume that $pa = 0$ and if $pb = 0$ then we can assume that $pa = 0$ or baa. (4 algebras.)

If $ca = p^2c$, $cb = baa$ then we need $\varepsilon = \alpha = 1$. We can assume that $pb = xbaa$ with $0 \leq x < p$, and if $x \neq 0$ we can assume that $pa = 0$. If $pb = 0$ then we can assume that $pa = 0$ or baa. ($p+1$ algebras.)

If $ca = 0$, $cb = p^2c$ then we need $\beta = 0$, $\delta = 1$ and so we can assume that $pa = 0$ or baa. If $pa = 0$ then we can assume that $pb = 0$, baa or ωbaa, and if $pa = baa$ then we can assume that $pb = xbaa$ with $0 \leq x < p$. ($p+3$ algebras.)

Finally, consider the case when $ca = 0$, $cb = baa + p^2c$. Then we need $\beta = 0$, $\delta = 1$, $\varepsilon = \alpha^2$, and so (as in the case above) we can assume that $pa = 0$ or baa. If $pa = 0$ then we can assume that $pb = 0$, baa or ωbaa, and if $pa = baa$ then we can assume that $pb = xbaa$ with $0 \leq x < p$. ($p+3$ algebras.)

11.2 $p^2c = 0$

If $p^2c = 0$ there are $2p^2 + 9p + 29$ algebras.

If $p^2c = 0$ then L_3 is generated by baa and bab. If

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \gamma a + \delta b \text{ modulo } L_2 \\ c' &= \varepsilon c + \xi ba + \rho pc \text{ modulo } L_3 \end{aligned}$$

then

$$\begin{aligned} b'a'a' &= \alpha(\alpha\delta - \beta\gamma)baa + \beta(\alpha\delta - \beta\gamma)bab, \\ b'a'b' &= \gamma(\alpha\delta - \beta\gamma)baa + \delta(\alpha\delta - \beta\gamma)bab, \\ c'a' &= \alpha\varepsilon ca + \beta\varepsilon cb + \alpha\xi baa + \beta\xi bab, \\ c'b' &= \gamma\varepsilon ca + \delta\varepsilon cb + \gamma\xi baa + \delta\xi bab, \\ pa' &= \alpha pa + \beta pb, \\ pb' &= \gamma pa + \delta pb. \end{aligned}$$

So if we let

$$\begin{pmatrix} ca \\ cb \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some 2×2 matrix A over $\mathbb{Z}_p$, then

$$\begin{pmatrix} c'a' \\ c'b' \end{pmatrix} = (\alpha\delta - \beta\gamma)^{-1} \left(\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} A \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} + \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \right) \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}$$

So we can assume that the trace of A is 0 (though we then need $\xi = 0$). We can choose ε so that the characteristic polynomial of A is one of x^2, $x^2 - 1$, $x^2 - \omega$, and so we can take A to be one of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}.$$

If $ca = cb = 0$ then we let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = B \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some 2×2 matrix B over $\mathbb{Z}_p$. By Theorem 5 we can assume that B is one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & c \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - c$ is irreducible. Furthermore none of these matrices are equivalent to each other, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. ($p+7$ algebras.)

If $ca = 0$, $cb = baa$ then we need $\beta = 0$, $\varepsilon = \alpha^2$, $\xi = 0$. We then have

$$\begin{aligned} b'a'a' &= \alpha^2 \delta baa, \\ b'a'b' &= \alpha\gamma\delta baa + \alpha\delta^2 bab, \\ pa' &= \alpha pa, \\ pb' &= \gamma pa + \delta pb. \end{aligned}$$

so we can assume that $pa = 0$, baa, bab or ωbab. If $pa = 0$ we can assume that $pb = 0$, baa, ωbaa or bab. If $pa = baa$ then we need $\delta = \alpha^{-1}$, and if $pb = xbaa + ybab$ then

$$pb' = (\gamma + \alpha^{-1}x)baa + \alpha^{-1}ybab = \alpha^{-1}(\gamma - \gamma y + \alpha^{-1}x)b'a'a' + yb'a'b'.$$

So we can take $0 \leq y < p$, and if $y \neq 1$ we can take $x = 0$. But if $y = 1$ then we can take $x = 0, 1$ or ω. If $pa = bab$ or ωbab then we need $\gamma = 0$ and $\delta = \pm 1$. So if $pb = xbaa + ybab$ then

$$pb' = \pm xbaa \pm ybab = \alpha^{-2}xb'a'a' \pm \alpha yb'a'b',$$

so we can take $y = 1$ and $0 \leq x < p$, or $y = 0$ and $x = 0, 1, \omega$. ($3p + 12$ algebras.)

If $ca = bab$, $cb = baa$ then we need $\alpha = \pm\delta$, $\beta = \pm\gamma$, $\varepsilon = \alpha^2 - \beta^2$. If we let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = B \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \pm(\alpha^2 - \beta^2)^{-1} \begin{pmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{pmatrix} B \begin{pmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{pmatrix}^{-1} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}.$$

We compute the number of orbits of matrices B under this action using Burnside's Lemma. For each matrix $A = \begin{pmatrix} \alpha & \beta \\ \pm\beta & \pm\alpha \end{pmatrix}$ we compute the number of matrices B øxed by the action of A.

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} - (\alpha^2 - \beta^2) \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x + \beta z - \alpha^3 x + \alpha x\beta^2 - \beta y\alpha^2 + \beta^3 y & \alpha y + \beta t - \beta x\alpha^2 + \beta^3 x - \alpha^3 y + \alpha y\beta^2 \\ \beta x + \alpha z - \alpha^3 z + \alpha z\beta^2 - \beta t\alpha^2 + \beta^3 t & \beta y + \alpha t - \beta z\alpha^2 + \beta^3 z - \alpha^3 t + \alpha t\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \alpha^3 + \alpha\beta^2 & -\beta\alpha^2 + \beta^3 & \beta & 0 \\ -\beta\alpha^2 + \beta^3 & \alpha - \alpha^3 + \alpha\beta^2 & 0 & \beta \\ \beta & 0 & \alpha - \alpha^3 + \alpha\beta^2 & -\beta\alpha^2 + \beta^3 \\ 0 & \beta & -\beta\alpha^2 + \beta^3 & \alpha - \alpha^3 + \alpha\beta^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$$

$$\begin{pmatrix} \alpha-\alpha^3+\alpha\beta^2 & -\beta\alpha^2+\beta^3 & \beta & 0 \\ -\beta\alpha^2+\beta^3 & \alpha-\alpha^3+\alpha\beta^2 & 0 & \beta \\ \beta & 0 & \alpha-\alpha^3+\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & \beta & -\beta\alpha^2+\beta^3 & \alpha-\alpha^3+\alpha\beta^2 \end{pmatrix}$$

has determinant $(1+\beta-\alpha)(\beta+1+\alpha)(-1+\beta-\alpha)(\beta-1+\alpha)(\beta+\alpha)^2(\beta-\alpha)^2\left(\beta^2-\alpha^2+1\right)^2$.

If $\beta \neq 0$ this has the same nullity as

$$\begin{pmatrix} -\beta & 0 & \alpha-\alpha^3+\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & -\beta & -\beta\alpha^2+\beta^3 & \alpha-\alpha^3+\alpha\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha-\alpha^3+\alpha\beta^2 & -\beta\alpha^2+\beta^3 & \beta & 0 \\ -\beta\alpha^2+\beta^3 & \alpha-\alpha^3+\alpha\beta^2 & 0 & \beta \\ \beta & 0 & \alpha-\alpha^3+\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & \beta & -\beta\alpha^2+\beta^3 & \alpha-\alpha^3+\alpha\beta^2 \end{pmatrix}$$

$$= \begin{pmatrix} (\beta-\alpha)(\beta+\alpha)\left(\beta^2-1+\alpha^2\right)\left(\beta^2-\alpha^2+1\right) & 2\alpha\left(\beta^2-\alpha^2+1\right)\beta\left(\beta^2-\alpha^2\right) \\ 2\alpha\left(\beta^2-\alpha^2+1\right)\beta\left(\beta^2-\alpha^2\right) & (\beta-\alpha)(\beta+\alpha)\left(\beta^2-1+\alpha^2\right)\left(\beta^2-\alpha^2+1\right) \end{pmatrix}$$

Since $\alpha^2-\beta^2 \neq 0$ this has the same nullity as

$$\left(\beta^2-\alpha^2+1\right)\begin{pmatrix} \left(\beta^2-1+\alpha^2\right) & 2\alpha\beta \\ 2\alpha\beta & \left(\beta^2-1+\alpha^2\right) \end{pmatrix}$$

$$\begin{pmatrix} \left(\beta^2-1+\alpha^2\right) & 2\alpha\beta \\ 2\alpha\beta & \left(\beta^2-1+\alpha^2\right) \end{pmatrix}$$

has determinant $(1+\beta-\alpha)(\beta+1+\alpha)(-1+\beta-\alpha)(\beta-1+\alpha)$.

So if $\beta \neq 0$ then the nullity is 2 if $\alpha^2-\beta^2=1$ ($p-3$ solutions when $\beta \neq 0$) or if $\alpha=0$, $\beta=\pm 1$, and the nullity is 1 if $\beta=\pm\alpha\pm 1$ and this is not covered by any of the nullity 2 cases. If $\beta=\pm(\alpha+1)$, $\alpha \neq 0,-1,-\frac{1}{2}$ then $\alpha^2-\beta^2 \neq 0,1$, and if $\beta=\pm(\alpha-1)$, $\alpha \neq 0,1,\frac{1}{2}$ then $\alpha^2-\beta^2 \neq 0,1$. So we get $4(p-3)$ pairs α,β giving rank 1.

If $\beta=0$ then we have

$$\begin{pmatrix} -\alpha^3+\alpha & 0 & 0 & 0 \\ 0 & -\alpha^3+\alpha & 0 & 0 \\ 0 & 0 & -\alpha^3+\alpha & 0 \\ 0 & 0 & 0 & -\alpha^3+\alpha \end{pmatrix}$$

so we get nullity 4 when $\alpha=\pm 1$.

So the contribution to Burnside's Lemma is

$$2(p^4-1)+(p-1)(p^2-1)+4(p-3)(p-1)+(p-1)^2=\left(2p^2+5p+12\right)(p-1)^2.$$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}+(\alpha^2-\beta^2)\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\beta z+\alpha^3x-\alpha x\beta^2-\beta y\alpha^2+\beta^3y & \alpha y+\beta t+\beta x\alpha^2-\beta^3x-\alpha^3y+\alpha y\beta^2 \\ -\beta x-\alpha z+\alpha^3z-\alpha z\beta^2-\beta t\alpha^2+\beta^3t & -\beta y-\alpha t+\beta z\alpha^2-\beta^3z-\alpha^3t+\alpha t\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha^3-\alpha\beta^2 & -\beta\alpha^2+\beta^3 & \beta & 0 \\ \beta\alpha^2-\beta^3 & \alpha-\alpha^3+\alpha\beta^2 & 0 & \beta \\ -\beta & 0 & -\alpha+\alpha^3-\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & -\beta & \beta\alpha^2-\beta^3 & -\alpha-\alpha^3+\alpha\beta^2 \end{pmatrix}$$

If $\beta \neq 0$ this has the same nullity as

$$\begin{pmatrix} \beta & 0 & \alpha+\alpha^3-\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & \beta & \beta\alpha^2-\beta^3 & \alpha-\alpha^3+\alpha\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha+\alpha^3-\alpha\beta^2 & -\beta\alpha^2+\beta^3 & \beta & 0 \\ \beta\alpha^2-\beta^3 & \alpha-\alpha^3+\alpha\beta^2 & 0 & \beta \\ -\beta & 0 & -\alpha+\alpha^3-\alpha\beta^2 & -\beta\alpha^2+\beta^3 \\ 0 & -\beta & \beta\alpha^2-\beta^3 & -\alpha-\alpha^3+\alpha\beta^2 \end{pmatrix}$$

$$-\left(\beta-\alpha\right)\left(\beta+\alpha\right)\left(\beta^2+1-\alpha^2\right)\left(\beta^2-1-\alpha^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we get nullity 2 when $\alpha^2-\beta^2 = \pm 1$.

If $\beta = 0$ we have

$$\begin{pmatrix} \alpha+\alpha^3 & 0 & 0 & 0 \\ 0 & \alpha-\alpha^3 & 0 & 0 \\ 0 & 0 & -\alpha+\alpha^3 & 0 \\ 0 & 0 & 0 & -\alpha-\alpha^3 \end{pmatrix}$$

which again gives nullity 2 when $\alpha^2-\beta^2 = \pm 1$. So the contribution to Burnside's Lemma is

$$2(p-1)(p^2-1)+(p-1)^2 = (2p+3)\,(p-1)^2$$

So the number of orbits is $p^2+\frac{7}{2}p+\frac{15}{2}$.

If $ca = \omega bab$, $cb = baa$ then we need $\alpha = \pm\delta$, $\omega\beta = \pm\gamma$, $\varepsilon = \alpha^2-\omega\beta^2$. If we let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = B\begin{pmatrix} baa \\ bab \end{pmatrix}$$

then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \pm(\alpha^2-\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \omega\beta \\ \pm\beta & \pm\alpha \end{pmatrix} B \begin{pmatrix} \alpha & \omega\beta \\ \pm\beta & \pm\alpha \end{pmatrix}^{-1}\begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}.$$

$$\begin{pmatrix} \alpha & \omega\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} - (\alpha^2-\omega\beta^2)\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & \omega\beta \\ \beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\omega\beta z-\alpha^3x+\alpha x\omega\beta^2-\beta y\alpha^2+\beta^3y\omega & \alpha y+\omega\beta t-x\omega\beta\alpha^2+x\omega^2\beta^3-\alpha^3y+\alpha y\omega\beta^2 \\ \beta x+\alpha z-\alpha^3z+\alpha z\omega\beta^2-t\beta\alpha^2+t\beta^3\omega & \beta y+\alpha t-\omega\beta z\alpha^2+\omega^2\beta^3z-\alpha^3t+\alpha t\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha^3+\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega & \omega\beta & 0 \\ -\omega\beta\alpha^2+\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 & 0 & \omega\beta \\ \beta & 0 & \alpha-\alpha^3+\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega \\ 0 & \beta & -\omega\beta\alpha^2+\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 \end{pmatrix}$$

If $\beta \neq 0$ then this matrix has the same nullity as

$$\begin{pmatrix} -\beta & 0 & \alpha-\alpha^3+\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega \\ 0 & -\beta & -\omega\beta\alpha^2+\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha-\alpha^3+\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega & \omega\beta & 0 \\ -\omega\beta\alpha^2+\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 & 0 & \omega\beta \\ \beta & 0 & \alpha-\alpha^3+\alpha\omega\beta^2 & -\beta\alpha^2+\beta \\ 0 & \beta & -\omega\beta\alpha^2+\omega^2\beta^3 & \alpha-\alpha^3+ \end{pmatrix}$$

which has the same nullity as

$$\left(-\alpha^2+\omega\beta^2\right)\left(1-\alpha^2+\omega\beta^2\right)\begin{pmatrix} \left(\omega\beta^2-1+\alpha^2\right) & 2\alpha\beta \\ 2\omega\alpha\beta & \left(\omega\beta^2-1+\alpha^2\right) \end{pmatrix}$$

So the nullity is 0 unless $\alpha^2 - \omega\beta^2 = 1$ ($p-1$ solutions with $\beta \neq 0$) when the nullity is 2.

If $\beta = 0$ we have

$$\begin{pmatrix} -\alpha^3+\alpha & 0 & 0 & 0 \\ 0 & -\alpha^3+\alpha & 0 & 0 \\ 0 & 0 & -\alpha^3+\alpha & 0 \\ 0 & 0 & 0 & -\alpha^3+\alpha \end{pmatrix}$$

so we have nullity 4 when $\alpha = \pm 1$.

So the contribution to Burnside's Lemma is

$$2(p^4-1)+(p-1)(p^2-1)+(p^2-1) = (p-1)\,(p+1)\left(2p^2+p+2\right)$$

$$\begin{pmatrix} \alpha & \omega\beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} + (\alpha^2-\omega\beta^2)\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & \omega\beta \\ -\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\omega\beta z+\alpha^3x-\alpha x\omega\beta^2-\beta y\alpha^2+\beta^3y\omega & \alpha y+\omega\beta t+x\omega\beta\alpha^2-x\omega^2\beta^3-\alpha^3y+\alpha y\omega\beta^2 \\ -\beta x-\alpha z+\alpha^3z-\alpha z\omega\beta^2-t\beta\alpha^2+t\beta^3\omega & -\beta y-\alpha t+\omega\beta z\alpha^2-\omega^2\beta^3z-\alpha^3t+\alpha t\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha^3-\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega & \omega\beta & 0 \\ \omega\beta\alpha^2-\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 & 0 & \omega\beta \\ -\beta & 0 & -\alpha+\alpha^3-\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega \\ 0 & -\beta & \omega\beta\alpha^2-\omega^2\beta^3 & -\alpha-\alpha^3+\alpha\omega\beta^2 \end{pmatrix}$$

If $\beta \neq 0$ this has the same nullity as

$$\begin{pmatrix} \beta & 0 & \alpha+\alpha^3-\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega \\ 0 & \beta & \omega\beta\alpha^2-\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha+\alpha^3-\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega & \omega\beta & 0 \\ \omega\beta\alpha^2-\omega^2\beta^3 & \alpha-\alpha^3+\alpha\omega\beta^2 & 0 & \omega\beta \\ -\beta & 0 & -\alpha+\alpha^3-\alpha\omega\beta^2 & -\beta\alpha^2+\beta^3\omega \\ 0 & -\beta & \omega\beta\alpha^2-\omega^2\beta^3 & -\alpha-\alpha^3+ \end{pmatrix}$$

$$-\left(\omega\beta^2+1-\alpha^2\right)\left(\omega\beta^2-\alpha^2\right)\left(\omega\beta^2-1-\alpha^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the nullity is 2 if $\alpha^2 - \omega\beta^2 = \pm 1$.

If $\beta = 0$ we have

$$\begin{pmatrix} \alpha+\alpha^3 & 0 & 0 & 0 \\ 0 & \alpha-\alpha^3 & 0 & 0 \\ 0 & 0 & -\alpha+\alpha^3 & 0 \\ 0 & 0 & 0 & -\alpha-\alpha^3 \end{pmatrix}$$

and again the nullity is 2 if $\alpha^2 - \omega\beta^2 = \pm 1$. so the contribution to Burnside's Lemma is

$$2(p+1)(p^2-1)+(p^2-1) = (p-1)\,(2p+3)\,(p+1)$$

So there are $p^2 + \frac{3}{2}p + \frac{5}{2}$ orbits.

11.3 $\quad p^2c = baa$

The total number of algebras with $p^2c = baa$ is

$$p^2 + 4p + 8 + \gcd(p-1,3) + \gcd(p-1,4).$$

If $p^2c = baa$ then L_3 is generated by baa and bab, and we need $\beta = 0$ and $\varepsilon = \alpha^2\delta$. We then have

$$\begin{aligned}
b'a'a' &= \alpha^2\delta baa,\\
b'a'b' &= \alpha\gamma\delta baa + \alpha\delta^2 bab,\\
c'a' &= \alpha^3\delta ca + \alpha\xi baa,\\
c'b' &= \alpha^2\gamma\delta ca + \alpha^2\delta^2 cb + \gamma\xi baa + \delta\xi bab,\\
pa' &= \alpha pa,\\
pb' &= \gamma pa + \delta pb.
\end{aligned}$$

Adding suitable scalar multiples of pc to a and b we can assume that pa, pb are scalar multiples of bab. We can assume that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$ (though we then need $\gamma = 0$), and if $pa = 0$ then we can assume that $pb = 0$ or bab.

If $pa = pb = 0$ then we can assume that $ca = 0$ or bab. If $ca = 0$ then we need $\xi = 0$ and if $cb = xbaa + ybab$ then

$$c'b' = \alpha^2\delta^2 xbaa + \alpha^2\delta^2 ybab$$

so we can assume that $cb = 0$, baa or bab. And if $ca = bab$ then we need $\delta = \alpha^2$, $\xi = \alpha^2\gamma$, and if $cb = xbaa + ybab$ then

$$\begin{aligned}
b'a'a' &= \alpha^4 baa,\\
b'a'b' &= \alpha^3\gamma baa + \alpha^5 bab,\\
c'b' &= (\alpha^6 x + \alpha^2\gamma^2)baa + (\alpha^6 y + 2\alpha^4\gamma)bab
\end{aligned}$$

and so we can assume that $cb = 0$, baa or ωbaa. (6 algebras.)

If $pa = 0$ and $pb = bab$ we need $\delta = \alpha^{-1}$ and we have

$$\begin{aligned}
b'a'a' &= \alpha baa,\\
b'a'b' &= \gamma baa + \alpha^{-1} bab,\\
c'a' &= \alpha^2 ca + \alpha\xi baa,\\
c'b' &= \alpha\gamma ca + cb + \gamma\xi baa + \alpha^{-1}\xi bab.
\end{aligned}$$

So we can assume that $ca = 0$ or bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. If $ca = 0$ then we need $\xi = 0$, and if $cb = xbaa + ybab$ then we have

$$c'b' = xbaa + ybab = \alpha^{-1}(x - \alpha\gamma y)b'a'a' + \alpha y b'a'b'$$

so we can assume that $cb = 0$, baa or bab. If $ca = kbab$ (with $k = 1, \omega$ or ω^2) then we need $\alpha^3 = 1$ and $\xi = \alpha^{-1}k\gamma$ and

$$c'b' = cb + \alpha^{-1}k\gamma^2 baa + 2\alpha k\gamma bab.$$

So we can assume that $cb = 0$ or $xbaa$, where x lies in a transversal for the cube roots of unity. ($p + 2 + \gcd(p-1, 3)$ algebras.)

If $pa = bab$ or ωbab, $pb = 0$ then we need $\gamma = 0$ and $\delta = \pm 1$. We then have

$$\begin{aligned}
b'a'a' &= \pm\alpha^2 baa,\\
b'a'b' &= \alpha bab,\\
c'a' &= \pm\alpha^3 ca + \alpha\xi baa,\\
c'b' &= \alpha^2 cb \pm \xi bab.
\end{aligned}$$

So we can assume that $ca = 0$, bab or (if $p = 1 \bmod 4$) ωbab, though we then need $\xi = 0$. Let $cb = xbaa + ybab$, so that

$$c'b' = \alpha^2 xbaa + \alpha^2 ybab = \pm xb'a'a' + \alpha yb'a'b'.$$

If $ca = 0$ then we can take $0 \le x \le (p-1)/2$ and $y = 0, 1$. If $ca = bab$ or ωbab then we need $\alpha^2 = \delta = \pm 1$, giving

$$\begin{aligned} b'a'a' &= baa, \\ b'a'b' &= \alpha bab, \\ c'a' &= \alpha ca, \\ c'b' &= cb \end{aligned}$$

with $\alpha = \pm 1$, or (when $p = 1 \bmod 4$)

$$\begin{aligned} b'a'a' &= baa, \\ b'a'b' &= \alpha bab, \\ c'a' &= \alpha ca, \\ c'b' &= -cb \end{aligned}$$

with $\alpha^2 = -1$. This gives

$$c'b' = xbaa + ybab = xb'a'a' + \alpha^{-1} yb'a'b'$$

with $\alpha = \pm 1$ or (when $p = 1 \bmod 4$)

$$c'b' = -xbaa - ybab = -xb'a'a' - \alpha^{-1} yb'a'b'$$

with $\alpha^2 = -1$. So if $p \ne 1 \bmod 4$ we can take $0 \le x < p$ and $0 \le y \le (p-1)/2$ and if $p = 1 \bmod 4$ then we can take $y = 0$ and $0 \le x \le (p-1)/2$, or $0 \le x < p$ and y in a transversal for the fourth roots of unity.

The total number of algebras with $pa = bab$ or ωbab, $pb = 0$ is $p^2 + 3p + \gcd(p-1, 4)$.

12 Immediate descendants of algebra 31 (5.14)

Algebra 5.14 has

$$\begin{aligned} &2p^5 + 7p^4 + 19p^3 + 49p^2 + 128p + 256 + (p^2 + 7p + 29)\gcd(p-1, 3) \\ &+(p^2 + 7p + 24)\gcd(p-1, 4) + (p+3)\gcd(p-1, 5) \end{aligned}$$

immediate descendants of order p^7 and p-class 2.

Algebra 5.14 has presentation

$$\langle a, b, c \mid cb,\ pa,\ pb,\ pc,\ \text{class } 2 \rangle,$$

and so if L is an immediate descendant of 5.14 of order p^7 then L_2 is generated by ba, ca modulo L_3, and L_3 has order p^2 and is generated by baa, bab, bac, caa, cab. And cb, pa, pb, $pc \in L_3$. The commutator structure is the same as one of 7.65 ˘ 7.88 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$. So we may assume that one of the following holds:

$$cb = caa = cab = cac = 0, \tag{7.65}$$

$$caa = cab = cac = 0,\ cb = baa, \tag{7.66}$$

$$cb = bab = bac = cab = cac = 0, \tag{7.67}$$

$$cb = baa,\ bab = bac = cab = cac = 0, \tag{7.68}$$

$$cb = bac = cac = 0,\ caa = bab, \tag{7.69}$$

$$cb = baa,\ bac = cac = 0,\ caa = bab, \tag{7.70}$$

$$cb = baa = bac = cac = 0, \tag{7.71}$$

$$baa = bac = cac = 0,\ cb = caa, \tag{7.72}$$

$$cb = bac = caa = 0,\ cac = bab, \tag{7.73}$$

$$cb = bac = caa = 0,\ cac = \omega bab, \tag{7.74}$$

$$bac = caa = 0,\ cb = baa,\ cac = bab, \tag{7.75}$$

$$bac = caa = 0,\ cb = baa,\ cac = \omega bab, \tag{7.76}$$

$$cb = bac = 0,\ caa = baa,\ cac = -bab, \tag{7.77}$$

$$bac = 0,\ cb = caa = baa,\ cac = -bab, \tag{7.78}$$

$$cb = baa = bac = caa = 0, \tag{7.79}$$

$$cb = bac = caa = 0,\ baa = cac, \tag{7.80}$$

$$cb = bac = 0,\ baa = cac,\ caa = bab, \tag{7.81}$$

$$cb = bac = 0,\ baa = cac,\ caa = \omega bab,\ (p = 1 \bmod 3) \tag{7.82}$$

$$cb = baa = caa = cac = 0, \tag{7.83}$$

$$cb = baa = cac = 0,\ caa = bab, \tag{7.84}$$

$$cb = caa = cac = 0,\ baa = bab, \tag{7.85}$$

$$cb = baa = caa = 0,\ cac = \omega bab, \tag{7.86}$$

$$cb = baa = 0,\ caa = bac,\ cac = \omega bab, \tag{7.87}$$

$$cb = baa = 0,\ caa = kbab + bac,\ cac = \omega bab,\ (p = 2 \bmod 3), \tag{7.88}$$

where k is any element of $\mathbb{Z}_p$ which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)}.$$

12.1 Case 1

First consider the case when

$$cb = caa = cab = cac = 0.$$

Then L_3 is generated by baa and bab, and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

We can assume that $pc = 0$, baa or bab.

If $pc = 0$ we can assume that $pb = 0$, baa, ωbaa or bab.

If $pb = pc = 0$ then we can assume that $pa = 0$, baa, bab or ωbab. (4 algebras.)

If $pb = baa$ or ωbaa, $pc = 0$ then we need $\alpha = \pm 1$, $\beta = 0$ and so if $pa = xbaa + ybab$ then we have

$$pa' = \pm xbaa \pm ybab = \pm\lambda^{-1}xb'a'a' + \lambda^{-2}yb'a'b'$$

and so we can take $x = 0$, $y = 0, 1, \omega$ or $x = 1$, $0 \leq y < p$. ($2p + 6$ algebras.)

If $pb = bab$, $pc = 0$ then we need $\lambda = \alpha^{-1}$, and if $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y + \beta)bab = xb'a'a' + \alpha(\alpha y + \beta - \beta x)b'a'b'.$$

So we can take $0 \leq x < p$ and $y = 0$, or $x = 1$ and $y = 1$ or ω. ($p + 2$ algebras.)

If $pc = baa$ then we need $\beta = 0$, and we can assume that $pa = xbab$, $pb = ybab$. This gives

$$\begin{aligned} pa' &= \lambda^{-2}xb'a'b', \\ pb' &= \alpha^{-1}\lambda^{-1}yb'a'b' \end{aligned}$$

so we can take $x = 0, 1, \omega$ and $y = 0, 1$. (6 algebras.)

If $pc = bab$ then we can assume that pa, pb are scalar multiples of baa, and if $pb \neq 0$ we can assume that $pa = 0$. so we can take $pa = 0$, $pb = 0$, baa or ωbaa, or $pa = baa$, $pb = 0$. (4 algebras.)

So there are $3p + 22$ agebras in all in this case.

12.2 Case 2

Next consider the case when

$$caa = cab = cac = 0,\ cb = baa.$$

Then L_3 is generated by baa and bab, and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

We can assume that $pc = 0$, baa or bab.

If $pc = 0$ then we can assume that $pb = 0$, baa, ωbaa or bab.

If $pb = pc = 0$ then we can assume that $pa = 0$, baa, bab or ωbab. (4 algebras.)

If $pc = 0$, $pb = baa$ or ωbaa then we need $\alpha = \pm 1$, $\beta = 0$, and so if $pa = xbaa + ybab$ then

$$pa' = \pm xbaa \pm ybab = \pm\lambda^{-1}xb'a'a' + \lambda^{-2}yb'a'b'$$

and so we can take $x = 0$ or 1; if $x = 0$ we can take $y = 0, 1, \omega$ and if $x = 1$ we can take $0 \leq y < p$. ($2p + 6$ algebras.)

If $pc = 0$, $pb = bab$ then we need $\lambda = \alpha^{-1}$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y + \beta)bab = xb'a'a' + \alpha(\alpha y + \beta - \beta x)b'a'b'$$

and so we can take $0 \leq x < p$, and if $x \neq 1$ we can take $y = 0$, but if $x = 1$ we can take $y = 0, 1, \omega$. ($p + 2$ algebras.)

If $pc = baa$ then we need $\beta = 0$ and $\lambda = 1$. We can assume that $pa = xbab$, $pb = ybab$ and then

$$\begin{aligned} pa' &= \alpha xbab = xb'a'b', \\ pb' &= ybab = \alpha^{-1}yb'a'b', \end{aligned}$$

so we can assume that $0 \leq x < p$, $y = 0, 1$. ($2p$ algebras.)

If $pc = bab$ then we need $\alpha = \lambda^2$. We can assume that $pa = xbaa$, $pb = ybaa$, where we can take $x = 0$ if $y \neq 0$. If $y = 0$ we can take $x = 0$, 1 or (if $p = 1 \bmod 3$) ω or ω^2, and if $y \neq 0$ we can take $y = 1$, ω or (if $p = 1 \bmod 4$) ω^2 or ω^3. ($1 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.)

So the total number of algebras in this case is $5p + 13 + \gcd(p-1, 3) + \gcd(p-1, 4)$.

12.3 Case 3

$$cb = bab = bac = cab = cac = 0.$$

L_3 is generated by baa and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & \nu & \xi \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu \\ \alpha^2\nu & \alpha^2\xi \end{pmatrix}^{-1}.$$

First consider the case when pb, pc span $\langle baa, caa \rangle$, so that we can assume that $pa = 0$. If we consider transformations with $\alpha = 1$ then we see that we can take

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = B \begin{pmatrix} baa \\ caa \end{pmatrix}$$

where B is a matrix of the form $\begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix}$ with $x \neq 0$, or a matrix of the form $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ with $x \neq 0$. Now allowing transformations with arbitrary α we see that in matrices $\begin{pmatrix} 0 & x \\ 1 & y \end{pmatrix}$ we can take $y = 0, 1, \omega$. If $y = 0$ we can take $x = 1$, ω or (if $p = 1 \bmod 4$) ω^2 or ω^3 and if $y \neq 0$ we can take $0 < x < p$. And in matrices $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ we can take $x = 1, \omega$. ($2p + \gcd(p-1, 4)$ algebras.)

Next consider the case when pb, pc span a one dimensional space. Then we can assume that $pb = 0$ and that $pc = baa$, bab or ωbab. If $pb = 0$ and $pc = baa$ then we need $\mu = 0$ and $\xi = \alpha^2\lambda$. We can assume that $pa = 0$ or bab. If $pb = 0$ and $pc = bab$ or ωbab then we need $\mu = \nu = 0$, $\alpha = \pm 1$. We can assume that $pa = 0$ or baa. (6 algebras.)

And ønally consider the case when $pb = pc = 0$. Then we can assume that $pa = 0$ or baa. (2 algebras.)

So the total number of algebras in this case is $2p + 8 + \gcd(p-1, 4)$.

12.4 Case 4

$$cb = baa,\ bab = bac = cab = cac = 0.$$

L_3 is generated by baa and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & 0 \\ 0 & \nu & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & 0 \\ \alpha^2\nu & \alpha^4 \end{pmatrix}^{-1}.$$

We can assume that $pb = 0$, baa, ωbaa or caa.

If $pb = 0$ we can assume that $pc = 0$, baa, caa or ωcaa.

If $pb = pc = 0$ then we can assume that $pa = 0$, baa, caa or (if $p = 1 \bmod 3$) ωcaa or $\omega^2 caa$. ($2 + \gcd(p-1, 3)$ algebras.)

If $pb = 0$, $pc = baa$ then we need $\lambda = 1$. We can take $pa = 0$, caa or (if $p = 1 \bmod 3$) ωcaa or $\omega^2 caa$. ($1 + \gcd(p-1, 3)$ algebras.)

If $pb = 0$, $pc = caa$ or ωcaa then we need $\nu = 0$, $\alpha = \pm 1$. We can take $pa = 0$ or baa. (4 algebras.)

If $pb = kbaa$ where $k = 1$ or ω then we need $\alpha = \pm 1$. If $pc = xbaa + ycaa$ then

$$pc' = (x + k\nu)baa + ycaa = \lambda^{-1}(x + k\nu - y\nu)b'a'a' + yb'a'b'$$

so we can take $0 \leq y < p$ and if $y \neq k$ we can take $x = 0$. If $y = k$ then we can take $x = 0$ or 1. If $y \neq 0$ we can take $pa = 0$. If $pb = kbaa$ and $pc = 0$ then we need $\nu = 0$, so we can take $pa = zcaa$ where $0 \leq z \leq (p-1)/2$. ($3p + 1$ algebras.)

If $pb = caa$ then we need $\lambda = \alpha^4$ and $\nu = 0$. If $pc = xbaa + ycaa$ then

$$pc' = \alpha^2 xbaa + \alpha^2 ycaa = \alpha^{-4}xb'a'a' + \alpha^{-2}yc'a'a'$$

so we can take $y = 0, 1$ or ω; if $y = 0$ we can take $x = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, and if $y \neq 0$ we can take $0 \leq x < p$. If $x \neq 0$ we can take $pa = 0$, and if $x = y = 0$

then we can take $pa = zbaa$ where $z = 0, 1$ or (if $p = 1 \bmod 5$) ω, ω^2, ω^3 or ω^4, and if $x = 0$, $y = 1, \omega$ then we need $\alpha = \pm 1$ so we can take $pa = zbaa$ where $0 \leq z \leq (p-1)/2$ ($3p + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.)

The total number of algebras in this case is $6p + 8 + 2\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$.

12.5 Case 5

$$cb = bac = cac = 0, \; caa = bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \lambda & \mu \\ 0 & 0 & \alpha^{-1}\lambda^2 \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha^2\mu + \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

We can take $pc = 0$, baa, bab or ωbab.

If $pc = 0$ we can take $pb = 0$, baa, ωbaa or bab.

If $pb = pc = 0$ then we can take $pa = 0$, baa, bab or ωbab. (4 algebras.)

If $pb = baa$ or ωbaa and $pc = 0$ then we need $\alpha = \pm 1$, $\mu = -\alpha\beta\lambda$. We can take $pa = 0$, bab or ωbab. (6 algebras.)

If $pb = bab$, $pc = 0$ then we need $\lambda = \alpha^{-1}$. We can take $pa = xbaa$ with $0 \leq x < p$. (p algebras.)

If $pc = baa$ then we need $\lambda = \alpha^3$, $\mu = -\alpha^2\beta$ and we can assume that $pb = 0$, bab, ωbab or (if $p = 1 \bmod 4$) $\omega^2 bab$ or $\omega^3 bab$, though we then need $\beta = \mu = 0$, and $\alpha^4 = 1$ if $pb \neq 0$. If $pb = 0$ we can take $pa = 0$, bab, ωbab or (if $p = 1 \bmod 3$) $\omega^2 bab$, $\omega^3 bab$, $\omega^4 bab$ or $\omega^5 bab$ and if $pb \neq 0$ we can take $pa = xbab$ with $0 \leq x < p$ when $p = 3 \bmod 4$ and $0 \leq x \leq (p-1)/2$ when $p = 1 \bmod 4$. ($2p - 1 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.)

If $pc = kbab$ with $k = 1$ or ω, then we need $\alpha = \pm 1$. We can take $pb = xbaa$ with $0 \leq x < p$. If $pb = 0$ then we need $\mu = 0$ and we can take $pa = 0$ or baa. If $pb = xbaa$ with $x \neq 0$ then we need $\mu(k - x) = \alpha\beta\lambda x$, so that if $x \neq k$ we can take $pa = 0$. But if $pc = kbab$, $pb = kbaa$ then we need $\beta = 0$ so we have to take $pa = 0$ or baa. ($2p + 4$ algebras.)

The total number of algebras in this case is $5p + 13 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$.

12.6 Case 6

$$cb = baa, \; bac = cac = 0, \; caa = bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha^2 & \beta & \gamma \\ 0 & \pm\alpha^3 & \mu \\ 0 & 0 & \alpha^4 \end{pmatrix} A \begin{pmatrix} \pm\alpha^7 & \alpha^4\mu \pm \alpha^5\beta \\ 0 & \alpha^8 \end{pmatrix}^{-1}.$$

We can assume that $pc = 0$, baa (or if $p = 1 \bmod 3$ ωbaa or $\omega^2 baa$) or bab, ωbab (or if $p = 1 \bmod 4$ $\omega^2 bab$ or $\omega^3 bab$).

If $pc = 0$ we can assume that $pb = 0$, baa, ωbaa or (if $p = 1 \bmod 4$ $\omega^2 baa$ or $\omega^3 baa$) or bab (or if $p = 1 \bmod 5$ ωbab, $\omega^2 bab$, $\omega^3 bab$ or $\omega^4 bab$).

If $pb = pc = 0$ then we can assume that $pa = 0$, baa (or if $p = 1 \bmod 5$ ωbaa, $\omega^2 baa$, $\omega^3 baa$ or $\omega^4 baa$), bab, ωbab (or if $p = 1 \bmod 3$ $\omega^2 bab$, $\omega^3 bab$, $\omega^4 bab$ or $\omega^5 bab$). $(1 + \gcd(p-1,5) + 2\gcd(p-1,3)$ algebras.)

If $pb = kbaa$ with $k = 1, \omega, \omega^2$ or ω^3 and $pc = 0$ then we need $\alpha^4 = 1$ and $\mu \pm \alpha\beta = 0$. We can take $pa = xbab$ where $0 \le x < p$ if $p = 3 \bmod 4$ and $0 \le x \le (p-1)/2$ if $p = 1 \bmod 4$. $(2p - 2 + \gcd(p-1,4)$ algebras.)

If $pb = kbab$ with $k = 1, \omega, \omega^2, \omega^3$ or ω^4 and $pc = 0$ then we need $\alpha^5 = \pm 1$, and we can take $pa = xbaa$ with $0 \le x < p$. $(p\gcd(p-1,5)$ algebras.)

If $pc = kbaa$ (with $k = 1, \omega$ or ω^2) then we need $\alpha^3 = \pm 1$ and $\pm\mu + \alpha^2\beta = 0$. We can take $pb = xbab$ where $x = 0$, or where x lies in a transversal for the cube roots of unity, though we then need $\beta = \mu = 0$. We can take $pa = ybab$ where $0 \le y < p$. $(p^2 - p + p\gcd(p-1,3)$ algebras.)

If $pc = kbab$ with $k = 1, \omega, \omega^2$ or ω^3 then we need $\alpha^4 = 1$. We can take $pb = xbaa$ with $0 \le x < p$. If $x = 0$ we need $\mu = 0$, and if $x \ne 0$ then we need $\mu k = \mu x \pm \alpha\beta x$. So if $pc = kbab$ and $pb = 0$ we can take $pa = ybaa$ where $y = 0$ or y lies in a transversal for the fourth roots of unity. If $pc = kbab$ and $pb = xbaa$ with $x \ne 0, k$ then we can take $pa = 0$. And if $pb = kbaa$, $pc = kbab$ then we can take $pa = ybaa$ where $y = 0$ or y lies in a transversal for the fourth roots of unity. $(2p - 2 + p\gcd(p-1,4)$ algebras.)2

The total number of algebras in this case is

$$p^2 + 3p - 3 + (p+2)\gcd(p-1,3) + (p+1)\gcd(p-1,4) + (p+1)\gcd(p-1,5).$$

12.7 Case 7

$$cb = baa = bac = cac = 0.$$

L_3 is generated by bab and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}.$$

Now

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\alpha u + \gamma y}{\alpha\lambda^2} & \frac{\alpha v + \gamma z}{\alpha^2\xi} \\ \frac{1}{\lambda}\frac{w}{\alpha} & \lambda\frac{x}{\alpha^2\xi} \\ \xi\frac{y}{\alpha\lambda^2} & \frac{z}{\alpha^2} \end{pmatrix}$$

and we distinguish three cases: $y=z=0$, $y=0$, $z\neq 0$, $y\neq 0$.

If $y=z=0$ we have

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{u}{\lambda^2} & \frac{1}{\alpha}\frac{v}{\xi} \\ \frac{1}{\lambda}\frac{w}{\alpha} & \lambda\frac{x}{\alpha^2\xi} \\ 0 & 0 \end{pmatrix}.$$

So we can take $v,w=0,1$. If $v=w=0$ we can take $u=0,1,\omega$, $x=0,1$. If $v=0$, $w=1$ then again we can take $u=0,1,\omega$, $x=0,1$. If $v=1$, $w=0$ again we can take $u=0,1,\omega$, $x=0,1$. If $v=w=1$ then we need $\alpha\xi=\alpha\lambda=1$ and so we can take $u=0$, $x=0,1,\omega$ or $u=1,\omega$ and $0\leq x<p$. ($2p+21$ algebras.)

If $y=0$, $z\neq 0$ then we can take $z=1,\omega$, $v=0$, though we then require $\alpha^2=1$, $\gamma=0$. This gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & 0 \\ w & x \\ 0 & z \end{pmatrix} \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{u}{\lambda^2} & 0 \\ \pm\frac{w}{\lambda} & \lambda\frac{x}{\xi} \\ 0 & z \end{pmatrix}$$

so we can take $w,x=0,1$. If $w=0$ we can take $u=0,1,\omega$, and if $w=1$ we can take $0\leq u<p$. ($4p+12$ algebras.)

If $y\neq 0$ we can take $y=1$ and $u=0$, though we then need $\gamma=0$, $\xi=\alpha\lambda^2$. This gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \alpha\lambda^2 \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \\ 1 & z \end{pmatrix} \begin{pmatrix} \alpha\lambda^2 & 0 \\ 0 & \alpha^3\lambda^2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\alpha^2}\frac{v}{\lambda^2} \\ \frac{1}{\lambda}\frac{w}{\alpha} & \frac{1}{\lambda}\frac{x}{\alpha^3} \\ 1 & \frac{z}{\alpha^2} \end{pmatrix}.$$

So we can take $w=0,1$. If $w=0$ then we can take $x=0,1$. If $x=w=0$ then we can take $v,z=0,1,\omega$. If $w=0$, $x=1$ then we need $\lambda=\alpha^{-3}$ so we can take $z=0$ and $v=0,1,\omega$ or (if $p=1 \bmod 4$) ω^2 or ω^3 or we can take $z=1,\omega$ and $0\leq v<p$. If $w=1$ then we need $\lambda=\alpha^{-1}$ so we can take $0\leq v<p$ and either $x=0$, $z=0,1,\omega$ or $x=1,\omega$ and $0\leq z<p$. ($2p^2+5p+10+\gcd(p-1,4)$ algebras.)

The total number of algebras in this case is $2p^2+11p+43+\gcd(p-1,4)$.

12.8 Case 8

$$baa=bac=cac=0,\ cb=caa.$$

L_3 is generated by bab and caa and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A\begin{pmatrix} bab \\ caa \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \xi \end{pmatrix} A \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1}.$$

Now

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\alpha u+\gamma y}{\alpha^5} & \frac{\alpha v+\gamma z}{\alpha^2\xi} \\ \frac{1}{\alpha^3}w & \frac{x}{\xi} \\ \xi\frac{y}{\alpha^5} & \frac{z}{\alpha^2} \end{pmatrix}$$

and we distinguish three cases: $y = z = 0$, $y = 0$, $z \neq 0$, $y \neq 0$.

If $y = z = 0$ we have

$$\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^2\xi \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\alpha^4}u & \frac{1}{\alpha}\frac{v}{\xi} \\ \frac{1}{\alpha^3}w & \frac{x}{\xi} \\ 0 & 0 \end{pmatrix}$$

so we can take $v, x = 0, 1$. If $v = 0$ or $x = 0$ we can take $w = 1$ or (if $p = 1 \bmod 3$) ω or ω^2 and we can take $u = 0$, or u in a transversal for the cube roots of unity, or we can take $w = 0$ and $u = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. If $v = x = 1$ then we can take $0 \leq u, w < p$. ($p^2 + 3p + 3\gcd(p-1, 3) + 3\gcd(p-1, 4)$ algebras.)

If $y = 0$, $z \neq 0$ we can take $v = 0$, $z = 1, \omega$ though we then need $\gamma = 0$ and $\alpha = \pm 1$. This gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u & 0 \\ w & x \\ 0 & z \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \xi \end{pmatrix}^{-1} = \begin{pmatrix} u & 0 \\ \frac{w}{\alpha} & \frac{x}{\xi} \\ 0 & z \end{pmatrix}$$

so we can take $0 \leq u < p$, $0 \leq w \leq (p-1)/2$, $x = 0, 1$. ($2p(p+1)$ algebras.)

If $y \neq 0$ we can take $u = 0$, $y = 1$ though we then need $\gamma = 0$ and $\xi = \alpha^5$. This gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^5 \end{pmatrix} \begin{pmatrix} 0 & v \\ w & x \\ 1 & z \end{pmatrix} \begin{pmatrix} \alpha^5 & 0 \\ 0 & \alpha^7 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\alpha^6}v \\ \frac{1}{\alpha^3}w & \frac{1}{\alpha^5}x \\ 1 & \frac{z}{\alpha^2} \end{pmatrix}.$$

So we can take $z = w = x = 0$, $v = 0, 1, \omega$ or (if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5. Or we can take $z = w = 0$ and $x = 1$ or (if $p = 1 \bmod 5$) ω, ω^2, ω^3 or ω^4 and $v = 0$ or v in a transversal for the øfth roots of unity. Or we can take $z = 0$, $w = 1$ or (if $p = 1 \bmod 3$) ω or ω^2, $x = 0$ or x in a transversal for the cube roots of unity, and $0 \leq v < p$. Or we can take $z = 1, \omega$ and $0 \leq v < p$ with either $w = 0$ and $0 \leq x \leq (p-1)/2$ or $1 \leq w \leq (p-1)/2$ and $0 \leq x < p$. ($p^3 + p^2 + p + (p+2)\gcd(p-1, 3) + \gcd(p-1, 5)$ algebras.)

The total number of algebras in this case is

$$p^3 + 4p^2 + 6p + (p+5)\gcd(p-1, 3) + 3\gcd(p-1, 4) + \gcd(p-1, 5).$$

12.9 Case 9

$$cb = bac = caa = 0, \; cac = bab.$$

L_3 is generated by bab and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \pm\lambda \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

We can take $pb = 0$, baa, ωbaa or bab.

If $pb = 0$ we can take $pc = 0$, baa or (if $p = 1 \bmod 4$) ωbaa, or bab.

If $pb = pc = 0$ then we can take $pa = 0$, baa, bab or ωbab. (4 algebras.)

If $pb = 0$ and $pc = kbaa$ where $k = 1, \omega$ then we need $\alpha^2 = \pm 1$ and $\beta = 0$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = \alpha^{-1}\lambda^{-1}xb'a'a' + \lambda^{-2}yb'a'b'$$

so we can take $x = 0$ or 1. If $x = 0$ then we can take $y = 0, 1, \omega$. If $x = 1$ then we need $\lambda = \alpha^{-1}$ so that if $p = 1 \bmod 4$ we can take $0 \leq y \leq (p-1)/2$ and if $p = 3 \bmod 4$ we can take $0 \leq y < p$. ($p - 1 + 2\gcd(p-1, 4)$ algebras.)

If $pb = 0$ and $pc = bab$ then we need $\lambda = \pm\alpha^{-1}$, so we can take $pa = xbaa$ with $0 \leq x \leq (p-1)/2$ or bab or ωbab. ($(p+5)/2$ algebras.)

If $pb = kbaa$ where $k = 1, \omega$ then we need $\alpha = \pm 1$ and $\beta = 0$. If $pa = xbaa + ybab$ and $pc = zbaa + tbab$ then

$$\begin{aligned} pb' &= \alpha xbaa + \alpha ybab = \alpha\lambda^{-1}xb'a'a' + \lambda^{-2}yb'a'b', \\ pc' &= \pm\lambda xbaa \pm \lambda ybab = \pm zb'a'a' \pm \alpha\lambda^{-1}tb'a'b'. \end{aligned}$$

So we can take $x = 0$ or 1. If $x = 0$ then we can take $t = 0$ or 1. If $x = t = 0$ then we can take $y = 0, 1, \omega$ and $0 \leq z \leq (p-1)/2$. If $x = 0$ and $t = 1$ then we can take $0 \leq y < p$ and $0 \leq z \leq (p-1)/2$. If $x = 1$ then we can take $0 \leq y < p$ and take $z = 0$ and $0 \leq t \leq (p-1)/2$ or take $1 \leq z \leq (p-1)/2$ and take $0 \leq t < p$. ($p^3 + p^2 + 5p + 3$ algebras.)

If $pb = bab$ then we need $\lambda = \alpha^{-1}$. We can take $pc = 0$, baa or (if $p = 1 \bmod 4$) ωbaa, or $xbab$ where $0 \leq x \leq (p-1)/2$.

If $pb = bab$ and $pc = xbab$ with $0 \leq x \leq (p-1)/2$ and $pa = ybaa + zbab$ then

$$pa' = \alpha ybaa + (\alpha z + \beta)bab = yb'a'a' + \alpha(\alpha z + \beta - \beta y)b'a'b'$$

so we can take $0 \leq y < p$. We can take $z = 0$ unless $y = 1$ when we can take $z = 0, 1, \omega$. ($(p+2)(p+1)/2$ algebras.)

If $pb = bab$ and $pc = baa$ or ωbaa then we need $\alpha^2 = \pm 1$ and $\beta = 0$. So if $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = xb'a'a' + \alpha^2 yb'a'b'$$

and so we can take $0 \leq x < p$ and if $p = 1 \bmod 4$ can take $0 \leq y \leq (p-1)/2$ and if $p = 3 \bmod 4$ we can take $0 \leq y < p$. ($p^2 - p + p\frac{\gcd(p-1,4)}{2}$ algebras.)

The total number of algebras in this case is

$$p^3 + \frac{5}{2}p^2 + 7p + \frac{19}{2} + \frac{p+4}{2}\gcd(p-1, 4).$$

12.10 Case 10

$$cb = bac = caa = 0,\ cac = \omega bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \pm\lambda \end{pmatrix} A \begin{pmatrix} \alpha^2\lambda & \alpha\beta\lambda \\ 0 & \alpha\lambda^2 \end{pmatrix}^{-1}.$$

This case is identical to Case 9 and so there are

$$p^3+\frac{5}{2}p^2+7p+\frac{19}{2}+\frac{p+4}{2}\gcd(p-1,4).$$

algebras here.

12.11 Case 11

$$bac=caa=0,\ cb=baa,\ cac=bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A\begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A\to\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \pm\alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} A\begin{pmatrix} \pm\alpha^4 & \pm\alpha^3\beta \\ 0 & \alpha^5 \end{pmatrix}^{-1}.$$

We can assume that $pb=0$, baa, ωbaa, bab or (if $p=1 \bmod 3$) ωbab or $\omega^2 bab$.

If $pb=0$ we can take $pc=0$, baa (or ωbaa if $p=1 \bmod 4$) or bab (or ωbab or $\omega^2 bab$ if $p=1 \bmod 3$).

If $pb=pc=0$ then we can assume that $pa=0$, baa (or ωbaa or $\omega^2 baa$ if $p=1 \bmod 3$), bab, ωbab (or $\omega^2 bab$ or $\omega^3 bab$ if $p=1 \bmod 4$). ($1+\gcd(p-1,3)+\gcd(p-1,4)$ algebras.)

If $pb=0$ and $pc=kbaa$ with $k=1,\omega$ then we need $\alpha^2=\pm 1$ and $\beta=0$. If $pa=xbaa+ybab$ then

$$pa'=\alpha xbaa+\alpha ybab=\pm\alpha xb'a'a'+yb'a'b'$$

so we can take $x=0$ or x in a transversal for the fourth roots of unity and $0\le y<p$. ($p^2/2-p/2+p\gcd(p-1,4)/2$ algebras.)

If $pb=0$ and $pc=kbab$ where $k=1,\omega,\omega^2$ then we need $\alpha^3=1$ and we can assume that $pa=xbaa$ where $0\le x\le(p-1)/2$ or that $pa=xbab$ where x lies in a transversal for the cube roots of unity. ($p-1+\frac{p+1}{2}\gcd(p-1,3)$ algebras.)

If $pb=kbaa$ where $k=1,\omega$ then we need $\alpha^2=1$ and $\beta=0$. If we let $pa=xbaa+ybab$, $pc=zbaa+tbab$ then

$$\begin{aligned} pa' &= \alpha xbaa+\alpha ybab=\pm\alpha xb'a'a'+yb'a'b', \\ pc' &= zbaa+tbab=\pm zb'a'a'+\alpha tb'a'b'. \end{aligned}$$

So we can assume $0\le y<p$. If $x=0$ we can assume that $0\le z,t\le(p-1)/2$. If $x\ne 0$ and $t=0$ then we can assume that $1\le x\le(p-1)/2$ and that $0\le z\le(p-1)/2$. And if $x,t\ne 0$ we can assume that $1\le x,t\le(p-1)/2$ and that $0\le z<p$. ($\frac{1}{2}p^2\left(p^2+3\right)$ algebras.)

If $pb=kbab$ where $k=1,\omega,\omega^2$ then we need $\alpha^3=\pm 1$, so we have

$$A\to\begin{pmatrix} \alpha & \beta & 0 \\ 0 & \pm\alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} A\begin{pmatrix} \alpha & \beta \\ 0 & \pm\alpha^2 \end{pmatrix}^{-1}.$$

So we can take $pc = xbaa$ where $x = 0$ or x lies in a transversal for the sixth roots of unity or we can take $pc = xbab$ where $1 \leq x \leq (p-1)/2$.

If $pb = kbab$ and $pc = 0$ and $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y + \beta k)bab = xb'a'a' \pm \alpha^{-2}(\alpha y + \beta k - \beta x)b'a'b'$$

so we can take $0 \leq x < p$. If $x \neq k$ we can take $y = 0$ and if $x = k$ we can take $y = 0$ or y in a transversal for the cube roots of unity. ($p - 1 + p\gcd(p-1, 3)$ algebras.)

If $pb = kbab$ and $pc = xbaa$ with $x \neq 0$ then we need the + version of the transformation with $\alpha = 1$ and $\beta = 0$ (in other words the relevant automorphism group is trivial) so we can take $pa = ybaa + zbab$ with $0 \leq y, z < p$. ($p^2(p-1)/2$ algebras.)

If $pb = kbab$ and $pc = xbab$ (with $1 \leq x \leq (p-1)/2$) then we need the + version of the transformation with $\alpha^3 = 1$:

$$A \to \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} A \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^2 \end{pmatrix}^{-1}.$$

If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y + \beta k)bab = xb'a'a' + \alpha^{-2}(\alpha y + \beta k - \beta x)b'a'b'$$

so we can take $0 \leq x < p$. If $x \neq k$ we can take $y = 0$, and if $x = k$ then we can take $y = 0$ or y in a transversal for the cube roots of unity. ($(p-1)(p-1+p\gcd(p-1,3))/2$ algebras.)

The total number of algebras in this case is

$$(p^4 + p^3 + 4p^2 + p - 1 + (p^2 + 2p + 3)\gcd(p-1, 3) + (p+2)\gcd(p-1, 4))/2$$

12.12 Case 12

$$bac = caa = 0,\ cb = baa,\ cac = \omega bab.$$

This case is identical to Case 11, so again there are

$$(p^4 + p^3 + 4p^2 + p - 1 + (p^2 + 2p + 3)\gcd(p-1, 3) + (p+2)\gcd(p-1, 4))/2$$

algebras here.

12.13 Case 13

$$cb = bac = 0,\ caa = baa,\ cac = -bab.$$

In this case there are $2p^2 + 11p + 27 + \gcd(p-1, 4)$ immediate descendants of order p^7 and p-class 3.

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \lambda & \mu \\ 0 & \mu & \lambda \end{pmatrix} A \begin{pmatrix} \alpha^2(\lambda + \mu) & \alpha\beta(\lambda + \mu) \\ 0 & \alpha(\lambda^2 - \mu^2) \end{pmatrix}^{-1}.$$

Considering transformations with $\mu = 0$, we see that we can take $pb = 0$, baa, ωbaa or bab.

If $pb = kbaa$ with $k = 1, \omega$ and $pc = xbaa + ybab$ with $x \neq \pm k$ then we can reduce to the case when $pb = 0$ or bab. If $pb = kbaa$ and $pc = kbaa + ybab$ then

$$\begin{aligned} pb' &= (\lambda+\mu)kbaa + \mu ybab = \alpha^{-2}kb'a'a' + zb'a'b', \\ pc' &= (\lambda+\mu)kbaa + \lambda ybab = \alpha^{-2}kb'a'a' + tb'a'b' \end{aligned}$$

for some z, t. Taking $\mu = 0$, $\alpha = 1$, $\beta = 0$ we have $z = 0$, $t = \lambda^{-1}y$ so we can take $y = 0, 1$. If $pb = kbaa$, $pc = -kbaa + ybab$ then

$$\begin{aligned} pb' &= (\lambda-\mu)kbaa + \mu ybab = \alpha^{-2}\frac{\lambda-\mu}{\lambda+\mu}kb'a'a' + zb'a'b', \\ pc' &= -(\lambda-\mu)kbaa + \lambda ybab = -\alpha^{-2}\frac{\lambda-\mu}{\lambda+\mu}kb'a'a' + tb'a'b' \end{aligned}$$

for some z, t. Since $\frac{\lambda-\mu}{\lambda+\mu}$ can take any value other than 0, we see that the case $k = \omega$ reduces to $k = 1$, and so we can take $pb = baa$, $pc = -baa + ybab$ with $y = 0, 1$.

Now consider the case when $pb = bab$. If $pc = ybab$ with $y \neq \pm 1$ then we can reduce to the case $pb = 0$. If $pb = pc = bab$ then

$$pb' = pc' = (\lambda+\mu)bab = \alpha^{-1}(\lambda-\mu)b'a'b'$$

so we cannot reduce this case further. Similarly, we cannot reduce the case $pb = -pc = bab$ further.

If $pb = bab$ and $pc = xbaa + ybab$ with $x \neq 0$ then we are restricted to transformations with $\mu = 0$ to preserve the condition that $pb = 0$ or bab. We then have

$$\begin{aligned} pb' &= \lambda bab = \alpha^{-1}\lambda^{-1}b'a'b', \\ pc' &= \lambda xbaa + \lambda ybab = \alpha^{-2}xb'a'a' + zb'a'b' \end{aligned}$$

for some z. With suitable choice of β we can take $z = 0$, and clearly we can take $x = 1$ or ω.

If $pb = 0$ then we can take $pc = 0$, baa, ωbaa or bab.

So altogether we can assume that the pair (pb, pc) takes one of the following 14 values: $(0,0)$, $(0, baa)$, $(0, \omega baa)$, $(0, bab)$, (bab, bab), $(bab, -bab)$, (bab, baa), $(bab, \omega baa)$, (baa, baa), $(baa, baa+bab)$, $(baa, -baa)$, $(baa, -baa+bab)$, $(\omega baa, \omega baa)$, $(\omega baa, \omega baa + bab)$.

If $pb = pc = 0$ we can take $pa = 0$, baa or bab. (3 algebras.)

If $pb = 0$ and $pc = baa$ or ωbaa then we need $\alpha = \pm 1$, $\beta = \mu = 0$. So if $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = \alpha^{-1}\lambda^{-1}xb'a'a' + \lambda^{-2}yb'a'b'$$

so we can take $x = 0$ and $y = 0, 1, \omega$, or $x = 1$ and $0 \leq y < p$. ($2p + 6$ algebras.)

If $pb = 0$ and $pc = bab$ then we need $\mu = 0$ and $\lambda = \alpha^{-1}$. So if $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y - \beta)bab = xb'a'a' + \alpha(\alpha y - \beta - \beta x)b'a'b'$$

so we can take $0 \leq x < p$. If $x \neq -1$ we can take $y = 0$, and if $x = -1$ we can take $y = 0, 1, \omega$. ($p + 2$ algebras.)

If $pb = pc = bab$ then we need $\alpha = (\lambda - \mu)^{-1}$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab$$

and so we can take $pa = 0$, baa or bab. (3 algebras.)

If $pb = bab$, $pc = -bab$ then we need $\alpha = (\lambda + \mu)^{-1}$ so if $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y + 2\beta)bab = xb'a'a' + (\lambda - \mu)^{-1}(\alpha y + 2\beta - \beta x)bab.$$

So we can take $0 \leq x < p$. If $x \neq 2$ we can take $y = 0$, and if $x = 2$ we can take $y = 0, 1$. ($p + 1$ algebras.)

If $pb = bab$, $pc = baa$ or ωbaa then we need $\alpha = \lambda = \pm 1$, $\beta = \mu = 0$, and so we can take $pa = xbaa + ybab$ with $0 \leq x, y < p$. ($2p^2$ algebras.)

If $pb = pc = baa$ then we need $\beta = 0$ and $\alpha = \pm 1$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = \alpha^{-1}(\lambda + \mu)^{-1}xb'a'a' + (\lambda^2 - \mu^2)^{-1}yb'a'b',$$

so we can take $x, y = 0, 1$. (4 algebras.)

If $pb = baa$, $pc = baa + bab$ then we need $\alpha = \lambda + \mu = \pm 1$, $\beta = \mu$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y - \beta)bab = xb'a'a' + (\lambda - \mu)^{-1}(\alpha y - \beta - \beta x)b'a'b'$$

so we can take $0 \leq x < p$. If $x = -1$ we can take $y = 0$ or 1. If $x \neq -1$ then we can take $y = 0$ provided we can choose $\beta = \mu$ so that $\beta(x + 1) = (\lambda + \mu)y$, with $\lambda + \mu = \pm 1$. But if $\lambda + \mu = 1$ then $\mu = \frac{1}{2}$ is impossible, and if $\lambda + \mu = -1$ then $\mu = -\frac{1}{2}$ is impossible. So we cannot take $y = 0$ if $y = \frac{x+1}{2}$, and in this case

$$pa' = xb'a'a' + \frac{x + 1}{2}b'a'b'$$

whatever the values of λ, μ. ($2p$ algebras.)

If $pb = baa$, $pc = -baa$ then we need $\alpha^2(\lambda + \mu) = \lambda - \mu$ and $\beta = 0$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = \alpha^{-1}(\lambda + \mu)^{-1}xb'a'a' + \alpha^{-2}(\lambda - \mu)^{-2}yb'a'b'$$

so we can take $x = 0$ or 1, and if $x = 0$ we can take $y = 0, 1, \omega$ and if $x = 1$ we can take $0 \leq y < p$. ($p + 3$ algebras.)

If $pb = baa$, $pc = -baa + bab$ then we need $\alpha^2(\lambda + \mu) = \lambda - \mu$, $\alpha\beta(\lambda + \mu) = \mu$ and $\alpha(\lambda - \mu) = 1$, which gives $\lambda + \mu = \alpha^{-3}$, $\lambda - \mu = \alpha^{-1}$, $\beta = \alpha^2\mu = \frac{\alpha^{-1} - \alpha}{2}$. If $pa = xbaa + ybab$ then

$$pa' = (\alpha x + 2\beta)baa + (\alpha y - \beta)bab = (\alpha^2 x + 1 - \alpha^2)b'a'a' + (\alpha^4 y + \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha^2 x + \frac{1}{2}\alpha^4 x - \frac{1}{2})b'a'b'.$$

If $x = 1$ then

$$pa' = b'a'a' + (\alpha^4 y + \frac{1}{2}\alpha^4 - \frac{1}{2})b'a'b'$$

and so we can take $y = -\frac{1}{2}$, $\frac{1}{2}$, $\omega - \frac{1}{2}$ or (if $p = 1 \bmod 4$) $\omega^2 - \frac{1}{2}$ or $\omega^3 - \frac{1}{2}$.

If $1 - x$ is a non-zero square then we can choose α so that $\alpha^2 x + 1 - \alpha^2 = 0$, and if $1 - x$ is not a square then we can choose α so that $\alpha^2 x + 1 - \alpha^2 = 1 - \omega$.

So we can take $x = 0, 1, 1 - \omega$. If $x = 0$ then we need $\alpha^2 = 1$ and so we can take $0 \leq y < p$, and if $x = 1 - \omega$ then again we need $\alpha^2 = 1$ and so again we can take $0 \leq y < p$.

So there are $2p + 1 + \gcd(p - 1, 4)$ algebras here.

If $pb = \omega baa$ and $pc = \omega baa$ then we need $\alpha^2 = 1$ and $\beta = 0$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + \alpha ybab = \alpha^{-1}(\lambda + \mu)^{-1}xb'a'a' + (\lambda^2 - \mu^2)^{-1}yb'a'ab'$$

so we can take $x, y = 0, 1$. (4 algebras.)

Finally, if $pb = \omega baa$ and $pc = \omega baa + bab$ then we need $\alpha^2 = 1$, $\alpha = \lambda + \mu = \pm 1$, $\beta\omega = \mu$. If $pa = xbaa + ybab$ then

$$pa' = \alpha xbaa + (\alpha y - \beta)bab = xb'a'a' + (\lambda - \mu)^{-1}(\alpha y - \beta - \beta x)b'a'b'$$

so we can take $0 \leq x < p$. If $x = -1$ we can take $y = 0$ or 1. If $x \neq -1$ then we can take $y = 0$ provided we can choose $\beta = \omega^{-1}\mu$ so that $\beta(x+1) = (\lambda+\mu)y$, with $\lambda+\mu = \pm 1$. But if $\lambda+\mu = 1$ then $\mu = \frac{1}{2}$ is impossible, and if $\lambda+\mu = -1$ then $\mu = -\frac{1}{2}$ is impossible. So we cannot take $y = 0$ if $y = \frac{x+1}{2\omega}$, and in this case

$$pa' = xb'a'a' + \frac{x+1}{2\omega}b'a'b'$$

whatever the values of λ, μ. ($2p$ algebras.)

12.14 Case 14

$$bac = 0,\ cb = caa = baa,\ cac = -bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \lambda & \lambda-\alpha^2 \\ 0 & \lambda-\alpha^2 & \lambda \end{pmatrix} A \begin{pmatrix} 2\alpha^2\lambda-\alpha^4 & 2\alpha\beta\lambda-\alpha^3\beta \\ 0 & 2\alpha^3\lambda-\alpha^5 \end{pmatrix}^{-1}.$$

If we set $\xi = 2\lambda - \alpha^2$ then we can express this as

$$A \to \begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \frac{\xi+\alpha^2}{2} & \frac{\xi-\alpha^2}{2} \\ 0 & \frac{\xi-\alpha^2}{2} & \frac{\xi+\alpha^2}{2} \end{pmatrix} A \begin{pmatrix} \alpha^2\xi & \alpha\beta\xi \\ 0 & \alpha^3\xi \end{pmatrix}^{-1}.$$

We compute the number of orbits of matrices A under this action using Burnside's Lemma. For each choice of α, β, ξ we compute the number of matrices A which are øxed.

$$\begin{pmatrix} \alpha & \beta & -\beta \\ 0 & \frac{\xi+\alpha^2}{2} & \frac{\xi-\alpha^2}{2} \\ 0 & \frac{\xi-\alpha^2}{2} & \frac{\xi+\alpha^2}{2} \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha^2\xi & \alpha\beta\xi \\ 0 & \alpha^3\xi \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\beta z-\beta u-x\alpha^2\xi & \alpha y+\beta t-\beta v-x\alpha\beta\xi-y\alpha^3\xi \\ \frac{1}{2}z\xi+\frac{1}{2}z\alpha^2+\frac{1}{2}u\xi-\frac{1}{2}u\alpha^2-z\alpha^2\xi & \frac{1}{2}t\xi+\frac{1}{2}t\alpha^2+\frac{1}{2}v\xi-\frac{1}{2}v\alpha^2-z\alpha\beta\xi-t\alpha^3\xi \\ \frac{1}{2}z\xi-\frac{1}{2}z\alpha^2+\frac{1}{2}u\xi+\frac{1}{2}u\alpha^2-u\alpha^2\xi & \frac{1}{2}t\xi-\frac{1}{2}t\alpha^2+\frac{1}{2}v\xi+\frac{1}{2}v\alpha^2-u\alpha\beta\xi-v\alpha^3\xi \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha^2\xi & 0 & \beta & 0 & -\beta & 0 \\ -\alpha\beta\xi & \alpha-\alpha^3\xi & 0 & \beta & 0 & -\beta \\ 0 & 0 & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi & 0 & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 & 0 \\ 0 & 0 & -\alpha\beta\xi & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^3\xi & 0 & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 \\ 0 & 0 & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 & 0 & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi & 0 \\ 0 & 0 & 0 & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 & -\alpha\beta\xi & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^3\xi \end{pmatrix}$$

Determinant: $\alpha^6\xi^2(\alpha+1)\left(\alpha^2+\alpha+1\right)(\alpha-1)^2(-1+\xi)\left(\alpha^2\xi-1\right)(-1+\alpha\xi)^2$

If $\alpha = 1$ we have

$$\begin{pmatrix} 1-\xi & 0 & \beta & 0 & -\beta & 0 \\ -\beta\xi & 1-\xi & 0 & \beta & 0 & -\beta \\ 0 & 0 & -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 \\ 0 & 0 & -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 & -\frac{1}{2}\xi+\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\xi-\frac{1}{2} & -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} \end{pmatrix}$$

If in addition $\xi = 1$ we have

$$\begin{pmatrix} 0 & 0 & \beta & 0 & -\beta & 0 \\ -\beta & 0 & 0 & \beta & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta & 0 \end{pmatrix}$$

which has nullity 6 if $\beta = 0$, and nullity 3 if $\beta \neq 0$. If $\xi \neq 1$ then we have the same nullity as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 \\ -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} \\ \frac{1}{2}\xi-\frac{1}{2} & 0 & -\frac{1}{2}\xi+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\xi-\frac{1}{2} & -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 \\ -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\xi-\frac{1}{2} & -\beta\xi & -\frac{1}{2}\xi+\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta\xi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ \beta\xi & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta\xi & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -\beta\xi & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta\xi & -1 \end{pmatrix}$$

which is one if $\beta \neq 0$ and 2 if $\beta = 0$.

If $\alpha = -1$ we have

$$\begin{pmatrix} -1-\xi & 0 & \beta & 0 & -\beta & 0 \\ \beta\xi & -1+\xi & 0 & \beta & 0 & -\beta \\ 0 & 0 & -\frac{1}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 \\ 0 & 0 & \beta\xi & \frac{3}{2}\xi+\frac{1}{2} & 0 & \frac{1}{2}\xi-\frac{1}{2} \\ 0 & 0 & \frac{1}{2}\xi-\frac{1}{2} & 0 & -\frac{1}{2}\xi+\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\xi-\frac{1}{2} & \beta\xi & \frac{3}{2}\xi+\frac{1}{2} \end{pmatrix}$$

If $\xi = 1$ we have

$$\begin{pmatrix} -2 & 0 & \beta & 0 & -\beta & 0 \\ \beta & 0 & 0 & \beta & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \beta & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & \beta & 0 & -\beta & 0 \\ \beta & 0 & 0 & \beta & 0 & -\beta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\beta & 0 & \beta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta^2 & 2\beta & -\beta^2 & -2\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 2 \end{pmatrix}$$

which has nullity 3.

If $\xi = -1$ we have

$$\begin{pmatrix} 0 & 0 & \beta & 0 & -\beta & 0 \\ -\beta & -2 & 0 & \beta & 0 & -\beta \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -\beta & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -\beta & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -\beta & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has nullity 3.

If $\xi \neq \pm 1$ we have

$$\begin{pmatrix} \xi & 0 & \frac{1}{2}\xi - \frac{1}{2} \\ 0 & 0 & 0 \\ \xi & 0 & \frac{3}{2}\xi + \frac{1}{2} \end{pmatrix}$$

which has nullity 1.

If $\xi = 1$, $\alpha \neq \pm 1$ we have

$$\begin{pmatrix} \alpha - \alpha^2 & 0 & \beta & 0 & -\beta & 0 \\ -\alpha\beta & \alpha - \alpha^3 & 0 & \beta & 0 & -\beta \\ 0 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 & 0 \\ 0 & 0 & -\alpha\beta & \frac{1}{2} + \frac{1}{2}\alpha^2 - \alpha^3 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 \\ 0 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} - \frac{1}{2}\alpha^2 & -\alpha\beta & \frac{1}{2} + \frac{1}{2}\alpha^2 - \alpha^3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\alpha^2 - \alpha^3 & \alpha\beta & \frac{1}{2} - \frac{1}{2}\alpha^2 \\ 0 & 0 & 0 \\ \frac{1}{2} - \frac{1}{2}\alpha^2 & -\alpha\beta & \frac{1}{2} + \frac{1}{2}\alpha^2 - \alpha^3 \end{pmatrix}$$

$$\begin{pmatrix} 1-\alpha^3 & 0 & 1-\alpha^3 \\ 0 & 0 & 0 \\ \frac{1}{2}-\frac{1}{2}\alpha^2 & -\alpha\beta & \frac{1}{2}+\frac{1}{2}\alpha^2-\alpha^3 \end{pmatrix}$$

So we have nullity 1 if $\alpha^2+\alpha+1 \neq 0$, and nullity 2 if $\alpha^2+\alpha+1=0$.

If $\xi = \alpha^{-2}$ with $\alpha \neq \pm 1$ we have

$$\begin{pmatrix} \alpha-1 & 0 & \beta & 0 & -\beta & 0 \\ -\frac{1}{\alpha}\beta & 0 & 0 & \beta & 0 & -\beta \\ 0 & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 \\ 0 & 0 & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} \\ 0 & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\alpha}\beta & \alpha-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha-1 & 0 & \beta & 0 & -\beta & 0 \\ -\frac{1}{\alpha}\beta & 0 & 0 & \beta & 0 & -\beta \\ 0 & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 \\ 0 & 0 & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} \\ 0 & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{\alpha}\beta^2 & (\alpha-1)\,\beta & -\frac{1}{\alpha}\beta^2 & -(\alpha-1)\,\beta \\ 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 \\ 0 & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} \\ 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha}\beta^2 & -(\alpha-1)\,\beta \\ 0 & -\frac{-1+\alpha^2}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} & 0 \\ 0 & -\frac{1}{\alpha}\beta & -\frac{\alpha^3-1}{\alpha^2} & 0 & -\frac{1}{2}\frac{-1+\alpha^4}{\alpha^2} \\ 0 & -\frac{-1+\alpha^2}{\alpha^2} & 0 & \frac{1}{2}\frac{1+\alpha^4-2\alpha^2}{\alpha^2} & 0 \\ 0 & -\frac{1}{\alpha}\beta & -\frac{\alpha^3-1}{\alpha^2} & -\frac{1}{\alpha}\beta & \frac{1}{2}\frac{1+\alpha^4-2\alpha^3}{\alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha^3-1}{\alpha^2} \end{pmatrix}$$

So we have nullity 1 if $\alpha^2+\alpha+1 \neq 0$ and nullity 2 if $\alpha^2+\alpha+1=0$.

If $\xi = \alpha^{-1}$ and $\alpha \neq \pm 1$ then we have

$$\begin{pmatrix} 0 & 0 & \beta & 0 & -\beta & 0 \\ -\beta & \alpha-\alpha^2 & 0 & \beta & 0 & -\beta \\ 0 & 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 \\ 0 & 0 & -\beta & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \\ 0 & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & -\beta & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} 0 & \beta & 0 & -\beta & 0 \\ 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 \\ 0 & -\beta & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \\ 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & -\beta & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \end{pmatrix}$$

If $\alpha^2+\alpha+1=0$ we have

$$\begin{pmatrix} 0 & \beta & 0 & -\beta & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\beta & 0 \end{pmatrix}$$

which has nullity 3, and if $\alpha^2+\alpha+1\neq 0$ we have

$$\begin{pmatrix} 0 & \beta & 0 & -\beta \\ 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \\ 0 & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & 0 & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} & -\frac{1}{2}\frac{\alpha^3-1}{\alpha} \\ -\frac{1}{2}\frac{\alpha^3-1}{\alpha} & \frac{1}{2}\frac{1+\alpha^3-2\alpha^2}{\alpha} \end{pmatrix}$$

Determinant: $\alpha-1-\alpha^3+\alpha^2=-(\alpha+1)(\alpha-1)^2$. So we have nullity 2.

Finally, if $\alpha^2+\alpha+1=0$ with $\alpha\neq\pm1$, $\xi\neq 1,\alpha^{-1},\alpha^{-2}$ we have

$$\begin{pmatrix} \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi & 0 & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 \\ \frac{1}{2}\xi-\frac{1}{2}\alpha^2 & 0 & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi \\ -\alpha\beta\xi & 0 & -\alpha\beta\xi \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi & \frac{1}{2}\xi-\frac{1}{2}\alpha^2 \\ \frac{1}{2}\xi-\frac{1}{2}\alpha^2 & \frac{1}{2}\xi+\frac{1}{2}\alpha^2-\alpha^2\xi \end{pmatrix}$$

Determinant: $\alpha^2\xi-\xi^2\alpha^2-\alpha^4\xi+\alpha^4\xi^2=\xi\alpha^2(\alpha-1)(\alpha+1)(-1+\xi)$. So we have nullity 1.

So a Burnside's Lemma count gives
$\alpha=1$, $\xi=1$, $\beta=0$: p^6-1
$\alpha=1$, $\xi=1$, $\beta\neq0$: $(p-1)(p^3-1)$
$\alpha=1$, $\xi\neq1$, $\beta=0$: $(p-2)(p^2-1)$
$\alpha=1$, $\xi\neq1$, $\beta\neq0$: $(p-2)(p-1)(p-1)$
$\alpha=-1$, $\xi=\pm1$: $2p(p^3-1)$
$\alpha=-1$, $\xi\neq\pm1$: $(p-3)p(p-1)$
$\alpha\neq\pm1$, $\xi=1$: $p((p-2-\gcd(p-1,3))(p-1)+(\gcd(p-1,3)-1)(p^2-1))$
$\alpha\neq\pm1$, $\xi=\alpha^{-2}$: $p((p-2-\gcd(p-1,3))(p-1)+(\gcd(p-1,3)-1)(p^2-1))$
$\alpha\neq\pm1$, $\xi=\alpha^{-1}$: $p((p-2-\gcd(p-1,3))(p^2-1)+(\gcd(p-1,3)-1)(p^3-1))$
$\alpha^2+\alpha+1=0$, $\xi\neq1,\alpha^{-1},\alpha^{-2}$: $p(\gcd(p-1,3)-1)(p-4)(p-1)$

$$p^6-1+(p-1)(p^3-1)+(p-2)(p^2-1)+(p-2)(p-1)(p-1)+2p(p^3-1)+(p-3)p(p-1)$$

$$p^6+3p^4+4p+2p^3-10p^2+2p((p-2-g)(p-1)+(g-1)(p^2-1))$$

$$p^6+3p^4+10p+2p^3-16p^2-2gp^2+2gp^3+p((p-2-g)(p^2-1)+(g-1)(p^3-1))$$

$$p^6+3p^4+13p-17p^2-2gp^2+gp^3+gp^4+p(g-1)(p-4)(p-1)+(p-1)^2p$$

$$=p(p-1)^2\left(p^3+2p^2+6p+gp+10+4g\right)$$

so there are $p^3+2p^2+6p+10+(p+4)\gcd(p-1,3)$ algebras.

$$cb = baa = bac = caa = 0.$$

L_3 is generated by bab and cac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ cac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1}$$

and

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} A \begin{pmatrix} 0 & \alpha\beta^2 \\ \alpha\gamma^2 & 0 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\beta^2} & \frac{y}{\gamma^2} \\ \frac{1}{\beta}\frac{z}{\alpha} & \beta\frac{t}{\alpha\gamma^2} \\ \gamma\frac{u}{\alpha\beta^2} & \frac{1}{\gamma}\frac{v}{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & \alpha\beta^2 \\ \alpha\gamma^2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{y}{\beta^2} & \frac{x}{\gamma^2} \\ \frac{1}{\beta}\frac{v}{\alpha} & \beta\frac{u}{\alpha\gamma^2} \\ \gamma\frac{t}{\alpha\beta^2} & \frac{1}{\gamma}\frac{z}{\alpha} \end{pmatrix}$$

We can take $(z, v) = (0, 0)$, $(0, 1)$ or $(1, 1)$.

If $z = v = 0$ we can take $(x, y) = (0, 0)$, $(0, 1)$, $(0, \omega)$, $(1, 1)$, $(1, \omega)$, (ω, ω).

If $x = y = z = v = 0$ we can take $(t, u) = (0, 0)$, $(0, 1)$, $(1, 1)$ or (if $p = 1 \bmod 3$) $(1, \omega)$. ($(5 + \gcd(p - 1, 3))/2$ algebras.)

If $x = z = v = 0$, $y = 1, \omega$ then we need $\gamma = \pm 1$, and we can take $(t, u) = (0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ or (if $p = 1 \bmod 3$) $(1, \omega)$ or $(1, \omega^2)$. ($6 + 2\gcd(p - 1, 3)$ algebras.)

If $z = v = 0$ and $x = y = 1, \omega$ then we need $\beta^2 = \gamma^2 = 1$ so we can take $(t, u) = (0, 0)$, $(0, 1)$, $(1, u)$ with $u \neq 0$ and $u \sim -u$, $u \sim u^{-1}$. ($(p + 7 + \gcd(p - 1, 4))/2$ algebras.)

If $z = v = 0$, $x = 1$, $y = \omega$ then we need $\beta^2 = \gamma^2 = 1$ so we can take $(t, u) = (0, 0)$, $(0, 1)$, $(1, 0)$ or $(1, u)$ with $1 \leq u \leq (p - 1)/2$. ($(p + 5)/2$ algebras.)

So there are a total of

$$p + \frac{29}{2} + \frac{5}{2}\gcd(p - 1, 3) + \frac{1}{2}\gcd(p - 1, 4)$$

algebras with $z = v = 0$.

If $z = 0$, $v = 1$ then we need $\gamma = \alpha^{-1}$ and we have

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ 0 & t \\ u & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\beta^2} & \alpha^2 y \\ 0 & \beta t\alpha \\ \frac{1}{\alpha^2}\frac{u}{\beta^2} & 1 \end{pmatrix}$$

so we can take $t = 0, 1$. If $t = 0$ and at least one of x, y, u is zero then we can take the other two to be $0, 1, \omega$, bu if all are non-zero then we can take $x, y = 1, \omega$ and $1 \leq u < p$.

If $t = 1$ then we can take $0 \leq u < p$, $x = 0, 1, \omega$, $0 \leq y < p$ if $x \neq 0$ and $y = 0, 1, \omega$ if $x = 0$. ($2p^2 + 7p + 15$ algebras.)

If $z = v = 1$ then we need $\beta = \alpha^{-1}$, $\gamma = \alpha^{-1}$ and we have

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ 1 & t \\ u & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \alpha\gamma^2 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^2 x & \alpha^2 y \\ 1 & t \\ u & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 1 & t \\ u & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha\beta^2 \\ \alpha\gamma^2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^2 y & \alpha^2 x \\ 1 & u \\ t & 1 \end{pmatrix}$$

so we can take $0 \leq t < u < p$, $x = 0$ and $y = 0, 1, \omega$ or $x = 1, \omega$ and $0 \leq y < p$, or we can take $0 \leq t = u < p$ and $x = 0$, $y = 0, 1, \omega$ or $(x, y) = (1, k^2)$ or $(\omega, \omega k^2)$ with $k^2 \sim k^{-2}$ or $(x, y) = (1, \omega k^2)$.

$$p^3 + \frac{3}{2}p^2 + \frac{1}{2}p + p\gcd(p-1, 4)/2$$

algebras.

The total number of algebras in this case is

$$p^3 + \frac{7}{2}p^2 + \frac{17}{2}p + \frac{59}{2} + \frac{5}{2}\gcd(p-1, 3) + \frac{p+1}{2}\gcd(p-1, 4)$$

12.16 Case 16

$$cb = bac = caa = 0,\ baa = cac.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \alpha^2 \frac{y}{\gamma^4} \\ \frac{1}{\alpha^2} z & \frac{1}{\gamma^2} t \\ \frac{1}{\gamma} \frac{u}{\alpha} & \frac{1}{\gamma^3} v\alpha \end{pmatrix}.$$

We can take $u = 0, 1$.

If $u = 0$ we can take $v = 0, 1$.

If $u = v = 0$ we can take $z = 0, 1, \omega$.

If $u = v = z = 0$ we can take $y = 0, 1, \omega$ and we can take $x = 0$, $t = 0, 1, \omega$ or we can take $x = 1, \omega$ and $0 \leq t < p$. ($6p + 9$ algebras.)

If $u = v = 0$, $z = 1, \omega$ then we can take $x = t = 0$, $y = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, or we can take $x = 0$, $t = 1, \omega$, $0 \leq y < p$, or we can take $x = 1, \omega$, $0 \leq y, t < p$. ($4p^2 + 4p + 2 + 2\gcd(p-1, 4)$ algebras.)

If $u=0$, $v=1$ then we need $\alpha=\gamma^3$ which gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \gamma^2 y \\ \frac{1}{\gamma^6}z & \frac{1}{\gamma^2}t \\ 0 & 1 \end{pmatrix}.$$

So we can assume that $x=y=t=0$ and $z=0,1,\omega$ or (if $p=1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5, or that $x=y=0$, $t=1,\omega$, $0\le z<p$, or that $x=0$, $y=1,\omega$, $0\le z,t<p$, or that $x=1,\omega$, $0\le y,z,t<p$. ($2p^3+2p^2+2p+1+2\gcd(p-1,3)$ algebras.)

If $u=1$ we need $\gamma=\alpha^{-1}$ which gives

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ 1 & u \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^{-1}\gamma^4 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^2 x & \alpha^6 y \\ \frac{1}{\alpha^2}z & \alpha^2 t \\ 1 & \alpha^4 u \end{pmatrix}.$$

So we can assume that $x=z=t=u=0$ and $y=0,1,\omega$ or (if $p=1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5, or that $x=z=t=0$ and when $p=1 \bmod 4$ $u=1,\omega,\omega^2$ or ω^3 and $0\le y\le (p-1)/2$ and when $p=3 \bmod 4$ $u=1,\omega$ and $0\le y<p$, or that $x=z=0$, $t=1,\omega$ and $0\le u,y<p$, or that $x=0$, $z=1,\omega$ and $0\le t,u,y<p$, or that $x=1,\omega$ and $0\le y,z,t,u<p$. ($2p^4+2p^3+2p^2+2p-1+2\gcd(p-1,3)+\gcd(p-1,4)$ algebras.)

So the total number of algebras here is

$$2p^4+4p^3+8p^2+14p+11+4\gcd(p-1,3)+3\gcd(p-1,4).$$

12.17 Case 17

$$cb = bac = 0,\ baa = cac,\ caa = bab.$$

L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}$$

or

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \alpha^{-1}\gamma^2 \\ 0 & \gamma & 0 \end{pmatrix} A \begin{pmatrix} 0 & \alpha\gamma^2 \\ \alpha^2\gamma & 0 \end{pmatrix}^{-1}$$

with $\alpha^3=\gamma^3$.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \frac{1}{\alpha}\frac{y}{\gamma} \\ \frac{1}{\alpha^2}z & \frac{1}{\alpha^3}\gamma t \\ \frac{1}{\gamma}\frac{u}{\alpha} & \frac{v}{\alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \alpha^{-1}\gamma^2 \\ 0 & \gamma & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} 0 & \alpha\gamma^2 \\ \alpha^2\gamma & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{y}{\gamma^2} & \frac{1}{\alpha}\frac{x}{\gamma} \\ \frac{v}{\alpha^2} & \frac{1}{\alpha^3}\gamma u \\ \frac{1}{\gamma}\frac{t}{\alpha} & \frac{1}{\alpha^2}z \end{pmatrix}$$

If $p \neq 1 \bmod 3$ then $\alpha = \gamma$ and the number of orbits is

$$p^5 + p^4 + p^3 + p^2 + p + 2 + (p^2 + p + 1)\gcd(p-1, 4)/2.$$

If $p = 1 \bmod 3$ then $\alpha = \gamma$ or $\xi\gamma$ or $\xi^2\gamma$ where $\xi^3 = 1$. The number of orbits is then

$$(p^5 + p^4 + p^3 + p^2 + 7p + 10)/3 + (p^2 + p + 1)\gcd(p-1, 4)/2$$

So in general the number of orbits is

$$(p^4 + 2p^3 + 3p^2 + 4p + 2)\frac{p-1}{\gcd(p-1, 3)} + 3p + 4 + (p^2 + p + 1)\gcd(p-1, 4)/2$$

12.18 Case 18

$$cb = bac = 0,\ baa = cac,\ caa = \omega bab\,(p = 1 \bmod 3).$$

This case is very similar to Case 17, though we do not have as many automorphisms. L_3 is generated by baa and bab and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}$$

with $\alpha^3 = \gamma^3$.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1}\gamma^2 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\gamma^2 & 0 \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\gamma^2} & \frac{1}{\alpha}\frac{y}{\gamma} \\ \frac{1}{\alpha^2}z & \frac{1}{\alpha^3}\gamma t \\ \frac{1}{\gamma}\frac{u}{\alpha} & \frac{v}{\alpha^2} \end{pmatrix}$$

The number of algebras is

$$(2p^5 + 2p^4 + 2p^3 + 2p^2 + 14p + 17)/3$$

Combining Case 17 and Case 18, the total number of algebras in the two cases is

$$p^5 + p^4 + p^3 + p^2 - 2p - \frac{3}{2} + (3p + \frac{7}{2})\gcd(p-1, 3) + (p^2 + p + 1)\gcd(p-1, 4)/2$$

12.19 Case 19

$$cb = baa = caa = cac = 0.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & 0 & \delta \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 2\alpha\beta\gamma \\ 0 & \alpha\beta\delta \end{pmatrix}^{-1}$$

We can assume that $pc = 0$, bab or bac.

If $pc = 0$ then we can assume that $pb = 0$, bab or bac.

If $pb = pc = 0$ then we can assume that $pa = 0$, bab, ωbab or bac. (4 algebras.)

If $pb = bab$, $pc = 0$ then we need $\beta = \alpha^{-1}$, $\gamma = 0$ and so if $pa = xbab + ybac$ then

$$pa' = \alpha xbab + \alpha ybac = \alpha^2 xb'a'b' + \alpha\delta^{-1} yb'a'c'$$

so we can take $x = 0, 1, \omega$ and $y = 0, 1$. (6 algebras.)

If $pb = bac$, $pc = 0$ then we need $\delta = \alpha^{-1}$ and so we can assume that $pa = 0$, bab, ωbab or bac. (4 algebras.)

If $pc = bab$ then we need $\delta = \alpha\beta^2$ and $\gamma = 0$. so if $pa = xbab + ybac$, $pb = zbab + tbac$ then

$$\begin{aligned} pa' &= \alpha xbab + \alpha ybac = \beta^{-2} xb'a'b' + \alpha^{-1}\beta^{-3} yb'a'c', \\ pb' &= \beta zbab + \beta tbac = \alpha^{-1}\beta^{-1} zb'a'b' + \alpha^{-2}\beta^{-2} tb'a'c'. \end{aligned}$$

So if $z \neq 0$ we can take $z = 1$, $0 \leq t < p$ and $x = 0$, $y = 0, 1, \omega$ or $x = 1, \omega$ and $0 \leq y < p$. If $z = 0$ we can take $y = 0$ and $x, t = 0, 1, \omega$ or $y = 1$ and $x = 1, \omega$, $0 \leq t < p$ or $x = 0$, $y = 1$ and $t = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. ($2p^2 + 5p + 10 + \gcd(p-1, 4)$ algebras.)

Finally, if $pc = bac$ we need $\beta = \alpha^{-1}$. If $pb = xbab + ybac$ then

$$pb' = \beta xbab + (\beta y + \gamma)bac = xb'a'b' + \delta^{-1}(\beta y + \gamma - 2\gamma x)b'a'c'$$

so we can take $0 \leq x < p$. If $x \neq \frac{1}{2}$ we can take $y = 0$ (though we then need $\gamma = 0$), and if $x = \frac{1}{2}$ we can take $y = 0, 1$. If $pa' = zbab + tbac$ then

$$pa' = \alpha zbab + \alpha tbac.$$

If $x \neq \frac{1}{2}$ so that $\gamma = 0$ we have

$$pa' = \alpha^2 zb'a'b' + \alpha\delta^{-1} tb'a'c'$$

so we can take $z = 0, 1, \omega$ and $t = 0, 1$. But if $x = \frac{1}{2}$ then γ is arbitrary so we can take $z = 1, \omega$ and $t = 0$, or $z = 0$ and $t = 0, 1$ if $y = 0$ and $t = 0, 1, \omega$ if $y = 1$. ($6p + 3$ algebras.)

So the total number of algebras in this case is $2p^2 + 11p + 27 + \gcd(p - 1, 4)$.

12.20 Case 20

$$cb = baa = cac = 0,\ caa = bab.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} A \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{\beta^2} & \alpha\frac{y}{\beta^3} \\ \frac{1}{\beta}\frac{z}{\alpha} & \frac{1}{\beta^2}t \\ \frac{1}{\alpha^2}u & \frac{1}{\alpha\beta}v \end{pmatrix}$$

We can take $z = 0, 1$.

If $z = 0$ we can take $v = 0, 1$.

If $z = v = 0$ we can take $y = 0, 1$. If $z = v = y = 0$ we can take $u = 0, 1, \omega$ and we can take $x = 0$, $t = 0, 1, \omega$ or we can take $x = 1, \omega$, $0 \leq t < p$. If $z = v = 0$ and $y = 1$ then we can take $x = t = 0$, $u = 0, 1, \omega$ (or if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5, or we can take $x = 0$, $t = 1, \omega$, $0 \leq u < p$, or we can take $x = 1, \omega$, $0 \leq t, u < p$. ($2p^2 + 8p + 10 + 2\gcd(p-1, 3)$ algebras.)

If $z = 0$, $v = 1$ we need $\beta = \alpha^{-1}$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & t \\ u & 1 \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^2 x & \alpha^4 y \\ 0 & \alpha^2 t \\ \frac{1}{\alpha^2}u & 1 \end{pmatrix}.$$

So we can take $x = t = u = 0$, $y = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, or we can take $x = t = 0$, $u = 1, \omega$, $0 \leq y < p$, or $x = 0$, $t = 1, \omega$, $0 \leq y, u < p$, or $x = 1, \omega$, $0 \leq y, t, u < p$. ($2p^3 + 2p^2 + 2p + 1 + \gcd(p-1, 4)$ algebras.)

If $z = 1$ we need $\beta = \alpha^{-1}$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^{-1}\beta^2 \end{pmatrix} \begin{pmatrix} x & y \\ 1 & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha\beta^2 & 0 \\ 0 & \beta^3 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^2 x & \alpha^4 y \\ 1 & \alpha^2 t \\ \frac{1}{\alpha^2}u & v \end{pmatrix}$$

so we can take $0 \leq v < p$ and x, y, t, u as in the case when $z = 0$ and $v = 1$. ($2p^4 + 2p^3 + 2p^2 + p + p\gcd(p-1, 4)$ algebras.)

So the total number of algebras here is

$$2p^4 + 4p^3 + 6p^2 + 11p + 11 + 2\gcd(p-1, 3) + (p+1)\gcd(p-1, 4).$$

12.21 Case 21

$$cb = caa = cac = 0,\ baa = bab.$$

L_3 is generated by baa and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} baa \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 2\beta \\ 0 & \alpha & \beta \\ 0 & 0 & \gamma \end{pmatrix} A \begin{pmatrix} \alpha^3 & 2\alpha^2\beta \\ 0 & \alpha^2\gamma \end{pmatrix}^{-1}.$$

So we can assume that $pc = 0$, baa, bac or ωbac.

If $pc = 0$ we can take $pb = 0$, baa, ωbaa or bac.

If $pb = pc = 0$ we can take $pa = 0$, baa, ωbaa or bac. (4 algebras.)

If $pb = baa$ or ωbaa and $pc = 0$ then we need $\alpha = \pm 1$, $\beta = 0$ and so if $pa = xbaa + ybac$ then

$$pa' = \alpha xbaa + \alpha ybac = xb'a'a' + \alpha\gamma^{-1}yb'a'c'$$

so we cna take $0 \leq x < p$ and $y = 0, 1$. ($4p$ algebras.)

If $pb = bac$ and $pc = 0$ then we need $\gamma = \alpha^{-1}$ and so we can take $pa = baa$, ωbaa or $xbac$ with $0 \leq x < p$. ($p + 2$ algebras.)

If $pc = baa$ then we need $\beta = 0$ and $\gamma = \alpha^3$ and so if $pa = xbaa + ybac$, $pb = zbaa + tbac$ then

$$\begin{aligned} pa' &= \alpha xbaa + \alpha ybac = \alpha^{-2}xb'a'a' + \alpha^{-4}yb'a'c', \\ pb' &= \alpha zbaa + \alpha tbac = \alpha^{-2}zb'a'a' + \alpha^{-4}tb'a'c'. \end{aligned}$$

So we can take $x = 1, \omega$ and $0 \leq y, z, t < p$, or $x = 0$, $z = 1, \omega$ and $0 \leq y, t < p$, or $x = z = 0$, $y = 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3 and $0 \leq t < p$, or $x = y = z = 0$, $t = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. ($2p^3 + 2p^2 + 1 + (p+1)\gcd(p-1, 4)$ algebras.)

Finally, if $pc = kbac$ where $k = 1, \omega$ then we need $\alpha = \pm 1$. If $pa = xbaa + ybac$, $pb = zbaa + tbac$ then

$$\begin{aligned} pa' &= \alpha xbaa + (\alpha y + 2\beta k)bac = xb'a'a' + \gamma^{-1}(\alpha y + 2\beta k - 2\beta x)b'a'c', \\ pb' &= \alpha zbaa + (\alpha t + \beta k)bac = zb'a'a' + \gamma^{-1}(\alpha t + \beta k - 2\beta z)b'a'c'. \end{aligned}$$

So we can take $0 \leq x, z < p$. If $x \neq k$ we can take $y = 0$ (though we then need $\beta = 0$) and we can take $t = 0, 1$. If $x = k$ and $z \neq \frac{k}{2}$ we can take $y = 0, 1$ and $t = 0$. If $x = k$ and $z = \frac{k}{2}$ then we can take $y = 0$ and $t = 0, 1$ or $y = 1$ and $0 \leq t < p$. ($4p^2 + 2p$ algebras.)

The total number of algebras in this case is

$$2p^3 + 6p^2 + 7p + 7 + (p+1)\gcd(p-1, 4).$$

12.22 Case 22

$$cb = baa = caa = 0,\ cac = \omega bab.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \omega\beta & \pm\gamma \\ 0 & \omega\gamma & \pm\omega\beta \end{pmatrix} A \begin{pmatrix} \omega\alpha(\omega\beta^2 + \gamma^2) & \pm 2\omega\alpha\beta\gamma \\ 2\omega^2\alpha\beta\gamma & \pm\omega\alpha(\omega\beta^2 + \gamma^2) \end{pmatrix}^{-1}.$$

If we let $pa = xbab + ybac$ then

$$(x, y) \to (x, y) \begin{pmatrix} \omega(\omega\beta^2 + \gamma^2) & \pm 2\omega\beta\gamma \\ 2\omega^2\beta\gamma & \pm\omega(\omega\beta^2 + \gamma^2) \end{pmatrix}^{-1},$$

and a straightforward Burnside's Lemma count shows that there are 3 orbits for pa. Clearly $pa = 0$ and $pa = bac$ are in separate orbits, and by considering transformations with $\beta = 0$ or with $\gamma = 0$ it is easy to see that if $x \neq 0$ then $pa = xbac$ is in the same orbit as $pa = bac$. If $p = 1 \bmod 4$ then $\omega\beta^2 + \gamma^2 \neq 0$, and so $pa = bab$ gives the third orbit. If $p = 3 \bmod 4$ then it is possible to ønd non-zero β, γ so that $\omega\beta^2 + \gamma^2 = 0$ so that $pa = bab$ is in the same orbit as $pa = bac$, but there exists y such that $pa = bab + ybac$ gives the third orbit.

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \omega\beta & \gamma \\ 0 & \omega\gamma & \omega\beta \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \omega\alpha(\omega\beta^2+\gamma^2) & 2\omega\alpha\beta\gamma \\ 2\omega^2\alpha\beta\gamma & \omega\alpha(\omega\beta^2+\gamma^2) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x - x\omega^2\alpha\beta^2 - x\omega\alpha\gamma^2 - 2y\omega^2\alpha\beta\gamma & \alpha y - 2x\omega\alpha\beta\gamma - y\omega^2\alpha\beta^2 - y\omega\alpha\gamma^2 \\ \omega\beta z + \gamma u - z\omega^2\alpha\beta^2 - z\omega\alpha\gamma^2 - 2t\omega^2\alpha\beta\gamma & \omega\beta t + \gamma v - 2z\omega\alpha\beta\gamma - t\omega^2\alpha\beta^2 - t\omega\alpha\gamma^2 \\ \omega\gamma z + \omega\beta u - u\omega^2\alpha\beta^2 - u\omega\alpha\gamma^2 - 2v\omega^2\alpha\beta\gamma & \omega\gamma t + \omega\beta v - 2u\omega\alpha\beta\gamma - v\omega^2\alpha\beta^2 - v\omega\alpha\gamma^2 \end{pmatrix}$$

$$\left(\begin{array}{llllll} \alpha - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & 0 & 0 & 0 & 0 \\ -2\omega\alpha\beta\gamma & \alpha - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & \gamma & 0 \\ 0 & 0 & -2\omega\alpha\beta\gamma & \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & 0 & \gamma \\ 0 & 0 & \omega\gamma & 0 & \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & - \\ 0 & 0 & 0 & \omega\gamma & -2\omega\alpha\beta\gamma & \omega \end{array} \right.$$

Determinant: $\omega^2\alpha^2\left(-\omega\beta^2+\gamma^2\right)\left(\gamma^2\omega-(\omega\beta-1)^2\right)\left(\gamma^2\omega-(\omega\beta+1)^2\right)$
$\left(\gamma^4\omega\alpha^2 - \gamma^2 - 2\omega^2\alpha^2\beta^2\gamma^2 - 6\gamma^2\omega\alpha\beta + \omega\beta^2(\alpha\omega\beta-1)^2\right)\left(\omega\alpha^2\gamma^2 - (\alpha\omega\beta-1)^2\right)$
$\left(\gamma^4\omega - \gamma^2 - 2\omega^2\beta^2\gamma^2 - 6\gamma^2\omega\beta + \omega\beta^2(\omega\beta-1)^2\right)$

$$\begin{pmatrix} \alpha - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma \\ -2\omega\alpha\beta\gamma & \alpha - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 \end{pmatrix}$$

has determinant $\alpha^2\left(\omega\gamma^2-(\omega\beta-1)^2\right)\left(\omega\gamma^2-(\omega\beta+1)^2\right)$ and so has nullity 0 unless $\gamma = 0$. So if $\gamma \neq 0$ the nullity of the full 6×6 matrix is the same as the nullity of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & -\gamma & 0 \\ -2\omega\alpha\beta\gamma & \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & 0 & -\gamma \end{pmatrix} \left(\begin{array}{lll} \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & \gamma \\ -2\omega\alpha\beta\gamma & \omega\beta - \omega^2\alpha\beta^2 - \omega\alpha\gamma^2 & 0 \\ \omega\gamma & 0 & \omega\beta \\ 0 & \omega\gamma & -2 \end{array} \right.$$

$$\left(\begin{array}{cc} \omega^2\beta^2 - 2\omega^3\beta^3\alpha - 2\omega^2\beta\alpha\gamma^2 + \omega^4\alpha^2\beta^4 + 6\omega^3\alpha^2\beta^2\gamma^2 + \omega^2\alpha^2\gamma^4 - \omega\gamma^2 & 4\omega^3\left(-\beta + \omega\alpha\beta^2 + \alpha\gamma \right. \\ 4\omega^2\alpha\beta\gamma\left(-\beta + \omega\alpha\beta^2 + \alpha\gamma^2\right) & \omega^2\beta^2 - 2\omega^3\beta^3\alpha - 2\omega^2\beta\alpha\gamma^2 + \omega^4\alpha^2\beta^4 + 6 \end{array} \right.$$

Now this matrix has the form $\begin{pmatrix} a & \omega b \\ b & a \end{pmatrix}$ so it can only be singular if $a = b = 0$, and when $a = b = 0$ it has nullity 2. We are assuming that $\gamma \neq 0$, and we cannot have $a = b = 0$ with $\beta = 0$. So we have $\alpha\gamma^2 = (\beta - \omega\alpha\beta^2)$. Then

$$\begin{aligned} & \omega^2\beta^2 - 2\omega^3\beta^3\alpha - 2\omega^2\beta\alpha\gamma^2 + \omega^4\alpha^2\beta^4 + 6\omega^3\alpha^2\beta^2\gamma^2 + \omega^2\alpha^2\gamma^4 - \omega\gamma^2 \\ = \quad & -\omega\beta\frac{-\omega\beta\alpha - 4\omega^2\alpha^2\beta^2 + 4\omega^3\beta^3\alpha^3 + 1}{\alpha} \end{aligned}$$

So we need

$$(-1 + \omega\beta\alpha)(2\omega\beta\alpha + 1)(2\omega\beta\alpha - 1) = 0$$

If $-1+\omega\beta\alpha=0$ then $\beta=\frac{1}{\omega\alpha}$ and $(\beta-\omega\alpha\beta^2)=0$, giving $\gamma=0$. If $2\omega\beta\alpha+1=0$ then $\beta=-\frac{1}{2\omega\alpha}$ and $(\beta-\omega\alpha\beta^2)=-\frac{3}{4\omega\alpha}$ and so we get two non-zero solutions for γ if $p=2\bmod 3$. And if $2\omega\beta\alpha-1=0$ then $\beta=\frac{1}{2\omega\alpha}$ and $(\beta-\omega\alpha\beta^2)=\frac{1}{4\omega\alpha}$ so that there are no solutions for γ.

So if $p=2\bmod 3$ then there are $2(p-1)$ choices of α,β,γ with $\gamma\neq 0$ giving matrices with nullity 2.

If $\omega\beta=1$ and $\gamma=0$ then we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\alpha \end{pmatrix}$$

so we have nullity 2 if $\alpha\neq 1$ and nullity 6 if $\alpha=1$.

If $\omega\beta=-1$ and $\gamma=0$ then we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\alpha \end{pmatrix}$$

so we have nullity 2 if $\alpha\neq -1$ and nullity 6 if $\alpha=-1$.

If $\omega\alpha\beta=1$ and $\gamma=0$, with $\alpha\neq\pm 1$ then we have

$$\begin{pmatrix} \alpha-\alpha^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha-\alpha^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so we have nullity 4.

So the contribution to Burnside's Lemma from matrices of the ørst type is

$$\begin{aligned} &2(p^6-1)+(p-3)(p^4-1)+(2p-4+(3-\gcd(p-1,3))(p-1))(p^2-1)+(p-1)(p^2-1) \\ &\qquad = (p+1)\,(p-1)^2\,\bigl(2p^3+3p^2+2p+9-\gcd(p-1,3)\bigr) \end{aligned}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \omega\beta & -\gamma \\ 0 & \omega\gamma & -\omega\beta \end{pmatrix}\begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix}-\begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix}\begin{pmatrix} \omega\alpha(\omega\beta^2+\gamma^2) & -2\omega\alpha\beta\gamma \\ 2\omega^2\alpha\beta\gamma & -\omega\alpha(\omega\beta^2+\gamma^2) \end{pmatrix}$$

$$=\begin{pmatrix} \alpha x-x\omega^2\alpha\beta^2-x\omega\alpha\gamma^2-2y\omega^2\alpha\beta\gamma & \alpha y+2x\omega\alpha\beta\gamma+y\omega^2\alpha\beta^2+y\omega\alpha\gamma^2 \\ \omega\beta z-\gamma u-z\omega^2\alpha\beta^2-z\omega\alpha\gamma^2-2t\omega^2\alpha\beta\gamma & \omega\beta t-\gamma v+2z\omega\alpha\beta\gamma+t\omega^2\alpha\beta^2+t\omega\alpha\gamma^2 \\ \omega\gamma z-\omega\beta u-u\omega^2\alpha\beta^2-u\omega\alpha\gamma^2-2v\omega^2\alpha\beta\gamma & \omega\gamma t-\omega\beta v+2u\omega\alpha\beta\gamma+v\omega^2\alpha\beta^2+v\omega\alpha\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\omega^2\alpha\beta^2-\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & 0 & 0 & 0 \\ 2\omega\alpha\beta\gamma & \alpha+\omega^2\alpha\beta^2+\omega\alpha\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & \omega\beta-\omega^2\alpha\beta^2-\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & -\gamma \\ 0 & 0 & 2\omega\alpha\beta\gamma & \omega\beta+\omega^2\alpha\beta^2+\omega\alpha\gamma^2 & 0 \\ 0 & 0 & \omega\gamma & 0 & -\omega\beta-\omega^2\alpha\beta^2-\omega\alpha\gamma^2 \\ 0 & 0 & 0 & \omega\gamma & 2\omega\alpha\beta\gamma \end{pmatrix}$$

Determinant: $-\omega^2\alpha^2\left(\omega\gamma^2+1-\beta^2\omega^2\right)\left(\omega\gamma^2-1-\beta^2\omega^2\right)\left(-\omega\beta^2+\gamma^2\right)^2\left(\omega\alpha^2\gamma^2-\omega^2\alpha^2\beta^2+1\right)^2$

If $\omega\gamma^2+1-\beta^2\omega^2=0$ then we have $\beta^2\omega^2=(\omega\gamma^2+1)$ which gives

$$\begin{pmatrix} -2\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & 0 & 0 & 0 & 0 \\ 2\omega\alpha\beta\gamma & 2\alpha+2\omega\alpha\gamma^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega\beta-2\omega\alpha\gamma^2-\alpha & -2\omega^2\alpha\beta\gamma & -\gamma & 0 \\ 0 & 0 & 2\omega\alpha\beta\gamma & \omega\beta+2\omega\alpha\gamma^2+\alpha & 0 & -\gamma \\ 0 & 0 & \omega\gamma & 0 & -\omega\beta-2\omega\alpha\gamma^2-\alpha & -2\omega^2\alpha\beta\gamma \\ 0 & 0 & 0 & \omega\gamma & 2\omega\alpha\beta\gamma & -\omega\beta+2\omega\alpha\gamma^2+\alpha \end{pmatrix}$$

$$\begin{pmatrix} -2\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma \\ 2\omega\alpha\beta\gamma & 2\alpha+2\omega\alpha\gamma^2 \end{pmatrix}$$

has determinant: $4\omega\alpha^2\gamma^2\left(-1-\omega\gamma^2+\omega^2\beta^2\right)$ and so has nullity 1.

Suppose $\gamma\neq 0$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\omega\beta-2\omega\alpha\gamma^2-\alpha & -2\omega^2\alpha\beta\gamma & \gamma & 0 \\ 2\omega\alpha\beta\gamma & -\omega\beta+2\omega\alpha\gamma^2+\alpha & 0 & \gamma \end{pmatrix}\begin{pmatrix} \omega\beta-2\omega\alpha\gamma^2-\alpha & -2\omega^2\alpha\beta\gamma & -\gamma \\ 2\omega\alpha\beta\gamma & \omega\beta+2\omega\alpha\gamma^2+\alpha & 0 \\ \omega\gamma & 0 & -\omega\beta-2\omega\alpha\gamma^2-\alpha \\ 0 & \omega\gamma & 2\omega\alpha\beta\gamma \end{pmatrix}$$

$$\begin{pmatrix} -\omega^2\beta^2+4\omega^2\alpha^2\gamma^4+4\omega\alpha^2\gamma^2+\alpha^2-4\omega^3\alpha^2\beta^2\gamma^2+\omega\gamma^2 & 0 \\ 0 & -\omega^2\beta^2+4\omega^2\alpha^2\gamma^4+4\omega\alpha^2\gamma^2+\alpha^2-4\omega^3\alpha^2\beta^2\gamma^2+\omega\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} (\alpha-1)(\alpha+1) & 0 \\ 0 & (\alpha-1)(\alpha+1) \end{pmatrix}$$

So if $\gamma\neq 0$ the nullity of the full 6×6 matrix is 1 if $\alpha\neq\pm 1$, and 3 if $\alpha=\pm 1$.

If $\gamma=0$ we have

$$\begin{pmatrix} \omega\beta-\alpha & 0 & 0 & 0 \\ 0 & \omega\beta+\alpha & 0 & 0 \\ 0 & 0 & -\omega\beta-\alpha & 0 \\ 0 & 0 & 0 & -\omega\beta+\alpha \end{pmatrix}$$

with $\omega\beta=\pm 1$ so again if $\alpha\neq\pm 1$ we have nullity 1, and if $\alpha=\pm 1$ we have nullity three.

There are $p+1$ pairs β,γ with $\omega\gamma^2+1-\beta^2\omega^2=0$ and so the contribution to Burnside's Lemma is

$$2(p+1)(p^3-1)+(p-3)(p+1)(p-1).$$

If $\omega\gamma^2-1-\beta^2\omega^2=0$ we have $\beta^2\omega^2=(\omega\gamma^2-1)$ which gives

$$\begin{pmatrix} 2\alpha-2\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma & 0 & 0 & 0 & 0 \\ 2\omega\alpha\beta\gamma & 2\omega\alpha\gamma^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega\beta-2\omega\alpha\gamma^2+\alpha & -2\omega^2\alpha\beta\gamma & -\gamma & 0 \\ 0 & 0 & 2\omega\alpha\beta\gamma & \omega\beta+2\omega\alpha\gamma^2-\alpha & 0 & -\gamma \\ 0 & 0 & \omega\gamma & 0 & -\omega\beta-2\omega\alpha\gamma^2+\alpha & -2\omega^2\alpha\beta\gamma \\ 0 & 0 & 0 & \omega\gamma & 2\omega\alpha\beta\gamma & -\omega\beta+2\omega\alpha\gamma^2-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha-2\omega\alpha\gamma^2 & -2\omega^2\alpha\beta\gamma \\ 2\omega\alpha\beta\gamma & 2\omega\alpha\gamma^2 \end{pmatrix}$$

has determinant 0, so has nullity 1. If $\gamma\neq 0$ then the 4×4 block gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\omega\beta-2\omega\alpha\gamma^2+\alpha & -2\omega^2\alpha\beta\gamma & \gamma & 0 \\ 2\omega\alpha\beta\gamma & -\omega\beta+2\omega\alpha\gamma^2-\alpha & 0 & \gamma \end{pmatrix}\begin{pmatrix} \omega\beta-2\omega\alpha\gamma^2+\alpha & -2\omega^2\alpha\beta\gamma & -\gamma \\ 2\omega\alpha\beta\gamma & \omega\beta+2\omega\alpha\gamma^2-\alpha & 0 \\ \omega\gamma & 0 & -\omega\beta-2\omega\alpha\gamma^2+\alpha \\ 0 & \omega\gamma & 2\omega\alpha\beta\gamma \end{pmatrix}$$

$$\begin{pmatrix} -\omega^2\beta^2+4\omega^2\alpha^2\gamma^4-4\omega\alpha^2\gamma^2+\alpha^2-4\omega^3\alpha^2\beta^2\gamma^2+\omega\gamma^2 & 0 \\ 0 & -\omega^2\beta^2+4\omega^2\alpha^2\gamma^4-4\omega\alpha^2\gamma^2+\alpha^2-4\omega^3\alpha^2\beta^2\gamma^2+\omega\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2+1 & 0 \\ 0 & \alpha^2+1 \end{pmatrix}$$

so the nullity is 1 if $\alpha^2+1 \neq 0$ and 3 if $\alpha^2+1=0$.

If $\gamma = 0$ then $\beta^2\omega^2 = -1$ and we get

$$\begin{pmatrix} \omega\beta+\alpha & 0 & 0 & 0 \\ 0 & \omega\beta-\alpha & 0 & 0 \\ 0 & 0 & -\omega\beta+\alpha & 0 \\ 0 & 0 & 0 & -\omega\beta-\alpha \end{pmatrix}$$

so again the nullity is 1 if $\alpha^2+1 \neq 0$ and 3 if $\alpha^2+1=0$.

There are $p+1$ pairs β, γ with $\omega\gamma^2-1-\beta^2\omega^2=0$ and so the contribution to Burnside's Lemma is

$$(\gcd(p-1,4)-2)(p+1)(p^3-1)+(p+1-\gcd(p-1,4))(p+1)(p-1).$$

If $\omega\alpha^2\gamma^2-\omega^2\alpha^2\beta^2+1$ with $\alpha^2 \neq \pm 1$ we have $\omega^2\beta^2 = (\omega\gamma^2+\alpha^{-2})$ which gives

$$\begin{pmatrix} \alpha^2-2\omega\alpha^2\gamma^2-1 & -2\omega^2\alpha^2\beta\gamma & 0 & 0 & 0 & \\ 2\omega\alpha^2\beta\gamma & \alpha^2+2\omega\alpha^2\gamma^2+1 & 0 & 0 & 0 & \\ 0 & 0 & \omega\beta\alpha-2\omega\alpha^2\gamma^2-1 & -2\omega^2\alpha^2\beta\gamma & -\alpha\gamma & \\ 0 & 0 & 2\omega\alpha^2\beta\gamma & \omega\beta\alpha+2\omega\alpha^2\gamma^2+1 & 0 & - \\ 0 & 0 & \omega\alpha\gamma & 0 & -\omega\beta\alpha-2\omega\alpha^2\gamma^2-1 & -2\omega \\ 0 & 0 & 0 & \omega\alpha\gamma & 2\omega\alpha^2\beta\gamma & -\omega\beta\alpha+ \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2-2\omega\alpha^2\gamma^2-1 & -2\omega^2\alpha^2\beta\gamma \\ 2\omega\alpha^2\beta\gamma & \alpha^2+2\omega\alpha^2\gamma^2+1 \end{pmatrix}$$

has determinant $(\alpha-1)(\alpha+1)(\alpha^2+1)$ and so the nullity is 0.

If $\gamma \neq 0$ then the 4×4 block gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\omega\beta\alpha-2\omega\alpha^2\gamma^2-1 & -2\omega^2\alpha^2\beta\gamma & \alpha\gamma & 0 \\ 2\omega\alpha^2\beta\gamma & -\omega\beta\alpha+2\omega\alpha^2\gamma^2+1 & 0 & \alpha\gamma \end{pmatrix} \begin{pmatrix} \omega\beta\alpha-2\omega\alpha^2\gamma^2-1 & -2\omega^2\alpha^2\beta\gamma & - \\ 2\omega\alpha^2\beta\gamma & \omega\beta\alpha+2\omega\alpha^2\gamma^2+1 & \\ \omega\alpha\gamma & 0 & -\omega\beta\alpha-2 \\ 0 & \omega\alpha\gamma & 2\omega \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so the nullity of the 4×4 block is 2.

If $\gamma = 0$ then the 4×4 block gives

$$\begin{pmatrix} \omega\beta\alpha-1 & 0 & 0 & 0 \\ 0 & \omega\beta\alpha+1 & 0 & 0 \\ 0 & 0 & -\omega\beta\alpha-1 & 0 \\ 0 & 0 & 0 & -\omega\beta\alpha+1 \end{pmatrix}$$

which again has nullity 2 since $\omega\beta\alpha = \pm 1$.

There are $p-1-\gcd(p-1,4)$ values of α with $\alpha^2 \neq \pm 1$, and for each value there are $p+1$ pairs β, γ with $\omega\alpha^2\gamma^2 - \omega^2\alpha^2\beta^2 + 1 = 0$. So the contribution to Burnside's Lemma is

$$(p-1-\gcd(p-1,4))(p+1)(p^2-1).$$

So the full contribution to Burnside's Lemma from transformations of the second kind is

$$2(p+1)(p^3-1)+(p-3)(p+1)(p-1)+(g-2)(p+1)(p^3-1)+(p+1-g)(p+1)(p-1)+(p-1-g)(p+1)(p^2-1)+(p-1)(p^2-1)$$

$$= (p+1)(p-1)^2(p+4+(p+1)\gcd(p-1,4)).$$

So the total number of algebras in Case 22 is

$$(2p^3+3p^2+3p+13-\gcd(p-1,3)+(p+1)\gcd(p-1,4))/2.$$

12.23 Case 23

$$cb = baa = 0,\ caa = bac,\ cac = \omega bab.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \pm\alpha \end{pmatrix} A \begin{pmatrix} \alpha^3 & 0 \\ 0 & \pm\alpha^3 \end{pmatrix}^{-1}$$

or when $p = 2 \bmod 3$ and $12\omega\beta^2 = -1$,

$$A \to \begin{pmatrix} 4\omega\alpha\beta & -3\omega\alpha\beta & \frac{\alpha}{2} \\ 0 & -2\omega\alpha\beta & \alpha \\ 0 & \pm\omega\alpha & \mp 2\omega\alpha\beta \end{pmatrix} A \begin{pmatrix} \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega\alpha^3 \\ \pm\frac{4}{3}\omega^2\alpha^3 & \pm\frac{8}{3}\omega^2\alpha^3\beta \end{pmatrix}^{-1}$$

Now

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \pm\alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \alpha^3 & 0 \\ 0 & \pm\alpha^3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\alpha^2}x & \pm\frac{1}{\alpha^2}y \\ \frac{1}{\alpha^2}z & \pm\frac{1}{\alpha^2}t \\ \pm\frac{1}{\alpha^2}u & \frac{1}{\alpha^2}v \end{pmatrix}$$

and so if $p = 1 \bmod 3$ there are $p^5+p^4+p^3+p^2+p+2+(p^2+p+1)\gcd(p-1,4)/2$ algebras.

In the case when $p = 2 \bmod 3$ and $12\omega\beta^2 = -1$, we have

$$\begin{pmatrix} 4\omega\alpha\beta & -3\omega\alpha\beta & \frac{\alpha}{2} \\ 0 & -2\omega\alpha\beta & \alpha \\ 0 & \omega\alpha & -2\omega\alpha\beta \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega\alpha^3 \\ \frac{4}{3}\omega^2\alpha^3 & \frac{8}{3}\omega^2\alpha^3\beta \end{pmatrix}$$

$$= \begin{pmatrix} 4\omega\alpha\beta x - 3\omega\alpha\beta z + \frac{1}{2}\alpha u - \frac{8}{3}x\omega^2\alpha^3\beta - \frac{4}{3}y\omega^2\alpha^3 & 4\omega\alpha\beta y - 3\omega\alpha\beta t + \frac{1}{2}\alpha v - \frac{4}{3}x\omega\alpha^3 - \frac{8}{3}y\omega^2\alpha^3\beta \\ -2\omega\alpha\beta z + \alpha u - \frac{8}{3}z\omega^2\alpha^3\beta - \frac{4}{3}t\omega^2\alpha^3 & -2\omega\alpha\beta t + \alpha v - \frac{4}{3}z\omega\alpha^3 - \frac{8}{3}t\omega^2\alpha^3\beta \\ \omega\alpha z - 2\omega\alpha\beta u - \frac{8}{3}u\omega^2\alpha^3\beta - \frac{4}{3}v\omega^2\alpha^3 & \omega\alpha t - 2\omega\alpha\beta v - \frac{4}{3}u\omega\alpha^3 - \frac{8}{3}v\omega^2\alpha^3\beta \end{pmatrix}$$

$$\begin{pmatrix} 4\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & -\frac{4}{3}\omega^2\alpha^3 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha & 0 \\ -\frac{4}{3}\omega\alpha^3 & 4\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & 0 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha \\ 0 & 0 & -2\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & -\frac{4}{3}\omega^2\alpha^3 & \alpha & 0 \\ 0 & 0 & -\frac{4}{3}\omega\alpha^3 & -2\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & 0 & \alpha \\ 0 & 0 & \omega\alpha & 0 & -2\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & -\frac{4}{3}\omega^2\alpha^3 \\ 0 & 0 & 0 & \omega\alpha & -\frac{4}{3}\omega\alpha^3 & -2\omega\alpha\beta - \frac{8}{3}\omega^2 \end{pmatrix}$$

Determinant: $\frac{16}{729}\alpha^6\omega^4\left(4\omega\beta^2 - 1\right)\left(4\omega\alpha^2 + 3\right)^2\left(4\omega^2\alpha^4\beta^2 - 12\omega\alpha^2\beta^2 + 9\beta^2 - \alpha^4\omega\right)$
$\left(64\omega^3\alpha^4\beta^2 + 96\omega^2\alpha^2\beta^2 + 36\omega\beta^2 - 9 + 24\omega\alpha^2 - 16\omega^2\alpha^4\right)$
If we use the fact that $\beta^2 = \frac{-1}{12\omega}$ we see that the determinant is

$$-\frac{64}{19683}\alpha^6\omega^3\left(4\omega\alpha^2 + 3\right)^2\left(16\omega^2\alpha^4 - 12\omega\alpha^2 + 9\right)^2$$

Now $16\omega^2\alpha^4 - 12\omega\alpha^2 + 9$ has no roots, but $4\omega\alpha^2 + 3$ has 2 roots. The matrix

$$\begin{pmatrix} 4\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & -\frac{4}{3}\omega^2\alpha^3 \\ -\frac{4}{3}\omega\alpha^3 & 4\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta \end{pmatrix}$$

has determinant $-\frac{4}{27}\omega\alpha^2\left(16\omega^2\alpha^4 - 12\omega\alpha^2 + 9\right)$, and so the nullity of the matrix is the same as the nullity of

$$\begin{pmatrix} -2\omega\beta - \frac{8}{3}\omega^2\alpha^2\beta & -\frac{4}{3}\omega^2\alpha^2 & 1 & 0 \\ -\frac{4}{3}\omega\alpha^2 & -2\omega\beta - \frac{8}{3}\omega^2\alpha^2\beta & 0 & 1 \\ \omega & 0 & -2\omega\beta - \frac{8}{3}\omega^2\alpha^2\beta & -\frac{4}{3}\omega^2\alpha^2 \\ 0 & \omega & -\frac{4}{3}\omega\alpha^2 & -2\omega\beta - \frac{8}{3}\omega^2\alpha^2\beta \end{pmatrix}$$

If we substitute $\frac{-3}{4\omega}$ for α^2 we get

$$\begin{pmatrix} 0 & \omega & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \omega & 0 & 0 & \omega \\ 0 & \omega & 1 & 0 \end{pmatrix}$$

which has nullity 2.
So the contribution to Burnsides Lemma is $4(p^2-1) + 2(p-1)$.

$$\begin{pmatrix} 4\omega\alpha\beta & -3\omega\alpha\beta & \frac{\alpha}{2} \\ 0 & -2\omega\alpha\beta & \alpha \\ 0 & -\omega\alpha & 2\omega\alpha\beta \end{pmatrix}\begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix}\begin{pmatrix} \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega\alpha^3 \\ -\frac{4}{3}\omega^2\alpha^3 & -\frac{8}{3}\omega^2\alpha^3\beta \end{pmatrix}$$

$$= \begin{pmatrix} 4\omega\alpha\beta x - 3\omega\alpha\beta z + \frac{1}{2}\alpha u - \frac{8}{3}x\omega^2\alpha^3\beta + \frac{4}{3}y\omega^2\alpha^3 & 4\omega\alpha\beta y - 3\omega\alpha\beta t + \frac{1}{2}\alpha v - \frac{4}{3}x\omega\alpha^3 + \frac{8}{3}y\omega^2\alpha^3\beta \\ -2\omega\alpha\beta z + \alpha u - \frac{8}{3}z\omega^2\alpha^3\beta + \frac{4}{3}t\omega^2\alpha^3 & -2\omega\alpha\beta t + \alpha v - \frac{4}{3}z\omega\alpha^3 + \frac{8}{3}t\omega^2\alpha^3\beta \\ -\omega\alpha z + 2\omega\alpha\beta u - \frac{8}{3}u\omega^2\alpha^3\beta + \frac{4}{3}v\omega^2\alpha^3 & -\omega\alpha t + 2\omega\alpha\beta v - \frac{4}{3}u\omega\alpha^3 + \frac{8}{3}v\omega^2\alpha^3\beta \end{pmatrix}$$

$$\begin{pmatrix} 4\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega^2\alpha^3 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha & 0 \\ -\frac{4}{3}\omega\alpha^3 & 4\omega\alpha\beta + \frac{8}{3}\omega^2\alpha^3\beta & 0 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha \\ 0 & 0 & -2\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega^2\alpha^3 & \alpha & 0 \\ 0 & 0 & -\frac{4}{3}\omega\alpha^3 & -2\omega\alpha\beta + \frac{8}{3}\omega^2\alpha^3\beta & 0 & \alpha \\ 0 & 0 & -\omega\alpha & 0 & 2\omega\alpha\beta - \frac{8}{3}\omega^2\alpha^3\beta & \frac{4}{3}\omega^2\alpha^3 \\ 0 & 0 & 0 & -\omega\alpha & -\frac{4}{3}\omega\alpha^3 & 2\omega\alpha\beta + \frac{8}{3}\omega^2\alpha^3 \end{pmatrix}$$

Determinant: $\frac{64}{19683}\alpha^6\omega^3\left(4\omega\alpha^2-3\right)^3\left(4\omega\alpha^2+3\right)^3$

If $\alpha^2=\frac{-3}{4\omega}$ we have

$$\begin{pmatrix} 1 & -6\omega\beta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 6\omega\alpha\beta & -\omega\alpha & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha & 0 \\ \alpha & 2\omega\alpha\beta & 0 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha \\ 0 & 0 & 0 & -\omega\alpha & \alpha & 0 \\ 0 & 0 & \alpha & -4\omega\alpha\beta & 0 & \alpha \\ 0 & 0 & -\omega\alpha & 0 & 4\omega\alpha\beta & -\omega\alpha \\ 0 & 0 & 0 & -\omega\alpha & \alpha & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3\omega\beta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -4\omega\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -3\omega\alpha\beta & 18\omega^2\beta^2\alpha & \frac{1}{2}\alpha & -3\omega\alpha\beta \\ 0 & 0 & -\omega\alpha & \alpha & 0 \\ 0 & \alpha & -4\omega\alpha\beta & 0 & \alpha \\ 0 & 0 & -4\omega^2\alpha\beta & 4\omega\alpha\beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{-1}{2}\omega\alpha & \frac{1}{2}\alpha & 0 \\ 0 & -\omega\alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has nullity 3.

If $\alpha^2=\frac{3}{4\omega}$ we have

$$\begin{pmatrix} 1 & 2\omega\beta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\omega & 1 & 0 \\ 0 & 0 & 1 & -4\omega\beta & 0 & 1 \end{pmatrix}\begin{pmatrix} 2\omega\alpha\beta & \omega\alpha & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha & 0 \\ -\alpha & 6\omega\alpha\beta & 0 & -3\omega\alpha\beta & 0 & \frac{1}{2}\alpha \\ 0 & 0 & -4\omega\alpha\beta & \omega\alpha & \alpha & 0 \\ 0 & 0 & -\alpha & 0 & 0 & \alpha \\ 0 & 0 & -\omega\alpha & 0 & 0 & \omega\alpha \\ 0 & 0 & 0 & -\omega\alpha & -\alpha & 4\omega\alpha\beta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -3\omega\beta & 0 & 0 \\ 0 & 1 & -4\omega\beta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -3\omega\alpha\beta & -6\omega^2\beta^2\alpha & \frac{1}{2}\alpha & \omega\alpha\beta \\ 0 & -4\omega\alpha\beta & \omega\alpha & \alpha & 0 \\ 0 & -\alpha & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{2}\omega\alpha & \frac{1}{2}\alpha & -2\omega\alpha\beta \\ 0 & \omega\alpha & \alpha & -4\omega\alpha\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which also has nullity 3.

So the contribution to Burnside's Lemma is $2\gcd(p-1,4)(p^3-1)+2(p-1)$.

So the number of algebras here is

$$\frac{2(p^6-1)+3\gcd(p-1,4)(p^3-1)+4(p^2-1)+6(p-1)}{6(p-1)}$$
$$=\frac{1}{3}p^5+\frac{1}{3}p^4+\frac{1}{3}p^3+\frac{1}{3}p^2+p+2+(p^2+p+1)\gcd(p-1,4)/2$$

$$cb = baa = 0,\ caa = kbab + bac,\ cac = \omega bab,\ (p = 2 \bmod 3),$$

where k is any element of $\mathbb{Z}_p$ which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)}.$$

L_3 is generated by bab and bac and if we let

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = A \begin{pmatrix} bab \\ bac \end{pmatrix}$$

then the isomorphism classes of algebras satisfying these commutator relations correspond to the orbits of 3×2 matrices A under the action

$$A \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} A \begin{pmatrix} \alpha^3 & 0 \\ 0 & \alpha^3 \end{pmatrix}^{-1}$$

and

$$A \to \begin{pmatrix} -4\alpha & k\alpha\beta + 3\alpha & 3k\omega^{-1}\alpha + \alpha\beta \\ 0 & 2\alpha & 2\alpha\beta \\ 0 & 2\omega\alpha\beta & 2\alpha \end{pmatrix} A \begin{pmatrix} 32\alpha^3 & -32\alpha^3\beta \\ -32\omega\alpha^3\beta & 32\alpha^3 \end{pmatrix}^{-1}$$

where $\omega\beta^2 = -3$. For a proof that these are the only automorphisms see Appendix F.

The contribution to Burnside's Lemma from the ørst transformation is $2(p^6 - 1) + (p - 1)$.

$$\begin{pmatrix} -4\alpha & k\alpha\beta + 3\alpha & 3k\omega^{-1}\alpha + \alpha\beta \\ 0 & 2\alpha & 2\alpha\beta \\ 0 & 2\omega\alpha\beta & 2\alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \begin{pmatrix} 32\alpha^3 & -32\alpha^3\beta \\ -32\omega\alpha^3\beta & 32\alpha^3 \end{pmatrix}$$

$$\left(\begin{matrix} 4x\omega - z\omega k\beta - 3z\omega - 3uk - u\omega\beta + 32x\alpha^2\omega - 32y\omega^2\alpha^2\beta & -4y\omega + t\omega k\beta + 3t\omega + 3vk + v\omega\beta + 32x\alpha^2\beta\omega - 32y\alpha^2 \\ z + \beta u - 16z\alpha^2 + 16t\omega\alpha^2\beta & t + \beta v + 16z\alpha^2\beta - 16t\alpha^2 \\ \omega\beta z + u - 16u\alpha^2 + 16v\omega\alpha^2\beta & \omega\beta t + v + 16u\alpha^2\beta - 16v\alpha^2 \end{matrix} \right.$$

$$\begin{pmatrix} 4\omega + 32\alpha^2\omega & -32\omega^2\alpha^2\beta & -\omega k\beta - 3\omega & 0 & -3k - \omega\beta & 0 \\ 32\alpha^2\beta\omega & -4\omega - 32\alpha^2\omega & 0 & \omega k\beta + \omega & 0 & 3k + \omega\beta \\ 0 & 0 & 1 - 16\alpha^2 & 16\omega\alpha^2\beta & \beta & 0 \\ 0 & 0 & 16\alpha^2\beta & 1 - 16\alpha^2 & 0 & \beta \\ 0 & 0 & \omega\beta & 0 & 1 - 16\alpha^2 & 16\omega\alpha^2\beta \\ 0 & 0 & 0 & \omega\beta & 16\alpha^2\beta & 1 - 16\alpha^2 \end{pmatrix}$$

Now the matrix

$$\begin{pmatrix} 4\omega + 32\alpha^2\omega & -32\omega^2\alpha^2\beta \\ 32\alpha^2\beta\omega & -4\omega - 32\alpha^2\omega \end{pmatrix}$$

has determinant$-16\omega^2\left(16\alpha^2+4\alpha+1\right)\left(16\alpha^2-4\alpha+1\right)$ which is never zero when $p=2 \bmod 3$. So the nullity of the whole matrix is the same as the nullity of

$$\begin{pmatrix} 1-16\alpha^2 & 16\omega\alpha^2\beta & \beta & 0 \\ 16\alpha^2\beta & 1-16\alpha^2 & 0 & \beta \\ \omega\beta & 0 & 1-16\alpha^2 & 16\omega\alpha^2\beta \\ 0 & \omega\beta & 16\alpha^2\beta & 1-16\alpha^2 \end{pmatrix}$$

Determinant: $16\,(4\alpha-1)^2\,(4\alpha+1)^2\left(16\alpha^2+4\alpha+1\right)\left(16\alpha^2-4\alpha+1\right)$. So the determinant is only zero when $16\alpha^2=1$. This gives

$$\begin{pmatrix} 0 & \omega\beta & \beta & 0 \\ \beta & 0 & 0 & \beta \\ \omega\beta & 0 & 0 & \omega\beta \\ 0 & \omega\beta & \beta & 0 \end{pmatrix}$$

which has nullity 2. So the contribution to Burnside's Lemma from the two possible values of β is $4(p^2-1)+2(p-1)$.

So the number of orbits is

$$\frac{2(p^6-1)+(p-1)+4(p^2-1)+2(p-1)}{3(p-1)}=\frac{2}{3}p^5+\frac{2}{3}p^4+\frac{2}{3}p^3+\frac{2}{3}p^2+2p+3.$$

The total number of algebras from Case 23 and Case 24 is

$$p^5+p^4+p^3+p^2+4p+\frac{13}{2}-(p+\frac{3}{2})\gcd(p-1,3)+(p^2+p+1)\gcd(p-1,4)/2$$

The total number of algebras from cases 17, 18, 23 and 24 is

$$p^5+p^4+p^3+p^2+2p+5+(2p+2)\gcd(p-1,3)+(p^2+p+1)\gcd(p-1,4).$$

13 Immediate descendants of algebra 32 (5.15)

Algebra 5.15 has $3p^2+12p+14+(p+2)\gcd(p-1,4)$ descendants of order p^7 and p-class 3.

Algebra 5.15 has presentation

$$\langle a,b,c\,|\,cb,\ pa-ba,\ pb,\ pc,\ \text{class }2\rangle$$

and so if L is an immediate descendant of 5.15 of order p^7 then L_2 is generated by ba, ca modulo L_3 and L_3 has order p^2 and is generated by caa and cac. In addition, cb, $pa-ba$, pb and $pc\in L_3$. Adding a suitable scalar multiple of ca to b we can assume that cb is a scalar multiple of caa, and so we can assume that $cb=0$, caa or ωcaa.

13.1 $cb=0$

If a', b', c' generate L and satsify the same relations as a,b,c modulo L_3, and if in addition $c'b'=0$ then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= b, \\ c' &= \delta b+\varepsilon c \end{aligned}$$

modulo L_2, and

$$\begin{aligned}
c'a'a' &= \alpha^2\varepsilon caa + \alpha\gamma\varepsilon cac, \\
c'a'c' &= \alpha\varepsilon^2 cac, \\
pa' - b'a' &= \alpha(pa - ba) + \beta pb + \gamma pc, \\
pb' &= pb, \\
pc' &= \delta pb + \varepsilon pc.
\end{aligned}$$

So we can assume that $pb = 0$, caa or cac.

If $pb = 0$ we can assume that $pc = 0$, caa, ωcaa or cac.

If $pb = pc = 0$ then we can assume that $pa - ba = 0$, caa, cac or ωcac. (4 algebras.)

If $pb = 0$, $pc = kcaa$ with $k = 1$ or ω then we need $\alpha = \pm 1$ and $\gamma = 0$. If $pa - ba = xcaa + ycac$ then

$$pa' - b'a' = \alpha xcaa + \alpha ycac = \alpha\varepsilon^{-1}xc'a'a' + \varepsilon^{-2}yc'a'c'$$

and so we can assume that $x = 0$ and $y = 0, 1, \omega$ or $x = 1$ and $0 \le y < p$. ($2p + 6$ algebras.)

If $pb = 0$, $pc = cac$ then we need $\varepsilon = \alpha^{-1}$. If $pa - ba = xcaa + ycac$ then

$$pa' - b'a' = \alpha xcaa + (\alpha y + \gamma)cac = xc'a'a' + \alpha(\alpha y + \gamma - \gamma x)c'a'c'$$

so we can take $0 \le x < p$. If $x \ne 1$ we can take $y = 0$, and if $x = 1$ we can take $y = 0, 1, \omega$. ($p + 2$ algebras.)

If $pb = caa$ then we need $\gamma = 0$ and $\varepsilon = \alpha^{-2}$. We can asume that pc is a scalar multiple of cac, though we then need to take $\delta = 0$. So we can take $pc = 0$ or cac. If $pb = caa$ and $pc = 0$ we can assume that $pa - ba$ is a scalar multiple of cac (though we then need to take $\beta = 0$), and so we can take $pa - ba = 0$, cac, ωcac or (if $p = 1 \bmod 4$) $\omega^2 cac$ or $\omega^3 cac$. And if $pb = caa$, $pc = cac$ then we need $\alpha = \varepsilon = 1$. We can assume that $pa - ba$ is a scalar multiple of cac (though we then need to take $\beta = 0$), so we can take $pa - ba = xcac$ with $0 \le x < p$. ($p + 1 + \gcd(p - 1, 4)$ algebras.)

If $pb = cac$ then we need $\alpha = \varepsilon^{-2}$, and we can take $pc = 0$, caa, ωcaa or (if $p = 1 \bmod 4$) $\omega^2 caa$ or $\omega^3 caa$. If $pb = cac$ and $pc = 0$ then we need $\delta = 0$ and we can take $pa - ba = 0$ or caa. If $pb = cac$ and $pc = kcaa$ (with $k = 1, \omega, \omega^2$ or ω^3) then we need $\varepsilon^4 = 1$, $\alpha = \varepsilon^2$, $\delta = k\alpha\gamma\varepsilon$. If $pa - ba = xcaa + ycac$ then

$$pa' - b'a' = (k\gamma + \alpha x)caa + (\beta + \alpha y)cac$$

so we can take $pa - baa = 0$. ($2 + \gcd(p - 1, 4)$ algebras.)

So when $cb = 0$ there are a total of $4p + 15 + 2\gcd(p - 1, 4)$ algebras.

13.2 $cb = caa$ or ωcaa

Let $cb = kcaa$ where $k = 1$ or ω. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3, and if in addition $c'b' = kc'a'a'$ then

$$\begin{aligned}
a' &= \pm a + \beta b + \gamma c, \\
b' &= b, \\
c' &= \delta b + \varepsilon c
\end{aligned}$$

modulo L_2, and

$$\begin{aligned}
c'a'a' &= \varepsilon caa + \pm\gamma\varepsilon cac, \\
c'a'c' &= \pm\varepsilon^2 cac, \\
pa' - b'a' &= \pm(pa - ba) + \beta pb + \gamma pc + 2k\gamma caa \pm k\gamma^2 cac, \\
pb' &= pb, \\
pc' &= \delta pb + \varepsilon pc.
\end{aligned}$$

We can take $pb = 0$, caa, cac or (if $p = 1 \bmod 4$) ωcac.

If $pb = 0$ we can take $pc = xcaa$ with $0 \leq x < p$ or cac.

Let $pb = pc = 0$. Then we can take $pa - ba$ to be a scalar multiple of cac (though we then need to take $\gamma = 0$), and so we can take $pa - ba = 0$, cac or ωcac. (3 algebras.)

Let $pb = 0$, $pc = xcaa$ with $x \neq 0$. Then we need $\gamma = 0$ and so if $pa - ba = ycaa + zcac$ then

$$pa' - b'a' = \pm ycaa \pm zcac = \pm\varepsilon^{-1}yc'a'a' + \varepsilon^{-2}zc'a'c'.$$

So we can take $y = 0$ and $z = 0, 1, \omega$ or $y = 1$ and $0 \leq z < p$. ($(p-1)(p+3)$ algebras.)

Let $pb = 0$, $pc = cac$. Then we need $\alpha = \varepsilon = \pm 1$, and if $pa - ba = xcaa + ycac$ then

$$pa' - b'a' = (\pm x + 2k\gamma)caa + (\pm y \pm k\gamma^2 + \gamma)cac$$

so we can assume that $x = 0$ (though we then need $\gamma = 0$) and we can assume that $0 \leq y < p$. (p algebras.)

If $pb = caa$ then we need $\gamma = 0$ and $\varepsilon = 1$. We can assume that pc is a scalar multiple of cac, though we then need to take $\delta = 0$, and we can assume that $pc = xcac$ with $0 \leq x \leq (p-1)/2$. We can assume that $pa - ba = ycac$ with $0 \leq y < p$. ($p(p+1)/2$ algebras.)

If $pb = lcac$ wih $l = 1$ or ω then we need $\varepsilon^2 = \alpha = \pm 1$, and we can take $pc = xcaa$ with $0 \leq x < p$. This means we need to take $\delta = l^{-1}x\alpha\gamma\varepsilon$, and if $pa - ba = ycaa + zcac$ then

$$pa' - b'a' = (\alpha y + x\gamma + 2k\gamma)caa + (\alpha z + l\beta + k\alpha\gamma^2)cac.$$

So if $x \neq -2k$ we can take $pa - ba = 0$, and if $x = -2k$ we can take $pa - ba = 0$ or $ycaa$ where y lies in a transversal for the fourth roots of unity. ($(p-1)/2 + p\gcd(p-1,4)/2$ algebras.)

So the total number of algebras with $cb = caa$ or ωcaa is $3p^2 + 8p - 1 + p\gcd(p-1,4)$.

14 Immediate descendants of algebra 33 (5.16)

Algebra 5.16 has $p^4 + 2p^3 + 5p^2 + 14p$ descendants of order p^7 and p-class 3.

Algebra 5.16 has presentation

$$\langle a, b, c \mid cb,\ pa,\ pb - ba,\ pc,\ \text{class } 2\rangle$$

and so L_2 is generated by ba, ca modulo L_3 and L_3 has order p^2 and is generated by caa and cac. In addition, cb, pa, $pb - ba$ and $pc \in L_3$. Adding a suitable scalar multiple of ca to b we may assume that cb is a scalar multiple of caa. So we may take $cb = 0$ or caa.

14.1 $cb = 0$

If a', b', c' generate L and satsify the same relations as a, b, c modulo L_3, and if $c'b' = 0$ then

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= \delta b, \\ c' &= \varepsilon c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a' &= \varepsilon caa + \gamma\varepsilon cac, \\ c'a'c' &= \varepsilon^2 cac, \\ pa' &= pa + \gamma pc, \\ pb' - b'a' &= \delta(pb - ba), \\ pc' &= \varepsilon pc. \end{aligned}$$

So we can take $pc = xcaa$ with $0 \leq x < p$ or cac.

If $pc = 0$ we can take $pa = 0$, caa, cac or ωcac.

Let $pa = pc = 0$. Then we can take $pb - ba = 0$, caa or cac. (3 algebras.)

Let $pa = caa$, $pc = 0$. Then we need $\gamma = 0$ and $\varepsilon = 1$, and so we can take $pb - ba = 0$, cac or $caa + xcac$ with $0 \leq x < p$. ($p + 2$ algebras.)

Let $pa = kcac$ where $k = 1$ or ω, and let $pc = 0$. Then we need $\varepsilon = \pm 1$, and we can take $pb - ba = 0$, caa or cac. (6 algebras.)

If $pc = xcaa$ where $x \neq 0$ then we need $\gamma = 0$. Let $pa = ycaa + zcac$, $pb - ba = tcaa + ucac$. Then

$$\begin{aligned} pa' &= ycaa + zcac = \varepsilon^{-1} yc'a'a' + \varepsilon^{-2} zc'a'c', \\ pb' - b'a' &= \delta tcaa + \delta ucac = \delta\varepsilon^{-1} tc'a'a' + \delta\varepsilon^{-2} uc'a'c'. \end{aligned}$$

So we can take $y = z = 0$ and $t, u = 0, 1$, or we can take $y = 0$, $z = 1, \omega$ and $u = 0$, $t = 0, 1$ or $u = 1$ and $0 \leq t \leq (p-1)/2$, or we can take $y = 1$, $0 \leq z < p$ and $t = 0$, $u = 0, 1$ or $t = 1$ and $0 \leq u < p$. ($(p-1)(p^2 + 3p + 9)$ algebras.)

If $pc = cac$ we need $\varepsilon = 1$. If $pa = xcaa + ycac$, $pb - ba = zcaa + tcac$ then

$$\begin{aligned} pa' &= xcaa + (y + \gamma)cac = xc'a'a' + (y + \gamma - \gamma x)c'a'c', \\ pb' - b'a' &= \delta zcaa + \delta tcac = \delta zc'a'a' + (\delta t - \delta z\gamma)c'a'c'. \end{aligned}$$

So we can take $0 \leq x < p$. If $x \neq 1$ then we can take $y = 0$ and $z = 0$, $t = 0, 1$ or $z = 1$, $0 \leq t < p$. If $x = 1$ we can take $0 \leq y < p$ and $z = 0$, $t = 0, 1$ or $z = 1$ and $t = 0$. ($p^2 + 4p - 2$ algebras.)

So the total number of algebras when $cb = 0$ is $p^3 + 3p^2 + 11p$.

14.2 $cb = caa$

If a', b', c' generate L and satsify the same relations as a, b, c modulo L_3, and if $c'b' = c'a'a'$ then

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= b, \\ c' &= \varepsilon c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a' &= \varepsilon caa + \gamma\varepsilon cac, \\ c'a'c' &= \varepsilon^2 cac, \\ pa' &= pa + \gamma pc, \\ pb' - b'a' &= pb - ba + 2\gamma caa + \gamma^2 cac, \\ pc' &= \varepsilon pc. \end{aligned}$$

We can assume that $pb - ba = 0$, cac or ωcac, though we then need $\gamma = 0$.

If $pb - ba = 0$ and $pa = xcaa + ycac$, $pc = zcaa + tcac$ then

$$\begin{aligned} pa' &= xcaa + ycac = \varepsilon^{-1}xc'a'a' + \varepsilon^{-2}yc'a'c', \\ pc' &= \varepsilon zcaa + \varepsilon tcac = zc'a'a' + \varepsilon^{-1}tc'a'c'. \end{aligned}$$

So we can take $0 \leq z < p$, and $x = t = 0$ and $y = 0, 1, \omega$, or $x = 0$, $t = 1$, $0 \leq y < p$, or $x = 1$ and $0 \leq y, t < p$. ($p^3 + p^2 + 3p$ algebras.)

If $pb - ba = kcac$ with $k = 1$ or ω then we need $\varepsilon = \pm 1$ and $\gamma = 0$. So if $pa = xcaa + ycac$, $pc = zcaa + tcac$ then we can take $x = 0$, $0 \leq y, z < p$, $0 \leq t \leq (p-1)/2$ or $1 \leq x \leq (p-1)/2$, $0 \leq y, z, t < p$. ($p^4 + p^2$ algebras.)

So when $cb = caa$ there are $p^4 + p^3 + 2p^2 + 3p$ algebras.

15 Immediate descendants of algebra 34 (5.18)

Algebra 5.18 has

$$3p^3 + 6p^2 + 6p + 11 + (p+7)\gcd(p-1,3) + (p+1)\gcd(p-1,4) + \gcd(p-1,5)$$

descendants of order p^7 and p-class 3.

Algebra 5.18 has presentation

$$\langle a, b, c \mid cb,\ pa,\ pb - ca,\ pc,\ \text{class } 2\rangle$$

and so L_2 is generated by ba, ca modulo L_3 and L_3 has order p^2 and is generated by baa and bab. In addition, cb, pa, $pb - ca$ and $pc \in L_3$. Adding a suitable scalar multiple of ba to c we may assume that cb is a scalar multiple of baa. So we may assume that $cb = 0$ or baa.

15.1 $cb = 0$

If a', b', c' generate L and satsify the same relations as a, b, c modulo L_3, and if $c'b' = 0$, then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha\varepsilon b + \delta c, \\ c' &= \varepsilon c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \alpha^3\varepsilon baa, \\ b'a'b' &= \alpha^3\varepsilon^2 bab, \\ pa' &= \alpha pa + \gamma pc, \\ pb' - c'a' &= \alpha\varepsilon(pb - ca) + \delta pc, \\ pc' &= \varepsilon pc. \end{aligned}$$

If $pc = xbaa + ybab$ then

$$pc' = \varepsilon xbaa + \varepsilon ybab = \alpha^{-3}xb'a'a' + \alpha^{-3}\varepsilon^{-1}yb'a'b',$$

so we can assume that $x = 0$ or 1 or (if $p = 1 \bmod 3$) ω or ω^2 and that $y = 0$ or 1.

Consider the case when $pc = 0$. If $pa = xbaa + ybab$, $pb - ca = zbaa + tbab$ then

$$\begin{aligned} pa' &= \alpha xbaa + \alpha ybab = \alpha^{-2}\varepsilon^{-1}xb'a'a' + \alpha^{-2}\varepsilon^{-2}yb'a'b', \\ pb' - c'a' &= \alpha\varepsilon zbaa + \alpha\varepsilon tbab = \alpha^{-2}zb'a'a' + \alpha^{-2}\varepsilon^{-1}tb'a'b'. \end{aligned}$$

So we can take $x = t = 0$ and $y, z = 0, 1, \omega$, or we can take $x = 0$, $t = 1$ and $y = 0$, $z = 0, 1, \omega$ or $y = 1, \omega$, $0 \leq z < p$, or we can take $x = 1$, $0 \leq t < p$ and $y = 0$, $z = 0, 1, \omega$ or $y = 1, \omega$, $0 \leq z < p$. ($2p^2 + 5p + 12$ algebras.)

Next consider the case when $pc = kbaa$ where $k = 1$ or ω or ω^2. Then we need $\alpha^3 = 1$. We can assume that pa and $pb - ca$ are scalar multiples of bab, though we then need to take $\gamma = \delta = 0$. If $pa = xbab$ and $pb - ca = ybab$ then

$$\begin{aligned} pa' &= \alpha xbab = \alpha^{-2}\varepsilon^{-2}xb'a'b', \\ pb' - c'a' &= \alpha\varepsilon ybab = \alpha^{-2}\varepsilon^{-1}yb'a'b'. \end{aligned}$$

So we can take $y = 0$ and $x = 0, 1, \omega$ or $y = 1$ and $x = 0$ or x in a transversal for the cube roots of unity. ($p - 1 + 4\gcd(p - 1, 3)$ algebras.)

And now consider the case when $pc = bab$. Then we need $\varepsilon = \alpha^{-3}$. We can assume that pa and $pb - ca$ are scalar multiples of baa, though we then need to take $\gamma = \delta = 0$. If $pa = xbaa$ and $pb - ca = ybaa$ then

$$\begin{aligned} pa' &= \alpha xbaa = \alpha xb'a'a', \\ pb' - c'a' &= \alpha\varepsilon ybaa = \alpha^{-2}yb'a'a'. \end{aligned}$$

So we can take $x = 0$ and $y = 0, 1, \omega$ or $x = 1$ and $0 \leq y < p$. ($p + 3$ algebras.)

Finally consider the case when $pc = kbaa + bab$ where $k = 1, \omega$ or ω^2. Then we need $\alpha^3 = \varepsilon = 1$. We can assume that pa and $pb - ca$ are scalar multiples of baa, though we then need to take $\gamma = \delta = 0$. If $pa = xbaa$ and $pb - ca = ybaa$ then

$$\begin{aligned} pa' &= \alpha xbaa = \alpha xb'a'a', \\ pb' - c'a' &= \alpha ybaa = \alpha yb'a'a'. \end{aligned}$$

So if $p = 2 \bmod 3$ we can take $0 \leq x, y < p$ and if $p = 1 \bmod 3$ we can take $x = y = 0$, or $x = 0$, y in a transversal for the cube roots of unity, or x in a transversal for the cube roots of unity and $0 \leq y < p$. ($p^2 - 1 + \gcd(p - 1, 3)$ algebras.)

So when $cb = 0$ there are $3p^2 + 7p + 13 + 5\gcd(p - 1, 3)$ algebras.

15.2 $cb = baa$

If a', b', c' generate L and satsify the same relations as a, b, c modulo L_3, and if $c'b' = b'a'a'$, then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^3 b + \delta c, \\ c' &= \alpha^2 c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \alpha^5 baa, \\ b'a'b' &= \alpha^7 bab, \\ pa' &= \alpha pa + \gamma pc, \\ pb' - c'a' &= \alpha^3(pb - ca) + \delta pc, \\ pc' &= \alpha^2 pc. \end{aligned}$$

If $pc = xbaa + ybab$ then

$$pc' = \alpha^2 xbaa + \alpha^2 ybab = \alpha^{-3} xb'a'a' + \alpha^{-5} yb'a'b'$$

so we can take $x = 0$ and $y = 0, 1$ or (if $p = 1 \bmod 5$) ω, ω^2, ω^3 or ω^4, or we can take $x = 1$ or (if $p = 1 \bmod 3$) ω or ω^2 and $y = 0$ or y in a transversal for the cube roots of unity.

Let $pc = 0$ and let $pa = xbaa + ybab$, $pb = zbaa + tbab$. Then

$$\begin{aligned} pa' &= \alpha xbaa + \alpha ybab = \alpha^{-4} xb'a'a' + \alpha^{-6} yb'a'b', \\ pb' - c'a' &= \alpha^3 zbaa + \alpha^3 tbab = \alpha^{-2} zb'a'a' + \alpha^{-4} tb'a'b', \end{aligned}$$

and so if $p = 1 \bmod 4$ we can take $x = z = t = 0$, $y = 0, 1, \omega$ or (if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5, or $x = z = 0$, $t = 1, \omega, \omega^2, \omega^3$, $0 \leq y \leq (p-1)/2$, or $z = 0$, $x = 1, \omega, \omega^2, \omega^3$, $0 \leq y \leq (p-1)/2$, $0 \leq t < p$, or $z = 1, \omega$, $0 \leq x, y, t < p$, and if $p = 3 \bmod 4$ then we can take $x = z = t = 0$, $y = 0, 1, \omega$ or (if $p = 1 \bmod 3$) ω^2, ω^3, ω^4 or ω^5, or $x = z = 0$, $t = 1, \omega$, $0 \leq y < p$, or $z = 0$, $x = 1, \omega$, $0 \leq y, t < p$, or $z = 1, \omega$, $0 \leq x, y, t < p$. ($2p^3 + 2p^2 - 1 + 2\gcd(p-1, 3) + (p+1)\gcd(p-1, 4)$ algebras.)

Now let $pc = kbab$ where $k = 1, \omega, \omega^2, \omega^3$ or ω^4, so that we need $\alpha^5 = 1$. We can assume that pa and $pb - ca$ are scalar multiples of baa, though we then need to take $\gamma = \delta = 0$. If $pa = xbaa$, $pb - ca = ybaa$ then

$$\begin{aligned} pa' &= \alpha xbaa = \alpha^{-4} xb'a'a', \\ pb' - c'a' &= \alpha^3 ybaa = \alpha^{-2} yb'a'a', \end{aligned}$$

and so we can assume that $x = 0$ and that $y = 0$ or y lies in a transversal for the øfth roots of unity, or we can assume that x lies in a transversal for the øfth roots of unity and that $0 \leq y < p$. ($p^2 - 1 + \gcd(p-1, 5)$ algebras.)

Next let $pc = kbaa$ where $k = 1, \omega$ or ω^2, so that we need $\alpha^3 = 1$. We can assume that pa and $pb - ca$ are scalar multiples of bab, though we then need to take $\gamma = \delta = 0$. If $pa = xbab$, $pb - ca = ybab$ then

$$\begin{aligned} pa' &= \alpha xbab = xb'a'b', \\ pb' - c'a' &= ybab = \alpha^{-1} yb'a'b', \end{aligned}$$

and so we can take $0 \leq x < p$ and $y = 0$ or y in a transversal for the cube roots of unity. ($p(p - 1 + \gcd(p-1, 3))$ algebras.)

Finally let $pc = kbaa + ybab$ where $k = 1, \omega$ or ω^2 and y lies in a transversal for the cube roots of unitty, so that we need $\alpha = 1$. Then we can take If $pa = xbab$, $pb - ca = ybab$ where $0 \leq x, y < p$. ($(p-1)p^2$ algebras.)

So the total number of algebras when $cb = baa$ is

$$3p^3 + 3p^2 - p - 2 + (p+2)\gcd(p-1, 3) + (p+1)\gcd(p-1, 4) + \gcd(p-1, 5).$$

The number of descendants of 4.1 is 1361 if $p = 3$. For $p > 3$ it is $p^5 + 2p^4 + 7p^3 + 25p^2 + 88p + 270 + (p + 4)\gcd(p - 1, 3) + \gcd(p - 1, 4)$.

If L is an immediate descendant of 4.1 of order p^7 then L is generated by a, b, c, d, L_2 has order p^3, and $L_3 = \{0\}$.

16.1 L abelian

$$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pd, \text{ class } 2\rangle. \tag{7.193}$$

One algebra here. (See dec4.1 and group4.1.)

16.2 L^2 has order p

If L^2 has order p then we can assume that L^2 is generated by ba and that one of the following two sets of commutator relations hold:

$$\begin{aligned} ca &= da = cb = db = dc = 0, \\ ca &= da = cb = db = 0,\ dc = ba. \end{aligned}$$

(See dec4.1 and group4.1.)

Case 1 Let $ca = da = cb = db = dc = 0$. Then $C = \langle c, d\rangle + L_2$ is a characteristic subalgebra.

If pc, pd span a space of dimension 2 modulo L^2, then we may assume that $pa, pb \in L^2$.

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa, pb, \text{ class } 2\rangle, \tag{7.194}$$

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa - ba, pb, \text{ class } 2\rangle. \tag{7.195}$$

If pc, pd span a space of dimension 1 modulo L^2, then we may assume that $pc \in L^2$, $pd \notin L^2$. If $pc \neq 0$ then we may assume that $pc = ba$, and subtracting suitable multiples of c, d from a, b we may assume that pa, pb span a one dimensional subspace outside L^2. But then we can assume that $pb = 0$, and so we have

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pb, pc - ba, \text{ class } 2\rangle. \tag{7.196}$$

On the other hand, if $pc = 0$ then we can assume that pa, pd are linearly independant modulo L^2, and that $pb \in L^2$, giving

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pb, pc, \text{ class } 2\rangle, \tag{7.197}$$

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pb - ba, pc, \text{ class } 2\rangle. \tag{7.198}$$

Finally, consider the case when $pc, pd \in L^2$. Then we can assume that $pd = 0$ and that $pc = 0$ or ba. So we have

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pc, pd, \text{ class } 2\rangle, \tag{7.199}$$

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pc - ba, pd, \text{ class } 2\rangle. \tag{7.200}$$

Seven algebras in this case.

Case 2 Let $ca = da = cb = db = 0$, $dc = ba$. Since pL has order p^2 modulo L^2, there are elements $u, v \in L$ such that u, v are linearly independant modulo L^2 and such that $pu, pv \in L^2$. Clearly we can choose u, v so that $pv = 0$. If u and v commute, then we can take $a = u$, $c = v$, and then we have

$$\langle a, b, c, d \,|\, ca, da, cb, db, dc - ba, pa, pc, \text{ class } 2\rangle, \tag{7.201}$$

$$\langle a, b, c, d \,|\, ca, da, cb, db, dc - ba, pa - ba, pc, \text{ class } 2\rangle. \tag{7.202}$$

And if u and v do not commute then we can take $a = u$, $b = v$, giving

$$\langle a, b, c, d \,|\, ca, da, cb, db, dc - ba, pa, pb, \text{ class } 2\rangle, \tag{7.203}$$

$$\langle a, b, c, d \,|\, ca, da, cb, db, dc - ba, pa - ba, pb, \text{ class } 2\rangle. \tag{7.204}$$

Four algebras here.

16.3 L^2 has order p^2

If L^2 has order p^2 then we can assume that one of the following sets of commutator relations holds:

$$\begin{aligned} da &= cb = db = dc = 0, \\ ca &= da = cb = db = 0, \\ da &= cb = dc = 0,\ db = ca, \\ da &= cb = 0,\ db = ca,\ dc = \omega ba. \end{aligned}$$

Note that L^2 is generated by ba, ca in all but the second of these algebras. In the second algebra, L^2 is generated by ba, dc. We obtain $2p + 29$ algebras in the ørst case, $(p^2 - 1)/2 + 4p + 30$ in the second, $3p + 25$ in the third, and $(p^2 - 1)/2 + 2p + 6$ in the fourth.

Case 1 (See dec4.1a and group4.1a.)

Let $da = cb = db = dc = 0$. Note that d is central.

If $pd = 0$ the subalgebra generated by a, b, c is one of 6.90 ˇ 6.103. This gives

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pb,\ pc, pd, \text{ class } 2\rangle, \tag{7.205}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pa,\ pc, pd, \text{ class } 2\rangle, \tag{7.206}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pb - ba,\ pc, pd, \text{ class } 2\rangle, \tag{7.207}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pb - ca,\ pc, pd, \text{ class } 2\rangle, \tag{7.208}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pa,\ pc - ba, pd, \text{ class } 2\rangle, \tag{7.209}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pa,\ pc - ca, pd, \text{ class } 2\rangle, \tag{7.210}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pa - ba,\ pc, pd, \text{ class } 2\rangle, \tag{7.211}$$

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pa - ca,\ pc, pd, \text{ class } 2\rangle, \tag{7.212}$$

and $(p + 1)/2$ algebras

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pb - ba,\ pc - \mu ca, pd, \text{ class } 2\rangle \tag{7.213}$$

($\mu \neq 0$, μ, μ^{-1} give isomorphic algebras),

$$\langle a, b, c, d \,|\, cb, da, db, dc,\ pb - ba - ca,\ pc - ca, pd, \text{ class } 2\rangle, \tag{7.214}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pb-\omega ca,\ pc-ba,pd,\ \text{class } 2\rangle, \tag{7.215}$$

and $(p-1)/2$ algebras

$$\langle a,b,c,d \mid cb,da,db,dc,\ pb-\mu ca,\ pc-ba-ca,pd,\ \text{class } 2\rangle\ (1+4\mu \text{ not a square}), \tag{7.216}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa-ca,\ pc-ba,pd,\ \text{class } 2\rangle, \tag{7.217}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa-ba,\ pc-ca,pd,\ \text{class } 2\rangle. \tag{7.218}$$

If $pd \notin L^2$, then we can assume that $pa,pb,pc \in L^2$ and so the subalgebra generated by a,b,c is one of 5.14 ˇ 5.23. So we have

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb,\ pc,\ \text{class } 2\rangle, \tag{7.219}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa-ba,\ pb,\ pc,\ \text{class } 2\rangle, \tag{7.220}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-ba,\ pc,\ \text{class } 2\rangle, \tag{7.221}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa-ca,\ pb-ba,\ pc,\ \text{class } 2\rangle, \tag{7.222}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-ca,\ pc,\ \text{class } 2\rangle, \tag{7.223}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa-ba,\ pb-ca,\ pc,\ \text{class } 2\rangle, \tag{7.224}$$

and $(p+1)/2$ algebras

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-ba,\ pc-\lambda ca,\ \text{class } 2\rangle \tag{7.225}$$

with $\lambda \neq 0$, where λ and λ^{-1} give isomorphic algegbras,

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-ba-ca,\ pc-ca,\ \text{class } 2\rangle, \tag{7.226}$$

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-\omega ca,\ pc-ba,\ \text{class } 2\rangle, \tag{7.227}$$

and $(p-1)/2$ algebras

$$\langle a,b,c,d \mid cb,da,db,dc,\ pa,\ pb-\alpha ca,\ pc-ba-ca,\ \text{class } 2\rangle \tag{7.228}$$

where $1+4\alpha$ is not a square.

Finally, consider the case when $pd \neq 0$, $pd \in L^2$. Then we can assume that $pd = ca$. At least one of pa,pb,pc must lie outside L^2. If $pc \notin L^2$ then we can assume that pa,pb are scalar multiples of ba. So we have

$$\langle a,b,c,d \mid cb,da,db,dc,pa,pb,pd-ca,\ \text{class } 2\rangle, \tag{7.229}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pa,pb-ba,pd-ca,\ \text{class } 2\rangle, \tag{7.230}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pa-ba,pb,pd-ca,\ \text{class } 2\rangle. \tag{7.231}$$

If $pc \in L^2$, bu $pb \notin L^2$ then we can assume that pa,pc are scalar multiples of ba. So we have

$$\langle a,b,c,d \mid cb,da,db,dc,pa,pc,pd-ca,\ \text{class } 2\rangle, \tag{7.232}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pa,pc-ba,pd-ca,\ \text{class } 2\rangle, \tag{7.233}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pa-ba,pc,pd-ca,\ \text{class } 2\rangle. \tag{7.234}$$

And if $pb,pc \in L^2$ then $pa \notin L^2$, and we can assume that pb,pc are scalar multiples of ba. So we have

$$\langle a,b,c,d \mid cb,da,db,dc,pb,pc,pd-ca,\ \text{class } 2\rangle, \tag{7.235}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pb,pc-ba,pd-ca,\ \text{class } 2\rangle, \tag{7.236}$$

$$\langle a,b,c,d \mid cb,da,db,dc,pb-ba,pc,pd-ca,\ \text{class } 2\rangle. \tag{7.237}$$

Case 2 (See dec4.1b and group4.1b.)

Let $ca = da = cb = db = 0$, so that L^2 is generated by ba, dc. At least one of pa, pb, pc, pd must lie outside L^2, and there is no loss in generality in assuming that $pd \notin L^2$. We can then assume that $pc \in L^2$. We can either assume that $pa, pb \in L^2$, or assume that $pa \in L^2$, $pb - pd \in L^2$.

First consider the case when $pa, pb, pc \in L^2$, $pd \notin L^2$. Note that these conditions imply that $\langle a, b\rangle + L_2$ and $\langle c\rangle + L_2$ are characteristic subalgebras.

If $pa = pb = 0$ then we have

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb, pc, \text{ class } 2\rangle, \tag{7.238}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb, pc - ba, \text{ class } 2\rangle, \tag{7.239}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb, pc - dc, \text{ class } 2\rangle, \tag{7.240}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb, pc - ba - dc, \text{ class } 2\rangle. \tag{7.241}$$

If pa, pb span a subspace of L^2 of dimension one then we may suppose that $pa = 0$ and that $pb = ba$ or dc or $ba + dc$. So we have

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba, pc, \text{ class } 2\rangle, \tag{7.242}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba, pc - ba, \text{ class } 2\rangle, \tag{7.243}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba, pc - dc, \text{ class } 2\rangle, \tag{7.244}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba, pc - ba - dc, \text{ class } 2\rangle, \tag{7.245}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - dc, pc, \text{ class } 2\rangle, \tag{7.246}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - dc, pc - ba, \text{ class } 2\rangle, \tag{7.247}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - dc, pc - dc, \text{ class } 2\rangle, \tag{7.248}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - dc, pc - ba - dc, \text{ class } 2\rangle, \tag{7.249}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba - dc, pc, \text{ class } 2\rangle, \tag{7.250}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba - dc, pc - ba, \text{ class } 2\rangle, \tag{7.251}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa, pb - ba - dc, pc - \alpha ba - dc, \text{ class } 2\rangle\ (0 \leq \alpha < p). \tag{7.252}$$

Finally, if pa, pb span L^2 then we can assume that $pa = ba$, $pb = dc$. This gives

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa - ba, pb - dc, pc, \text{ class } 2\rangle, \tag{7.253}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa - ba, pb - dc, pc - ba, \text{ class } 2\rangle, \tag{7.254}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa - ba, pb - dc, pc - dc, \text{ class } 2\rangle, \tag{7.255}$$

$$\langle a,b,c,d \,|\, ca, da, cb, db, pa - ba, pb - dc, pc - ba - dc, \text{ class } 2\rangle. \tag{7.256}$$

Next assume that $pd \notin L^2$, and that pa, $pb - pd$, $pc \in L^2$. Note that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d, and if in addition $pa', pb' - pd', pc' \in L^2$ then either

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \gamma b, \\ c' &= \delta c, \\ d' &= \varepsilon c + \gamma d, \end{aligned}$$

or

$$
\begin{aligned}
a' &= \alpha c,\\
b' &= \beta c + \gamma d,\\
c' &= \delta a,\\
d' &= \varepsilon a + \gamma b
\end{aligned}
$$

for some non-zero α, γ, δ. We can take $pa = 0$, ba, dc or $ba + dc$.

If $pa = 0$ then we can take $pc = 0$, ba, dc or $ba + dc$.

First consider the situation when $pa = pc = 0$. Then we can let $pb - pd = 0$, ba, dc or $ba + dc$. However, if $pb - pd = dc$ then we can let $a' = -c$, $b' = d$, $c' = a$, $d' = b$ and then $pb' - pd' = b'a'$ So we have

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc, pd - pb, \text{ class } 2\rangle, \tag{7.257}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc, pd + ba - pb, \text{ class } 2\rangle, \tag{7.258}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc, pd + ba + dc - pb, \text{ class } 2\rangle. \tag{7.259}$$

Next suppose that $pa = 0$, $pc = ba$. Then replacing d by $d + \lambda c$ for suitable λ, we have

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - ba, pd - pb, \text{ class } 2\rangle, \tag{7.260}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - ba, pd + dc - pb, \text{ class } 2\rangle. \tag{7.261}$$

Similarly, if $pa = 0$, $pc = dc$ or $ba + dc$ then we have

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - dc, pd - pb, \text{ class } 2\rangle, \tag{7.262}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - dc, pd + ba - pb, \text{ class } 2\rangle, \tag{7.263}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - ba - dc, pd - pb, \text{ class } 2\rangle, \tag{7.264}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa, pc - ba - dc, pd + ba - pb, \text{ class } 2\rangle. \tag{7.265}$$

Now let $pa = ba$. Then we can assume that $pc \neq 0$, and so we can assume that $pc = ba$ or $pc = ba + \lambda dc$ $(\lambda \neq 0)$ or λdc $(\lambda \neq 0)$. If $pa = pc = ba$ then we can assume that $pb - pd = 0$ or dc. And if $pa = ba$, $pc = ba + \lambda dc$ $(\lambda \neq 0)$ or λdc $(\lambda \neq 0)$ then we can assume that $pb - pd = 0$. However, if $pa = ba$ and $pc = \lambda dc$, then swapping a and c and also swapping b and d, and then (further) replacing b, d by $\lambda b, \lambda d$ gives $pa = ba$, $pc = \lambda^{-1} dc$. So we have

$$\langle a, b, c, d \mid ca, da, cb, db, pa - ba, pc - ba, pd - pb, \text{ class } 2\rangle, \tag{7.266}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa - ba, pc - ba, pd + dc - pb, \text{ class } 2\rangle, \tag{7.267}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa - ba, pc - ba - \lambda dc, pd - pb, \text{ class } 2\rangle \ (\lambda \neq 0), \tag{7.268}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa - ba, pc - \lambda dc, pd - pb, \text{ class } 2\rangle \ (\lambda \neq 0,\ \lambda \sim \lambda^{-1}). \tag{7.269}$$

Next let $pa = dc$. We can assume that $pc = ba$, ωba, dc, or $\lambda ba + dc$ $(\lambda \neq 0)$. However, the case $pa = pc = dc$ reduces to the case $pa = pc = ba$ if we swap a and c and swap b and d. In all the other cases we can assume that $pb - pd = 0$. So we have

$$\langle a, b, c, d \mid ca, da, cb, db, pa - dc, pc - ba, pd - pb, \text{ class } 2\rangle, \tag{7.270}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa - dc, pc - \omega ba, pd - pb, \text{ class } 2\rangle, \tag{7.271}$$

$$\langle a, b, c, d \mid ca, da, cb, db, pa - dc, pc - \lambda ba - dc, pd - pb, \text{ class } 2\rangle \ (\lambda \neq 0). \tag{7.272}$$

Finally let $pa = ba + dc$. The cases $pc = \lambda ba$ and $pc = \lambda dc$ reduce to earlier cases if we swap a and c and swap b and d. So we can assume that $pc = \lambda ba + \mu dc$ with $\lambda, \mu \neq 0$. If we swap a and c and also swap b and d then we have

$$\begin{aligned} pa &= \mu ba + \lambda dc, \\ pc &= ba + dc. \end{aligned}$$

Then letting $a' = \alpha a$, $b' = \mu b$, $c' = \alpha\lambda\mu^{-1}c$, $d' = \mu d$ we have

$$\begin{aligned} pa' &= \alpha\mu ba + \alpha\lambda dc = b'a' + d'c' \\ pc' &= \alpha\lambda\mu^{-1}ba + \alpha\lambda\mu^{-1}dc = \lambda\mu^{-2}b'a' + \mu^{-1}d'c'. \end{aligned}$$

So we can take μ to lie in a set of representatives for equivalence classes of the set of integers $x \in \{1, 2, \ldots, p-1\}$ under the equivalence relation

$$x \sim y \text{ if } xy = 1 \bmod p.$$

If pa, pc are linearly independant then we can take $pb - pd = 0$, and if pa, pc are linearly dependant then we can assume that $pb - pd = 0$ or ba. So we have

$$\langle a, b, c, d \,|\, ca, da, cb, db, pa - ba - dc, pc - \lambda ba - \mu dc, pd - pb, \text{ class } 2\rangle \tag{7.273}$$

where $0 < \lambda, \mu < p$ and where if $0 < \nu < p$ and if $\mu\nu = 1 \bmod p$ then $\mu \leq \nu$ ($(p^2-1)/2$ algebras), and

$$\langle a, b, c, d \,|\, ca, da, cb, db, pa - ba - dc, pc - \mu ba - \mu dc, pd + ba - pb, \text{ class } 2\rangle \tag{7.274}$$

where $0 < \mu < p$ and where if $0 < \nu < p$ and if $\mu\nu = 1 \bmod p$ then $\mu \leq \nu$ ($(p+1)/2$ algebras).

Case 3 (See dec4.1c and group4.1c.)

Let $da = cb = dc = 0$, $db = ca$, so that L^2 is generated by ba, ca. At least one of pa, pb, pc, pd must lie outside L^2. It is straightforward to show that non-zero linear combinations of c and d all have breadth 1, and that all elements outside $C = \langle c, d\rangle + L_2$ all have breadth 2. So C is a characteristic subalgebra of L. Using this fact it is easy to see that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \varepsilon a + \zeta b + \eta c + \theta d, \\ c' &= \lambda(\zeta c - \varepsilon d), \\ d' &= \lambda(-\beta c + \alpha d). \end{aligned}$$

Furthermore, any a', b', c', d' of this form satisfy the same commutator relations as a, b, c, d.

At least one of pa, pb, pc, pd must lie outside L^2.

First consider the case when one (or both) of pc, pd lies outside L^2. Then we can assume that $pd \notin L^2$, and that $pa, pb, pc \in L^2$. If we consider a', b', c', d' as above, with $pa', pb', pc' \in L^2$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \zeta b + \eta c, \\ c' &= \lambda\zeta c, \\ d' &= \lambda(-\beta c + \alpha d), \end{aligned}$$

and

$$\begin{aligned} b'a' &= \alpha\zeta ba + \alpha\eta ca, \\ c'a' &= \lambda\alpha\zeta ca. \end{aligned}$$

So we can assume that $pc = 0$ or ba or ca.

Suppose $pc = 0$ and let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some 2×2 matrix A. Then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \zeta \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \zeta \end{pmatrix} A \begin{pmatrix} \alpha\zeta & \alpha\eta \\ 0 & \lambda\alpha\zeta \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix}.$$

It is straightforward to show that we can take A to be one of the following 7 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So we have the following algebras:

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb, pc, \text{ class } 2 \rangle, \tag{7.275}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ba, pb, pc, \text{ class } 2 \rangle, \tag{7.276}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ca, pb, pc, \text{ class } 2 \rangle, \tag{7.277}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ba, pc, \text{ class } 2 \rangle, \tag{7.278}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ca, pc, \text{ class } 2 \rangle, \tag{7.279}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ba, pb - ca, pc, \text{ class } 2 \rangle, \tag{7.280}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ca, pb - ba, pc, \text{ class } 2 \rangle. \tag{7.281}$$

Next, consider the case when $pc = ba$. Then if a', b', c', d' are as above, and if $pc' = b'a'$, then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \zeta b, \\ c' &= \alpha\zeta c, \\ d' &= \alpha(-\beta c + \alpha d), \end{aligned}$$

and

$$\begin{aligned} b'a' &= \alpha\zeta ba, \\ c'a' &= \alpha^2\zeta ca. \end{aligned}$$

If we let $pb = \rho ba + \sigma ca$ then

$$pb' = \rho\zeta ba + \sigma\zeta ca = \rho\alpha^{-1}b'a' + \sigma\alpha^{-2}c'a'.$$

So we can suppose that $pb = 0$, $ba + \sigma ca$ $(0 \leq \sigma < p)$, ca or ωca. If pb and pc are linearly independant then we can take $pa = 0$. If $pb = 0$ or ba then we can take $pa = 0$ or ca. So we have

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb, pc - ba, \text{ class } 2 \rangle, \tag{7.282}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ca, pb, pc - ba, \text{ class } 2\rangle, \tag{7.283}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ba, pc - ba, \text{ class } 2\rangle, \tag{7.284}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ca, pb - ba, pc - ba, \text{ class } 2\rangle, \tag{7.285}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ba - \sigma ca, pc - ba, \text{ class } 2\rangle \ (\sigma \neq 0), \tag{7.286}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ca, pc - ba, \text{ class } 2\rangle, \tag{7.287}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - \omega ca, pc - ba, \text{ class } 2\rangle. \tag{7.288}$$

Finally, consider the case when $pc = ca$. If a', b', c', d' are as above, and if $pc' = c'a'$, then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \zeta b + \eta c, \\ c' &= \lambda \zeta c, \\ d' &= \lambda(-\beta c + d), \end{aligned}$$

and

$$\begin{aligned} b'a' &= \zeta ba + \eta ca, \\ c'a' &= \lambda \zeta ca. \end{aligned}$$

So we can assume that pb is a scalar multiple of ba, or that $pb = ba + ca$. Furthermore, if $pb \neq 0$ then we can assume that $pa = 0$, and if $pb = 0$ then we can assume that $pa = 0$ or ba. So we have

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb, pc - ca, \text{ class } 2\rangle, \tag{7.289}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa - ba, pb, pc - ca, \text{ class } 2\rangle, \tag{7.290}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - \lambda ba, pc - ca, \text{ class } 2\rangle \ (\lambda \neq 0), \tag{7.291}$$

$$\langle a, b, c, d \mid da, cb, db - ca, dc, pa, pb - ba - ca, pc - ca, \text{ class } 2\rangle. \tag{7.291A}$$

Now consider the situation when $pc, pd \in L^2$. Then we may assume that $pa \notin L^2$, and that $pb \in L^2$. We can now consider new generators a', b', c', d' of the form

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \zeta b + \eta c + \theta d, \\ c' &= \lambda \zeta c, \\ d' &= \lambda(-\beta c + \alpha d), \end{aligned}$$

with

$$\begin{aligned} b'a' &= \alpha \zeta ba + (\alpha \eta + \beta \theta - \delta \zeta) ca, \\ c'a' &= \lambda \alpha \zeta ca. \end{aligned}$$

Let

$$\begin{pmatrix} pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some 2×2 matrix A. Then

$$\begin{pmatrix} pc' \\ pd' \end{pmatrix} = \begin{pmatrix} \lambda\zeta & 0 \\ -\lambda\beta & \lambda\alpha \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \begin{pmatrix} \lambda\zeta & 0 \\ -\lambda\beta & \lambda\alpha \end{pmatrix} A \begin{pmatrix} \alpha\zeta & \alpha\eta + \beta\theta - \delta\zeta \\ 0 & \lambda\alpha\zeta \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix}.$$

Note that whatever values we assign to $\alpha, \beta, \gamma, \zeta, \eta, \theta, \lambda$, we can choose the value of δ in such a way to give $\alpha\eta + \beta\theta - \delta\zeta$ any value we choose. It is straightforward to show that we can take A to be one of the following 7 matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And the remark above shows that we can take A to be one of these matrices, while still letting η, θ take arbitrary values. We need to work out the stabilizers of

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and they are given by $\lambda = \alpha$, $\beta = 0$; $\alpha = 1$, $\beta = 0$; $\lambda = \zeta$; $\zeta = 1$ respectively. (We take $\alpha\eta + \beta\theta - \delta\zeta = 0$ in every case.) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we can take $pb = 0$, ca or ωca. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we can take $pb = \rho ba$ for some ρ. If $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then we can take $pb = 0$ or ca. And if $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then we can take $pb = 0$ or ba. Finally, if $A = 0$ then we can take $pb = 0$, ba or ca. So we have

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc, pd, \text{ class } 2 \rangle, \tag{7.292}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - ba, pc, pd, \text{ class } 2 \rangle, \tag{7.293}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - ca, pc, pd, \text{ class } 2 \rangle, \tag{7.294}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc - ba, pd, \text{ class } 2 \rangle, \tag{7.295}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - ca, pc - ba, pd, \text{ class } 2 \rangle, \tag{7.296}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - \omega ca, pc - ba, pd, \text{ class } 2 \rangle, \tag{7.297}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - \rho ba, pc - ca, pd, \text{ class } 2 \rangle \; (0 \leq \rho < p), \tag{7.298}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc, pd - ba, \text{ class } 2 \rangle, \tag{7.299}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - ca, pc, pd - ba, \text{ class } 2 \rangle, \tag{7.300}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc, pd - ca, \text{ class } 2 \rangle, \tag{7.301}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb - ba, pc, pd - ca, \text{ class } 2 \rangle, \tag{7.302}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc - ba, pd - ca, \text{ class } 2 \rangle, \tag{7.303}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc, pb, pc - ca, pd - ba, \text{ class } 2 \rangle. \tag{7.304}$$

Case 4 (See dec4.1d and group4.1d.)

Let $da = cb = 0$, $db = ca$, $dc = \omega ba$, so that L^2 is generated by ba, ca. At least one of pa, pb, pc, pd must lie outside L^2. We consider possible generating sets a', b', c', d' for L, where a', b', c', d' satisfy the same commutator relations as a, b, c, d. As in the computation of nilpotent Lie rings of order p^6, it is straightforward to show if $a' = a$, $b' = b$, then we either have $c' = c$ and $d' = d$ modulo L^2 or we have $c' = -c$ and $d' = -d$ modulo L^2. It is also straightforward to show that if $a' = a$ then d' must centralize a, so that $d' = \alpha a + \beta d$ modulo L^2 for some α, β with $\beta \neq 0$. Finally, it is straightforward to show that if we let

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \lambda a + \mu b + \nu c + \xi d, \\ c' &= -\omega \xi a + \omega \nu b + \mu c - \lambda d, \\ d' &= \omega \delta a - \omega \gamma b - \beta c + \alpha d \end{aligned}$$

then a', b', c', d' satisfy the same commutator relations as a, b, c, d. It follows that we can take a' to be any element outside L^2, and (having øxed a') we can take b' to be any element outside the centralizer of a'. This øxes c', d' up to change of sign. Note that pL cannot be trivial.

Consider the case when pL has order p. Then we can ønd elements $u, v, w \in L$ such that u, v, w are linearly independant modulo L_2, but such that $pu = pv = pw = 0$. These three elements cannot all commute, and so we can assume that $pa = pb = 0$. There is some non-trivial linear combination e of c and d such that $pe = 0$, and so by considering a change of generators of the form

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \lambda a + \mu b, \\ c' &= \mu c - \lambda d, \\ d' &= -\beta c + \alpha d \end{aligned}$$

we can assume that $pa = pb = pc = 0$. So we have

$$\langle a, b, c, d \mid da, cb, db - ca, dc - \omega ba, pa, pb, pc, \text{ class } 2\rangle. \tag{7.305}$$

The argument above shows that even if pL has order greater than p, we can assume that $pa, pb, pc \in L^2$, and that $pd \notin L^2$. If a', b', c', d' satisfy the same commutator relations as a, b, c, d, and if $pa', pb', pc' \in L^2$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \mu b + \nu c, \\ c' &= \pm(\omega \nu b + \mu c), \\ d' &= \pm(-\omega \gamma b - \beta c + \alpha d), \end{aligned}$$

and

$$\begin{aligned} b'a' &= \alpha \mu ba + \alpha \nu ca, \\ c'a' &= \pm(\omega \alpha \nu ba + \alpha \mu ca). \end{aligned}$$

Let

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some 2×2 matrix A. Then

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \alpha^{-1} \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix} A \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix}.$$

If $pb = pc = 0$ then we can assume that $pa = 0$ or ba. The case $pa = pb = pc = 0$ is 7.305, and so we have

$$\langle a, b, c, d \mid da, cb, db - ca, dc - \omega ba, pa - ba, pb, pc, \text{ class } 2\rangle. \tag{7.306}$$

If pb, pc span a space of dimension 1 then we can assume that $pc = 0$, and we can assume that $pb = ba + \lambda ca$ ($0 \le \lambda < p$, with λ and $-\lambda$ giving isomorphic algebras) or that $pb = ca$. If $pb = ba + \lambda ca$ we can assume that $pa = 0$ or ca, and if $pb = ca$ we can assume that $pa = 0$ or ba. This gives

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc - \omega ba, pa, pb - ba - \lambda ca, pc, \text{ class } 2\rangle \ (0 \le \lambda < p,\ \lambda \sim -\lambda), \tag{7.307}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc - \omega ba, pa - ca, pb - ba - \lambda ca, pc, \text{ class } 2\rangle \ (0 \le \lambda < p,\ \lambda \sim -\lambda), \tag{7.308}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc - \omega ba, pa, pb - ca, pc, \text{ class } 2\rangle, \tag{7.309}$$

$$\langle a, b, c, d \,|\, da, cb, db - ca, dc - \omega ba, pa - ba, pb - ca, pc, \text{ class } 2\rangle. \tag{7.310}$$

If pb, pc span a space of dimension 2 then we can assume that $pa = 0$, and we can assume that the matrix A above lies in a set of representatives for the equivalence classes of rank 2 matrices under the equivalence relation given by

$$A \sim \alpha^{-1} \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix} A \begin{pmatrix} \mu & \nu \\ \pm\omega\nu & \pm\mu \end{pmatrix}^{-1}.$$

It is straightforward to show that

$$A = \alpha^{-1} \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} A \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix}^{-1}$$

under the following circumstances

- $\nu = 0$, $\alpha = 1$, any A,
- $\alpha = 1$, A has the form $\begin{pmatrix} u & v \\ \omega v & u \end{pmatrix}$,
- $\alpha = -1$, $\mu = 0$, A has the form $\begin{pmatrix} u & v \\ -\omega v & -u \end{pmatrix}$.

And it is also straightforward to show that

$$A = \alpha^{-1} \begin{pmatrix} \mu & \nu \\ -\omega\nu & -\mu \end{pmatrix} A \begin{pmatrix} \mu & \nu \\ -\omega\nu & -\mu \end{pmatrix}^{-1}$$

under the following circumstances

- $\nu = 0$, $\alpha = 1$, A has the form $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$,
- $\nu \ne 0$, $\alpha = 1$, A has the form $\begin{pmatrix} u & v \\ -\omega v & u - 2\mu\nu^{-1}v \end{pmatrix}$,
- $\nu = 0$, $\alpha = -1$, A has the form $\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$,
- $\nu \ne 0$, $\alpha = -1$, A has the form $\begin{pmatrix} u & v \\ \omega v - 2\mu\nu^{-1}u & -u \end{pmatrix}$.

So the contribution to the number of equivalence classes of rank 2 matrices A from transformations of the ørst kind is

$$\frac{1}{2(p-1)(p^2-1)}\left((p-1)(p^2-1)(p^2-p)+(p^2-p)(p^2-1)+(p-1)(p^2-1)\right)=\frac{p^2+1}{2},$$

and the contribution from transformations of the second kind is

$$\frac{1}{2(p-1)(p^2-1)}\left(2p(p-1)(p^2-1)\right)=p$$

So we obtain $(p+1)^2/2$ algebras

$$\langle a,b,c,d\,|\,da,cb,db-ca,dc-\omega ba,pa,pb-xba-yca,pc-zba-tca,\text{ class }2\rangle, \tag{7.311}$$

as $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ runs over a set of representatives for these equivalence classes.

16.4 L^2 has order p^3

If L^2 has order p^3 then L must have the same commutator structure as one of 7.15 ˘ 7.20 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$, so we can assume that one of the following sets of commutator relations holds:

$$\begin{aligned} da &= db=dc=0,\\ ca &= da=db=0,\\ ca &= da=dc=0,\\ ca &= da=0,\ dc=ba,\\ da &= 0,\ db=ca,\ dc=cb,\\ da &= 0,\ db=\omega ca,\ dc=ba. \end{aligned}$$

In case 1 we have $3p+18$ algebras. (See dec4.11 and group4.11.)

In case 2 we have $\frac{77}{2}p+\frac{173}{2}+11p^2+\frac{5}{2}p^3+\frac{1}{2}p^4$ algebras, but you need to add 2 if $p=1 \bmod 3$. (See group4.12.)

In case 3 we have $p^2+3p+15$, but again you need to add 2 if $p=1 \bmod 3$. (See group 4.13.)

In case 4 we have $3p^2+13p+31$ algebras, but we need to add 2 if $p=1 \bmod 4$ and add 2 if $p=1 \bmod 3$. (See group4.14.)

In case 5 we have 550 algebras when $p=3$ and

$$\begin{aligned} p^5+p^4+4p^3+6p^2+18p+19 \text{ if } p &= 1 \bmod 3,\\ p^5+p^4+4p^3+6p^2+16p+17 \text{ if } p &= 2 \bmod 3. \end{aligned}$$

(See orb4.1j3.)

In case 6 we have $\frac{9}{2}p+\frac{13}{2}+3p^2+\frac{1}{2}p^4+\frac{1}{2}p^3$ algebras. This gives 101 when $p=3$, 479 when $p=5$, and 1557 when $p=7$. (See group4.16.)

Case 1 Let L satisfy $da=db=dc=0$. Then L^2 is generated by ba, ca, cb and $pL\leq L^2$. The generator d is central, and we may assume that $pd=0$ or cb.

If $pd=0$ then the subalgebra generated by a,b,c is one of 6.104 ˘ 6.117. (There is no algebra 6.107.) This gives

$$\langle a,b,c,d\,|\,da,db,dc,pa,\ pb,\ pc,pd,\text{ class }2\rangle, \tag{7.312}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - cb,\ pb,\ pc, pd,\ \text{class } 2\rangle, \tag{7.313}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb,\ pc, pd,\ \text{class } 2\rangle, \tag{7.314}$$

$$\langle a,b,c,d \,|\ da, db, dc, pa - ca,\ pb - \mu cb,\ pc, pd,\ \text{class } 2\rangle\ (\mu \neq 0,\ \mu \sim \mu^{-1}), \tag{7.315}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ca - cb,\ pb - cb,\ pc, pd,\ \text{class } 2\rangle, \tag{7.316}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - \omega cb,\ pb - ca,\ pc, pd,\ \text{class } 2\rangle, \tag{7.317}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - \mu cb,\ pb - ca - cb,\ pc,\ pd,\ \text{class } 2\rangle\ (1 + 4\mu \text{ not a square}), \tag{7.318}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - ca,\ pc, pd,\ \text{class } 2\rangle, \tag{7.319}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - cb,\ pc, pd,\ \text{class } 2\rangle, \tag{7.320}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - cb,\ pc - kba - ca, pd,\ \text{class } 2\rangle\ (k = 0, 1, \ldots, p-1), \tag{7.321}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - ca,\ pc - cb, pd,\ \text{class } 2\rangle, \tag{7.322}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - ca,\ pc + cb, pd,\ \text{class } 2\rangle, \tag{7.323}$$

$$\langle a,b,c,d \,|\, da, db, dc, pa - ba,\ pb - ca,\ pc - \omega ba + cb, pd,\ \text{class } 2\rangle. \tag{7.324}$$

If $pd = cb$ then we can subtract suitable scalar multiples of d from a, b, c so that pa, pb, pc lie in the linear span of ba, ca. If a', b', c' span the same space as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \lambda b + \mu c, \\ c' &= \nu b + \xi c \end{aligned}$$

and

$$\begin{aligned} b'a' &= \alpha\lambda ba + \alpha\mu ca + (\beta\mu - \gamma\lambda)cb, \\ c'a' &= \alpha\nu ba + \alpha\xi ca + (\beta\xi - \gamma\nu)cb. \end{aligned}$$

Let

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}.$$

Then

$$\begin{pmatrix} pb' \\ pc' \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \alpha^{-1} \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} A \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix} \text{ modulo } \langle cb\rangle.$$

By Theorem 6 we can take A to be one of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & \lambda \\ 1 & 1 \end{pmatrix},$$

where $1 + 4\lambda$ is not a square. Furthermore none of these matrices are equivalent to each other, except that if $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then we can take $pa = 0$ or ba, giving

$$\langle a,b,c,d \,|\, da, db, dc, pa, pb, pc, pd - cb,\ \text{class } 2\rangle, \tag{7.325}$$

$$\langle a,b,c,d \mid da, db, dc, pa-ba, pb, pc, pd-cb, \text{ class } 2\rangle. \tag{7.326}$$

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then we can take $pa = 0$ or ca, giving

$$\langle a,b,c,d \mid da, db, dc, pa, pb-ba, pc, pd-cb, \text{ class } 2\rangle, \tag{7.327}$$

$$\langle a,b,c,d \mid da, db, dc, pa-ca, pb-ba, pc, pd-cb, \text{ class } 2\rangle. \tag{7.328}$$

If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then we can take $pa = 0$ or ba, giving

$$\langle a,b,c,d \mid da, db, dc, pa, pb-ca, pc, pd-cb, \text{ class } 2\rangle, \tag{7.329}$$

$$\langle a,b,c,d \mid da, db, dc, pa-ba, pb-ca, pc, pd-cb, \text{ class } 2\rangle. \tag{7.330}$$

In all other cases the matrix A has rank 2, and we can assume that $pa = 0$, giving

$$\langle a,b,c,d \mid da, db, dc, pa, pb-ba, pc-\lambda ca, pd-cb, \text{ class } 2\rangle\ (\lambda \neq 0,\ \lambda \sim \lambda^{-1}), \tag{7.331}$$

$$\langle a,b,c,d \mid da, db, dc, pa, pb-ba-ca, pc-ca, pd-cb, \text{ class } 2\rangle \tag{7.331A}$$

$$\langle a,b,c,d \mid da, db, dc, pa, pb-\omega ca, pc-ba, pd-cb, \text{ class } 2\rangle, \tag{7.332}$$

$$\langle a,b,c,d \mid da, db, dc, pa, pb-\lambda ca, pc-ba-ca, pd-cb, \text{ class } 2\rangle\ (1+4\lambda \text{ not a square}). \tag{7.333}$$

Case 2 Let L satisfy $ca = da = db = 0$. Then L^2 is generated by ba, cb, dc and $pL \leq L^2$. It is easy to see that the only elements of L of breadth 1 are (modulo L^2) non-trivial scalar multiples of a and d. It follows that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then (modulo L^2)

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \gamma b + \delta d, \\ c' &= \varepsilon a + \zeta c + \eta d, \\ d' &= \rho d \end{aligned} \tag{*}$$

or

$$\begin{aligned} a' &= \alpha d, \\ b' &= \beta a + \gamma c + \delta d, \\ c' &= \varepsilon a + \zeta b + \eta d, \\ d' &= \rho a. \end{aligned} \tag{**}$$

In the ørst case

$$\begin{aligned} b'a' &= \alpha\gamma ba, \\ c'b' &= -\gamma\varepsilon ba + \gamma\zeta cb - \delta\zeta dc, \\ d'c' &= \zeta\rho dc, \end{aligned}$$

and in the second case

$$\begin{aligned} b'a' &= -\alpha\gamma dc, \\ c'b' &= \beta\zeta ba - \gamma\zeta cb + \gamma\eta dc, \\ d'c' &= -\zeta\rho ba. \end{aligned}$$

It is straightforward to show that we can take (pa, pd) to be one of the following: $(0,0)$, $(0, dc)$, $(0, cb)$, $(0, ba)$, $(0, ba+dc)$, (dc, dc), (dc, cb), (dc, ba), $(dc, ba+dc)$, (cb, dc), (cb, cb), $(cb, cb+dc)$, $(cb, ba+dc)$, $(cb, ba+cb+dc)$, (ba, dc), $(ba, ba+dc)$, $(ba+dc, ba+\lambda dc)$ $(\lambda \neq 0)$. For each of these pairs we need to ønd the possible generating sets a', b', c', d' satisfying the same commutator relations as a, b, c, d and with pa', pd' corrsponding to pa, pd. In each case we obtain a subset of (*) and (**).

For $(pa, pd) = (0,0)$ we have no further restrictions.

For $(pa, pd) = (0, dc)$ we need (*) with $\zeta = 1$.

For $(pa, pd) = (0, cb)$ we need (*) with $\varepsilon = \delta = 0$ and $\rho = \gamma\zeta$.

For $(pa, pd) = (0, ba)$ we need (*) with $\alpha\gamma = \rho$.

For $(pa, pd) = (0, ba+dc)$ we need (*) with $\alpha\gamma = \rho$ and $\zeta = 1$.

For $(pa, pd) = (dc, dc)$ we need (*) with $\zeta = 1$ and $\rho = \alpha$.

For $(pa, pd) = (dc, cb)$ we need (*) with $\varepsilon = \delta = 0$, $\alpha = \zeta\rho$ and $\rho = \gamma\zeta$.

For $(pa, pd) = (dc, ba)$ we need (*) with $\alpha = \zeta\rho$ and $\rho = \alpha\gamma$ or (**) with $\alpha = -\zeta\rho$ and $\rho = -\alpha\gamma$.

For $(pa, pd) = (dc, ba+dc)$ we need (*) with $\alpha = \rho$, $\gamma = 1$ and $\zeta = 1$.

For $(pa, pd) = (cb, dc)$ we need (*) with $\delta = \varepsilon = 0$, $\alpha = \gamma$ and $\zeta = 1$.

For $(pa, pd) = (cb, cb)$ we need (*) with $\delta = \varepsilon = 0$, $\alpha = \rho = \gamma\zeta$ or (**) with $\beta = \eta = 0$ and $\alpha = \rho = -\gamma\zeta$.

For $(pa, pd) = (cb, cb+dc)$ we need (*) with $\delta = \varepsilon = 0$, $\alpha = \gamma = \rho$ and $\zeta = 1$.

For $(pa, pd) = (cb, ba+dc)$ we need (*) with $\delta = \varepsilon = 0$, $\alpha = \gamma$, $\rho = \alpha^2$ and $\zeta = 1$.

For $(pa, pd) = (cb, ba+cb+dc)$ we need (*) with $\delta = \varepsilon = 0$, $\alpha = \gamma = \zeta = \rho = 1$ or (**) with $\alpha = \beta = \eta = \rho = -1$, and $\gamma = \zeta = 1$.

For $(pa, pd) = (ba, dc)$ we need (*) with $\gamma = \zeta = 1$ or (**) with $\gamma = \zeta = -1$.

For $(pa, pd) = (ba, ba+dc)$ we need (*) with $\gamma = \zeta = 1$ and $\alpha = \rho$.

For $(pa, pd) = (ba+dc, ba+\lambda dc)$ $(\lambda \neq 0)$ we need (*) with $\gamma = \zeta = 1$ and $\alpha = \rho$ or (**) with $\gamma = -\lambda$, $\zeta = -\lambda^{-1}$ and $\alpha = \lambda^{-1}\rho$.

Now let

$$\begin{pmatrix} pa \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ cb \\ dc \end{pmatrix}$$

and let

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = B \begin{pmatrix} ba \\ cb \\ dc \end{pmatrix}$$

for 2×3 matrices A, B. We let A run over the possibilities described above, and for each A we consider transformations of the form (*) and (**) which preserve A (as described above), and we compute their eœect on B. We see that under a transformation of the form (*)

$$B \longmapsto \left(\begin{pmatrix} \gamma & 0 \\ 0 & \zeta \end{pmatrix} B + \begin{pmatrix} \beta & \delta \\ \varepsilon & \eta \end{pmatrix} A \right) \begin{pmatrix} \alpha\gamma & 0 & 0 \\ -\gamma\varepsilon & \gamma\zeta & -\delta\zeta \\ 0 & 0 & \zeta\rho \end{pmatrix}^{-1}$$

and that under a transformation of the form (**)

$$B \longmapsto \left(\begin{pmatrix} 0 & \gamma \\ \zeta & 0 \end{pmatrix} B + \begin{pmatrix} \beta & \delta \\ \varepsilon & \eta \end{pmatrix} A \right) \begin{pmatrix} 0 & 0 & -\alpha\gamma \\ \beta\zeta & -\gamma\zeta & \gamma\eta \\ -\zeta\rho & 0 & 0 \end{pmatrix}^{-1}$$

For each value of A we need to compute a set of representatives for the orbits of matrices B under this action.

If $pa = pd = 0$ then

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} \sim \alpha^{-1}\gamma^{-1}\rho^{-1}\zeta^{-1} \begin{pmatrix} \gamma\zeta\rho u + \gamma\rho v\varepsilon & \alpha\gamma\rho v & \alpha\gamma v\delta + \alpha\gamma^2 w \\ \rho\zeta^2 x + \rho\zeta y\varepsilon & \alpha\rho\zeta y & \alpha\zeta y\delta + \alpha\gamma\zeta z \end{pmatrix}$$

under transformations of the form (*), and

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} \sim \begin{pmatrix} z & y & x \\ w & v & u \end{pmatrix}$$

under the tranformation $a' = -d$, $b' = -c$, $c' - -b$, $d- = -a$. Considering just transformations of the form (*) we see that if $v \neq 0$ then we can take $v = 1$, $u = w = 0$, $x, y, z = 0, 1$, giving 8 algebras. If $v = 0$, $y \neq 0$ then we can take $y = 1$, $x = z = 0$, $u, w = 0, 1$, giving 4 more algebras. And if $v = y = 0$ then we can take $u, w, x, z = 0, 1$, except that if u, w, x, z are all non-zero then we can only take $u, w, x = 1$ and z then takes any non-zero value (giving an additional $p + 14$ algebras). this gives a total of $p + 26$ algebras. However if we consider

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} \sim \begin{pmatrix} z & y & x \\ w & v & u \end{pmatrix}$$

we see the following.

- If only one of v, y are non-zero, then we can take $v = x = z = 0$, $y = 1$, giving $pc = cb$, $pb = 0, dc, ba, ba + dc$ (4 algebras);
- If both of v, y are non-zero we can take $u = w = 0$, $y = v = 1$, $x, z = 0, 1$. But the transformation $a' = -d$, $b' = -c$, $c' - -b$, $d- = -a$ shows that $x = 0$, $z = 1$ is equivalent to $x = 1$, $z = 0$. So we have 3 algebras here.
- If $v = y = 0$, and we consider the 16 algebras given by $u, w, x, z = 0, 1$, then four of them are invariant under the transformation $a' = -d$, $b' = -c$, $c' - -b$, $d- = -a$, and the other twelve form six equivalence classes. So we get 10 algebras here.
- If $z \neq 0, 1$ then the transformation $a' = -d$, $b' = -c$, $c' - -b$, $d- = -a$ transforms

  $$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & z \end{pmatrix} \longmapsto \begin{pmatrix} z & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

  but further scaling changes this back to $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & z \end{pmatrix}$. So we have $p - 2$ algebras here.

Thus there are $p + 15$ algebras with $pa = pd = 0$.

If $pa = 0$ and $pd = dc$ then we can take B of the form $\begin{pmatrix} u & v & w \\ x & y & 0 \end{pmatrix}$ with

$$\begin{pmatrix} u & v & w \\ x & y & 0 \end{pmatrix} \sim \begin{pmatrix} \alpha^{-1}u + \alpha^{-1}v\varepsilon & v & \rho^{-1}v\delta + \rho^{-1}\gamma w + \rho^{-1}\delta \\ \alpha^{-1}\gamma^{-1}(x + y\varepsilon) & \gamma^{-1}y & 0 \end{pmatrix}.$$

So if $v \neq 0, -1$ we can take $u = w = 0$ and $x, y = 0, 1$. If $v = 0$ we can take $w = 0$ and if $y \neq 0$ we can take $x = 0$, giving 6 more algebras, with $u, x, y = 0, 1$. And if $v = -1$ we can take $u = 0$ and $w, x, y = 0, 1$. So we have $4p + 6$ algebras.

In the case when $pa = 0$, $pd = cb$ we can take B of the form $\begin{pmatrix} u & v & w \\ x & 0 & y \end{pmatrix}$ with

$$\begin{pmatrix} u & v & w \\ x & 0 & y \end{pmatrix} \sim \begin{pmatrix} \alpha^{-1}u & \zeta^{-1}v & \zeta^{-2}w \\ \alpha^{-1}\gamma^{-1}\zeta x & 0 & \gamma^{-1}\zeta^{-1}y \end{pmatrix}$$

so there are $p^2 + 8p + 21$ algebras.

If $pa = 0$, $pd = ba$ we can take B of the form $\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix}$ with

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & \zeta^{-1}x & \alpha^{-1}\zeta^{-2}(-x^2\varepsilon + \zeta y) \\ 0 & \gamma^{-1}z & \alpha^{-1}\gamma^{-1}\zeta^{-1}(-zx\varepsilon + \zeta t) \end{pmatrix}.$$

So if $x \neq 0$ we can take $x = 1$, $y = 0$, $z, t = 0, 1$, and if $x = 0$ we can take $y, z, t = 0, 1$, giving 12 algebras.

In the case when $pa = 0$, $pd = ba + dc$ we can take B of the form $\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix}$ with

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \alpha^{-1}\gamma^{-1} \begin{pmatrix} 0 & \alpha\gamma x & -\gamma x^2\varepsilon + \gamma y - \gamma x\varepsilon \\ 0 & \alpha z & -zx\varepsilon + t - z\varepsilon \end{pmatrix}$$

so there are $5p + 4$ algebras.

In the case when $pa = pd = dc$ then we can take B of the form $\begin{pmatrix} x & y & 0 \\ z & t & 0 \end{pmatrix}$ with

$$\begin{pmatrix} x & y & 0 \\ z & t & 0 \end{pmatrix} \sim \alpha^{-1}\gamma^{-1} \begin{pmatrix} \gamma x + \gamma y\varepsilon & \alpha\gamma y & 0 \\ z + t\varepsilon & \alpha t & 0 \end{pmatrix}$$

so there are $4p + 2$ algebras.

In the case when $pa = dc$, $pd = cb$, we can take B of the form $\begin{pmatrix} a & b & 0 \\ c & 0 & d \end{pmatrix}$ with

$$\begin{pmatrix} a & b & 0 \\ c & 0 & d \end{pmatrix} \sim \begin{pmatrix} \gamma^{-1}\zeta^{-2}a & \zeta^{-1}b & 0 \\ \gamma^{-2}\zeta^{-1}c & 0 & \gamma^{-1}\zeta^{-1}d \end{pmatrix}.$$

So there are $p^2 + 2p + 9$ algebras when $p \neq 1 \bmod 3$, and $p^2 + 2p + 11$ algebras when $p = 1 \bmod 3$.

If $pa = dc$, $pd = ba$ then we can take B of the form $\begin{pmatrix} 0 & x & 0 \\ 0 & y & z \end{pmatrix}$ with

$$\begin{pmatrix} 0 & x & 0 \\ 0 & y & z \end{pmatrix} \sim \begin{pmatrix} 0 & \gamma x & 0 \\ 0 & \gamma^{-1}y & \alpha^{-1}\gamma^{-1}(-y\gamma x\varepsilon + z + \gamma\varepsilon) \end{pmatrix}$$

under transformations of type (*). So we can take $x = 0$ or 1. If $x = 0$ then we can take $y = 0, 1$, $z = 0$. If $x = 1$ then y can take any value, and we can take $z = 0$ unless $x = y = 1$, in which case we can take $z = 0, 1$. So we have the following possibilities for B:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Under transformations of type (**) these are equivalent to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

respectively. But

$$\begin{pmatrix} 0 & y & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \end{pmatrix}$$

if $y \neq 0$ under transformations of the form (*). So we have the following possibilities for B:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

giving $p+2$ algebras.

In the case when $pa = dc$, $pd = ba + dc$ we can take B to be of the form $\begin{pmatrix} 0 & x & 0 \\ 0 & y & z \end{pmatrix}$ where $0 \leq x, y < p$ and $z = 0$ unless $(1+x)y = 1 \bmod p$, in which case we can take $z = 0$ or 1. So there are $p^2 + p - 1$ algebras.

In the case when $pa = cb$, $pd = dc$ we can take B of the form $\begin{pmatrix} x & 0 & y \\ z & t & 0 \end{pmatrix}$ where

$$\begin{pmatrix} x & 0 & y \\ z & t & 0 \end{pmatrix} \sim \begin{pmatrix} \gamma^{-1}x & 0 & \gamma\rho^{-1}y \\ \gamma^{-2}z & \gamma^{-1}t & 0 \end{pmatrix}.$$

So we can take $x, y = 0, 1$. If $x = 1$ then z, t can take any values. If $x = 0$ we can take $t = 0, 1$. If $x = 0, t = 1$ then z can take any value. And if $x = t = 0$ then $z = 0, 1, \omega$. So we have $2(p^2 + p + 3)$ algebras.

If $pa = pd = cb$ then we can take B of the form $\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix}$ where

$$\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix} \sim \begin{pmatrix} \gamma^{-1}\zeta^{-1}x & 0 & \zeta^{-2}y \\ \gamma^{-2}z & 0 & \gamma^{-1}\zeta^{-1}t \end{pmatrix}$$

under transformations of type (*), and where

$$\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix} \sim \begin{pmatrix} t & 0 & z \\ y & 0 & x \end{pmatrix}$$

under transformations of type (**). It $x = t = 0$ then we can take $y, z = 0, 1, \omega$, giving 9 possibilities in 6 equivalence classes under transformations of type (**). If one of x, t is zero, and the other non-zero then we can assume that $x = 0$, $t = 1$. We can then take one of y, z to be $0, 1, \omega$, but only both if one of them equals 0. So this gives $2p+3$ further algebras. If x, t are both non-zero then we can take $x = 1$, with $(x, t) = (1, t)$ equivalent to $(x, t) = (1, t^{-1})$. If we choose t from a set of representatives from the classes $\{t, t^{-1}\}$ $(t \neq 0)$, then if $t \neq 1, -1$ we get $2p+3$ algebras for this t. For $t = \pm 1$ the pairs y, z gives the same algebra as z, y. So we get $3p+3$ algebras with $t = \pm 1$. So the total number of algebras is $p^2 + (7p+15)/2$.

If $pa = cb$, $pd = cb + dc$ then we can take B of the form $\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix}$ where

$$\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix} \sim \begin{pmatrix} \gamma^{-1}x & 0 & y \\ \gamma^{-2}z & 0 & \gamma^{-1}t \end{pmatrix}$$

so we have $p^3 + p^2 + 3p$ algebras.

If $pa = cb$, $pd = ba + dc$ then We can take B of the form $\begin{pmatrix} x & 0 & y \\ 0 & z & t \end{pmatrix}$ where

$$\begin{pmatrix} x & 0 & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} \gamma^{-1}x & 0 & \gamma^{-1}y \\ 0 & \gamma^{-1}z & \gamma^{-2}t \end{pmatrix}$$

so there are p^3+p^2+p+3 algebras here.

If $pa=cb$, $pd=ba+cb+dc$ then we can take B of the form $\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix}$ where (under transformations of type (**))

$$\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix} \sim \begin{pmatrix} 1+t & 0 & z+1 \\ y-1 & 0 & x-1 \end{pmatrix}.$$

So matrices $\begin{pmatrix} 1+t & 0 & z+1 \\ y & 0 & t \end{pmatrix}$ are in orbits of size 1, and other matrices of the form $\begin{pmatrix} x & 0 & y \\ z & 0 & t \end{pmatrix}$ are in orbits of size 2. So there are $(p^4+p^2)/2$ algebras of this form.

If $pa=ba$, $pd=dc$ then we can take B of the form $\begin{pmatrix} 0 & x & y \\ z & t & 0 \end{pmatrix}$ where

$$\begin{pmatrix} 0 & x & y \\ z & t & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & x & \rho^{-1}(x\delta+y+\delta) \\ \alpha^{-1}(z+\varepsilon+t\varepsilon) & t & 0 \end{pmatrix}$$

Under transformations of type (*) and

$$\begin{pmatrix} 0 & x & y \\ z & t & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & t & z \\ y & x & 0 \end{pmatrix}$$

under transformations of type (**). So we can take $y=0$ unless $x=-1$, when we can take $y=0$ or 1. And we can take $z=0$ unless $t=-1$ in which case we can take $z=0$ or 1. We can also take $x \leq t$, and if $t=x=-1$ we can take $y \leq z$. So there are $(p+2)(p+1)/2$ algebras of this kind.

If $pa=ba$, $pd=ba+dc$ then we can take b of the form $\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix}$ with

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & x & \alpha^{-1}(x\delta+y+\delta) \\ 0 & z & \alpha^{-1}(z\delta+t-\varepsilon-z\varepsilon) \end{pmatrix}.$$

So if $x \neq -1$ we can take $y=0$ and if $z \neq -1$ then we can take $t=0$. If $x=z=-1$ we can still take $t=0$ and we can take $y=0,1$. So we have p^2 matrices with $y=t=0$. And we have p matrices with $x=-1$, $y=1$, $t=0$, and $p-1$ matrices with $x \neq -1$, $y=0$, $z=-1$, $t=1$. So we have p^2+2p-1 algebras here.

Finally, consider the case when $pa=ba+dc$, $pd=ba+\lambda dc$ with $\lambda \neq 0$. Then we can take B of the form $\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix}$ with

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & x & \alpha^{-1}(x\delta+y-\delta-x\varepsilon+\delta\lambda) \\ 0 & z & \alpha^{-1}(z\delta+t+\varepsilon-\lambda\varepsilon-\lambda z\varepsilon) \end{pmatrix}$$

under transformations of type (*) and

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & \lambda z & \alpha^{-1}(\beta-\lambda z\eta+t-\lambda^{-1}\beta+z\beta) \\ 0 & \lambda^{-1}x & \alpha^{-1}(-x\eta+\lambda^{-1}y-\eta\lambda+\eta+\lambda^{-2}x\beta) \end{pmatrix}$$

under transformations of type (**). We can take $y=t=0$ unless $\lambda=1$ or $(x+\lambda)(1+z)=1$.

If $\lambda=1$ then we have

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & x & \alpha^{-1}(x\delta+y-x\varepsilon) \\ 0 & z & \alpha^{-1}(z\delta+t-z\varepsilon) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & x & y \\ 0 & z & t \end{pmatrix} \sim \begin{pmatrix} 0 & z & \alpha^{-1}(-z\eta + t + z\beta) \\ 0 & x & \alpha^{-1}(-x\eta + y + x\beta) \end{pmatrix}.$$

So when $\lambda = 1$ we can take $0 \leq x \leq z < p$. If $x \neq 0$ we can take $y = 0$ and $t = 0$ or 1. If $x = 0$, $z \neq 0$ we can take $y = 0$ or 1 and $t = 0$, and if $x = z = 0$ then we can take $(y, t) = (0, 0)$, $(0, 1)$, or $(1, k)$ with $k \neq 0$, where $(1, k) \sim (1, k^{-1})$. So there are $p^2 + (3p + 1)/2$ algebras when $\lambda = 1$.

If $\lambda \neq 1$ then again we get $(p(p+1)/2$ algebras with $y = t = 0$. But if $(x+\lambda)(1+z) = 1$ then we cannot guarantee that $y = t = 0$. For each value of $z \neq -1$ there is a unique value of x such that $(x + \lambda)(1 + z) = 1$, and for this value of x we can still take either one of y, t to be 0, and the other to be 1. So for each value of z other than -1 we get one extra algbra. But we need to consider the eœect of a transformation of type (**) on these algebras. The pair $(x, z) = ((1+z)^{-1} - \lambda, z)$ transforms to $(\lambda z, \lambda^{-1}(1+z)^{-1} - 1)$. If λ is a square then there are two values of z (both distinct from -1) for which $((1 + z)^{-1} - \lambda, z)$ transforms into itself. And if λ is not a square there are no such values. So if λ is not a square then we get $(p - 1)/2$ algebras of this kind. And if λ is a square (other than 1) then there are $(p + 1)/2$ algebras of this kind.

So the total number of algebras for $pa = ba+dc$, $pd = ba+\lambda dc$ $(\lambda \neq 0)$ is $p^2+(3p+1)/2$ when $\lambda = 1$, $(p+1)^2/2$ when λ is square other than 1, and $p(p+1)/2+(p-1)/2$ when λ is not a square. So the total number of algebras of this kind (for all $\lambda \neq 0$) is $\frac{1}{2}p^3 + p^2 - \frac{1}{2}p$.

Case 3 Let L satisfy $ca = da = dc = 0$. If we swap a and b then $\langle b, c, d\rangle + L^2$ is a characteristic abelian subalgebra, L^2 is generated by ba, ca, da, and $pL \leq L^2$. We let

$$\begin{pmatrix} pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ da \end{pmatrix},$$

where A is a 3×3 matrix over $\mathbb{Z}_p$. If we let

$$a' = \alpha a + \beta b + \gamma c + \delta d \ (\alpha \neq 0)$$

and let

$$\begin{pmatrix} b' \\ c' \\ d' \end{pmatrix} = P \begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

for some non-singular matrix P over $\mathbb{Z}_p$, then a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d. Furthermore

$$\begin{pmatrix} pb' \\ pc' \\ pd' \end{pmatrix} = \alpha^{-1}PAP^{-1} \begin{pmatrix} b'a' \\ c'a' \\ d'a' \end{pmatrix}.$$

So we can take A to lie in a set of representatives for equivalence classes of 3×3 matrices under the equivalence relation

$$A \sim \alpha^{-1}PAP^{-1}.$$

So consider matrices A under this equivalence relation.

If A has rank 3 then A is similar to the companion matrix of a polynomial of degree 3 with non-zero constant term, or to a matrix of one of the following forms:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda, \mu \neq 0$, $\lambda \neq \mu$. We can choose α so that A is equivalent to one of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} (\mu \neq 0), \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x \\ 1 & 0 & y \\ 0 & 1 & 1 \end{pmatrix} (x \neq 0), \begin{pmatrix} 0 & 0 & x \\ 1 & 0 & y \\ 0 & 1 & 0 \end{pmatrix} (0 < x \leq (p-1)/2,\ y = 1 \text{ or } \omega), \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

or (when $p = 1 \bmod 3$)

$$\begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In each of these cases we can subtract suitable scalar multiples of b, c, d from a so that $pa = 0$. So we have $p^2 + p$ algebras (or $p^2 + p + 2$ algebras when $p = 1 \bmod 3$)

$$\langle a, b, c, d \mid cb, db, dc, pa, \begin{pmatrix} pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ da \end{pmatrix}, \text{ class } 2\rangle$$

where A is of this form.

If A has rank 2 then A is equivalent to a matrix of one of the following forms

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (x \neq 0), \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In all these cases $pd = 0$. In the ørst case $pb = ca$, $pc = da$. By subtracting suitable scalar multiples of b, c from a we may suppose that $pa = \lambda ba$. By scaling b, c, d by the same scale factor we can take $\lambda = 0, 1$. So we have

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ca, pc - da, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - ba, pb - ca, pc - da, pd, \text{ class } 2\rangle.$$

In the second case we have $pb = ba$, $pc = da$. As above, we can take $pa = \lambda ca$, where $\lambda = 0, 1$, giving

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ba, pc - da, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - ca, pb - ba, pc - da, pd, \text{ class } 2\rangle.$$

In a similar way, the other cases give

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ba, pc - ca, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - da, pb - ba, pc - ca, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - xca, pc - ba - ca, pd, \text{ class } 2\rangle \ (x \neq 0),$$

$$\langle a, b, c, d \mid cb, db, dc, pa - da, pb - xca, pc - ba - ca, pd, \text{ class } 2\rangle \ (x \neq 0),$$

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ca, pc - ba, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - da, pb - ca, pc - ba, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - \omega ca, pc - ba, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - da, pb - \omega ca, pc - ba, pd, \text{ class } 2\rangle.$$

If A has rank 1 then A is eqivalent to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the ørst case $pb = ba$, $pc = pd = 0$, and we can suppose that $pa = 0$ or ca. In the second case, $pb = ca$ and we can suppose that

$$pa = \lambda ba + \mu da$$

for some λ, μ. If $\lambda \neq 0$ then we can suppose that $pa = ba$, and if $\lambda = 0$ then we can suppose that $pa = 0$ or da. So we have

$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ba, pc, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - ca, pb - ba, pc, pd, \text{ class } 2\rangle,$$
$$\langle a, b, c, d \mid cb, db, dc, pa, pb - ca, pc, pd, \text{ class } 2\rangle,$$
$$\langle a, b, c, d \mid cb, db, dc, pa - ba, pb - ca, pc, pd, \text{ class } 2\rangle,$$
$$\langle a, b, c, d \mid cb, db, dc, pa - da, pb - ca, pc, pd, \text{ class } 2\rangle.$$

Finally, if $A = 0$, then we have $pb = pc = pd = 0$ and we can take $pa = 0$ or ba, giving

$$\langle a, b, c, d \mid cb, db, dc, pa, pb, pc, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid cb, db, dc, pa - ba, pb, pc, pd, \text{ class } 2\rangle.$$

Case 4 Let L satisfy $ca = da = 0$, $dc = ba$. Then L^2 is generated by ba, cb, db and $pL \leq L^2$. It is easy to see that the only elements of breadth 1 in the linear span of a, b, c, d are scalar multiples of a. It is also easy to see that the elements of breadth 2 are elements of the form

$$\alpha a + \gamma c + \delta d$$

with (at least) one of γ, δ non-zero. Every other non-trivial element in the linear span of a, b, c, d has breadth 3. From this it is straightforward to show that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then (modulo L^2)

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \alpha^{-1}(\zeta\nu - \eta\mu)b + \gamma c + \delta d, \\ c' &= \varepsilon a + \zeta c + \eta d, \\ d' &= \lambda a + \mu c + \nu d \end{aligned}$$

where $\alpha, \zeta\nu - \eta\mu \neq 0$. Furthermore any a', b', c', d' of this form satisfy the same commutator relations as a, b, c, d. If a', b', c', d' are as above then

$$\begin{aligned} b'a' &= (\zeta\nu - \eta\mu)ba, \\ c'b' &= (\gamma\eta - \delta\zeta - \alpha^{-1}\varepsilon(\zeta\nu - \eta\mu))ba + \alpha^{-1}\zeta(\zeta\nu - \eta\mu)cb + \alpha^{-1}\eta(\zeta\nu - \eta\mu)db, \\ d'b' &= (\gamma\nu - \delta\mu - \alpha^{-1}\lambda(\zeta\nu - \eta\mu))ba + \alpha^{-1}\mu(\zeta\nu - \eta\mu)cb + \alpha^{-1}\nu(\zeta\nu - \eta\mu)db. \end{aligned}$$

We can suppose that $pa = 0, ba$ or cb.

First consider the case when $pa = 0$. If $pc = pd = 0$ then we can suppose that $pb = 0$, ba or cb, giving

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb, pc, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - ba, pc, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - cb, pc, pd, \text{ class } 2\rangle.$$

If pc, pd span a space of dimension 1 then we can assume that $pd = 0$ and we can assume that $pc = ba$, cb or db. First consider the case when $pc = ba$. We can now consider changes of generating set as above, with $\mu = 0$, $\nu = 1$. Let $pb = \rho ba + \sigma cb + \tau db$. If a', b', c', d' are as above (with $\mu = 0$, $\nu = 1$) then

$$pb' = \alpha^{-1}\zeta pb + \gamma pc = (\alpha^{-1}\zeta\rho + \gamma)ba + \alpha^{-1}\zeta\sigma cb + \alpha^{-1}\zeta\tau db$$

and

$$\begin{aligned}
b'a' &= \zeta ba, \\
c'b' &= (\gamma\eta - \delta\zeta - \alpha^{-1}\varepsilon\zeta)ba + \alpha^{-1}\zeta\zeta cb + \alpha^{-1}\eta\zeta db, \\
d'b' &= (\gamma - \delta\mu - \alpha^{-1}\lambda\zeta)ba + \alpha^{-1}\zeta db.
\end{aligned}$$

So if $\sigma \neq 0$ we can take $pb = cb$ and if $\sigma = 0$ we can take $pb = \tau db$ $(0 \leq \tau < p)$.

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - cb, pc - ba, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - \tau db, pc - ba, pd, \text{ class } 2\rangle \ (0 \leq \tau < p).$$

Next, let $pa = pd = 0$, $pc = cb$, and let $pb = \rho ba + \sigma cb + \tau db$. We can now consider generators a', b', c', d' as above, with $\mu = \eta = 0$, $\alpha = \zeta\nu$, $\varepsilon = -\delta\zeta$. We then have

$$pb' = pb + \gamma pc = \rho ba + (\sigma + \gamma)cb + \tau db$$

and

$$\begin{aligned}
b'a' &= \zeta\nu ba, \\
c'b' &= \zeta cb, \\
d'b' &= (\gamma\nu - \lambda)ba + \nu db.
\end{aligned}$$

So we can take $pb = 0$, ba or db giving

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb, pc - cb, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - ba, pc - cb, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa, pb - db, pc - cb, pd, \text{ class } 2\rangle.$$

Finally, let $pa = pd = 0$, $pc = db$, and let $pb = \rho ba + \sigma cb + \tau db$. We can now consider generators a', b', c', d' as above, with $\mu = 0$, $\alpha = \nu^2$, $\gamma = \lambda\zeta\nu^{-2}$. We then have

$$pb' = \zeta\nu^{-1}pb + \lambda\zeta\nu^{-2}pc = \zeta\nu^{-1}\rho ba + \zeta\nu^{-1}\sigma cb + (\zeta\nu^{-1}\tau + \lambda\zeta\nu^{-2})db$$

and

$$\begin{aligned}
b'a' &= \zeta\nu ba, \\
c'b' &= (\lambda\zeta\nu^{-2}\eta - \delta\zeta - \varepsilon\zeta\nu^{-1})ba + \zeta\zeta\nu^{-1}cb + \eta\zeta\nu^{-1}db, \\
d'b' &= \zeta db.
\end{aligned}$$

So if $\sigma \neq 0$ we can take $pb = cb$, and if $\sigma = 0$ we can take $pb = 0$, ba or ωba giving

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb, pc - db, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - ba, pc - db, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - \omega ba, pc - db, pd, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - cb, pc - db, pd, \text{ class } 2\rangle.$$

Next, consider the case when $pa = 0$ and pc, pd span a space of dimension 2. If this subspace contains ba then we can suppose that $pc = ba$ and that $pd = cb$ or db. When $pa = 0$, $pc = ba$, $pd = cb$, we can take $pb = \tau db$ $(0 \leq \tau < p)$, and when $pa = 0$, $pc = ba$, $pd = db$ then we can take $pb = 0$ or cb. So when $pa = 0$ and $pc = ba$ we have

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - \tau db, pc - ba, pd - cb, \text{ class } 2\rangle \ (0 \leq \tau < p),$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb, pc - ba, pd - db, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - cb, pc - ba, pd - db, \text{ class } 2\rangle.$$

On the other hand if pc, pd span a 2-dimensional space which does not contain ba then we can write

$$\begin{pmatrix} pc \\ pd \end{pmatrix} = A \begin{pmatrix} cb \\ db \end{pmatrix} \text{ modulo } \langle ba\rangle$$

for some non-singular 2×2 matrix A over $\mathbb{Z}_p$. Replacing a, b, c, d by a', b', c', d' as above transforms A to

$$\alpha(\zeta\nu - \eta\mu)^{-1} \begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix} A \begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}^{-1},$$

so we can take A to be one of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 1 & 1 \end{pmatrix} (x \neq 0, -\tfrac{1}{4}), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}.$$

We can then take $pb = \tau ba$ for some τ. We now consider a further change of generating set, which preserves A. Thus we require

$$\alpha(\zeta\nu - \eta\mu)^{-1} \begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix} A \begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}^{-1} = A,$$

for which we need A to commute with $\begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}$, and we need $\alpha(\zeta\nu - \eta\mu)^{-1} = 1$. Provided this is satisøed we have

$$pb' = \alpha^{-1}\tau b'a' = (\zeta\nu - \eta\mu)^{-1}\tau b'a'.$$

So for $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we can take $\tau = 0, 1, \omega$ and for the other values of A we can take $\tau = 0, 1$. So we have

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb, pc - cb, pd - db, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - ba, pc - cb, pd - db, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb, pc - cb - db, pd - db, \text{ class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa, pb - ba, pc - cb - db, pd - db, \text{ class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb-\omega ba, pc-cb-db, pd-db,\ \text{class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb, pc-xdb, pd-cb-db,\ \text{class } 2\rangle\ (x\neq 0,-\frac{1}{4}),$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb-ba, pc-xdb, pd-cb-db,\ \text{class } 2\rangle\ (x\neq 0,-\frac{1}{4}),$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb, pc-db, pd-cb,\ \text{class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb-ba, pc-db, pd-cb,\ \text{class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb, pc-\omega db, pd-cb,\ \text{class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa, pb-ba, pc-\omega db, pd-cb,\ \text{class } 2\rangle.$$

Next, suppose that $pa = ba$. Then when considering a', b', c', d' as above, we require $\alpha = \zeta\nu - \eta\mu$, so if we write

$$\begin{pmatrix} pc \\ pd \end{pmatrix} = A \begin{pmatrix} cb \\ db \end{pmatrix} \text{ modulo } \langle ba\rangle$$

for some 2×2 matrix A over $\mathbb{Z}_p$, then we see that replacing a,b,c,d by a',b',c',d' transforms A to

$$\begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix} A \begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}^{-1}.$$

So we can take A to be of the form

$$\begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 1 & \rho \end{pmatrix}, \begin{pmatrix} 0 & \rho \\ 1 & \sigma \end{pmatrix}\ (\sigma^2+4\rho \text{ not a square}).$$

Furthermore, we can take

$$\begin{pmatrix} pc \\ pd \end{pmatrix} = A \begin{pmatrix} cb \\ db \end{pmatrix}$$

for any of these values of A. If $pc = pd = 0$ then we can take $pb = 0$ or cb, giving

$$\langle a,b,c,d \mid ca, da, dc-ba, pa-ba, pb, pc, pd,\ \text{class } 2\rangle,$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa-ba, pb-cb, pc, pd,\ \text{class } 2\rangle.$$

If pc, pd span a space of dimension 1 then we have A of the form $\begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and in both cases we can take $pb = 0$ or db. This gives

$$\langle a,b,c,d \mid ca, da, dc-ba, pa-ba, pb, pc-\rho cb, pd,\ \text{class } 2\rangle\ (\rho\neq 0),$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa-ba, pb-db, pc-\rho cb, pd,\ \text{class } 2\rangle\ (\rho\neq 0),$$

$$\langle a,b,c,d \mid ca, da, dc-ba, pa-ba, pb, pc, pd-cb,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - ba, pb - db, pc, pd - cb, \text{ class } 2\rangle.$$

Finally, if pc, pd span a space of dimension 2 then we can take $pb = 0$, but this means that we can only take

$$\begin{pmatrix} pc \\ pd \end{pmatrix} = A \begin{pmatrix} cb \\ db \end{pmatrix} + \begin{pmatrix} xba \\ yba \end{pmatrix}$$

for some x, y.

For each of the above possibilities for A we consider a', b', c', d' (as above) with $\beta = \gamma = \delta = 0$, $\alpha = \zeta\nu - \eta\mu$, and with $\begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}$ commuting with A.

First consider the case when $A = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$. Then A commutes with every matrix $\begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}$, and we let

$$\begin{aligned} a' &= (\zeta\nu - \eta\mu)a, \\ b' &= b, \\ c' &= \varepsilon a + \zeta c + \eta d, \\ d' &= \lambda a + \mu c + \nu d. \end{aligned}$$

Then

$$\begin{aligned} b'a' &= (\zeta\nu - \eta\mu)ba, \\ c'b' &= -\varepsilon ba + \zeta cb + \eta db, \\ d'b' &= -\lambda ba + \mu cb + \nu db, \end{aligned}$$

and

$$\begin{aligned} pc' &= (\varepsilon + \zeta x + \eta y)ba + \rho\zeta cb + \rho\eta db = (\varepsilon + \zeta x + \eta y + \rho\varepsilon)ba + \rho c'b', \\ pd' &= (\lambda + \mu x + \nu y)ba + \rho\mu cb + \rho\nu db = (\lambda + \mu x + \nu y + \rho\lambda)ba + \rho d'b'. \end{aligned}$$

If $\rho \neq -1$ we can choose ε, λ so that $pc' = \rho c'b'$, $pd' = \rho d'b'$. But if $\rho = -1$ and if x, y are not both zero, then the best we can do is to choose $\begin{pmatrix} \zeta & \eta \\ \mu & \nu \end{pmatrix}$ so that $pc' = b'a' - c'b'$, $pd' = -d'b'$. So we have

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - ba, pb, pc - \rho cb, pd - \rho db, \text{ class } 2\rangle \ (\rho \neq 0),$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - ba, pb, pc - ba + cb, pd + db, \text{ class } 2\rangle.$$

Next let $A = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$ with $0 < \rho < \sigma < p$. Then A commutes with diagonal matrices, and so we let

$$\begin{aligned} a' &= \zeta\nu a, \\ b' &= b, \\ c' &= \varepsilon a + \zeta c, \\ d' &= \lambda a + \nu d. \end{aligned}$$

Then

$$\begin{aligned} b'a' &= \zeta\nu ba, \\ c'b' &= -\varepsilon ba + \zeta cb, \\ d'b' &= -\lambda ba + \nu db, \end{aligned}$$

and

$$\begin{aligned} pc' &= (\varepsilon + \zeta x)ba + \rho\zeta cb = (\varepsilon + \zeta x + \rho\varepsilon)ba + \rho c'b', \\ pd' &= (\lambda + \nu y)ba + \sigma\nu db = (\lambda + \nu y + \sigma\lambda)ba + \sigma d'b'. \end{aligned}$$

Now $\rho \neq -1 \bmod p$, and so (for every ζ, ν) we can choose ε so that $pc' = \rho c'b'$. Similarly, if $\sigma \neq p-1$ then we can take $pd' = \sigma d'b'$. But if $\sigma = p-1$ and $y \neq 0$ then the best we can do is take $pd' = b'a' - d'b'$. So we have

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - ba, pb, pc - \rho cb, pd - \sigma db, \text{ class } 2\rangle \; (0 < \rho < \sigma < p),$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - ba, pb, pc - \rho cb, pd - ba + db, \text{ class } 2\rangle \; (0 < \rho < p-1).$$

Now let $A = \begin{pmatrix} \rho & 0 \\ 1 & \rho \end{pmatrix}$. Then A commutes with all matrices of the form $\begin{pmatrix} \zeta & 0 \\ \mu & \zeta \end{pmatrix}$, and so we let

$$\begin{aligned} a' &= \zeta^2 a, \\ b' &= b, \\ c' &= \varepsilon a + \zeta c, \\ d' &= \lambda a + \mu c + \zeta d. \end{aligned}$$

Then

$$\begin{aligned} b'a' &= \zeta^2 ba, \\ c'b' &= -\varepsilon ba + \zeta cb, \\ d'b' &= -\lambda ba + \mu cb + \zeta db, \end{aligned}$$

and

$$\begin{aligned} pc' &= (\varepsilon + \zeta x)ba + \rho\zeta cb = (\varepsilon + \zeta x + \rho\varepsilon)ba + \rho c'b', \\ pd' &= (\lambda + \mu x + \zeta y)ba + (\rho\mu + \zeta)cb + \rho\zeta db = (\lambda + \mu x + \zeta y + \rho\lambda + \varepsilon)ba + c'b' + \rho d'b'. \end{aligned}$$

If $\rho \neq -1$ then we can assume that $x = y = 0$. If $\rho = -1$ we can still take $y = 0$, but if $x \neq 0$ then the best we can do is take $x = 1$. so we have

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - ba, pb, pc - \rho cb, pd - cb - \rho db, \text{ class } 2\rangle \; (0 < \rho < p),$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - ba, pb, pc - ba + cb, pd - cb + db, \text{ class } 2\rangle.$$

Finally, let $A = \begin{pmatrix} 0 & \rho \\ 1 & \sigma \end{pmatrix}$ where $\sigma^2 + 4\rho$ is not a square. It turns out that it is suŒcient to consider

$$\begin{aligned} a' &= a, \\ b' &= b, \\ c' &= \varepsilon a + c, \\ d' &= \lambda a + d. \end{aligned}$$

Then

$$\begin{aligned} b'a' &= ba, \\ c'b' &= -\varepsilon ba + cb, \\ d'b' &= -\lambda ba + db, \end{aligned}$$

and

$$\begin{aligned} pc' &= (\varepsilon + x)ba + \rho db = (\varepsilon + x + \lambda\rho)ba + \rho d'b' \\ pd' &= (\lambda + y)ba + cb + \sigma db = (\lambda + y + \varepsilon + \sigma\lambda)ba + c'b' + \sigma d'b'. \end{aligned}$$

Now the matrix $\begin{pmatrix} 1 & \rho \\ 1 & 1+\sigma \end{pmatrix}$ is non-singular since $\sigma^2 + 4\rho$ is not a square, and so we can take $x = y = 0$, giving

$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - ba, pb, pc - \rho db, pd - cb - \sigma db,\ \text{class } 2\rangle$ ($\sigma^2 + 4\rho$ not a square).

Finally let $pa = cb$. If a', b', c', d' are as above, and if $pa' = c'b'$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \alpha\zeta^{-1} b + \gamma c - \alpha\varepsilon\zeta^{-2} d, \\ c' &= \varepsilon a + \zeta c, \\ d' &= \lambda a + \mu c + \alpha^2\zeta^{-2} d \end{aligned}$$

and

$$\begin{aligned} b'a' &= \alpha^2\zeta^{-1} ba, \\ c'b' &= \alpha cb, \\ d'b' &= (\gamma\alpha^2\zeta^{-2} + \alpha\varepsilon\zeta^{-2}\mu - \alpha\lambda\zeta^{-1})ba + \alpha\mu\zeta^{-1} cb + \alpha^3\zeta^{-3} db. \end{aligned}$$

We can assume that pc is a linear combination of ba and db. If pc is a scalar multiple of ba then we can assume that $pc = 0$, ba or ωba. And if pc is not a scalar multiple of ba then we can assume that $pc = db$.

So consider the case when $pc = 0$. Then we can assume that pd is a linear combination of ba and db, and hence that $pd = 0$, ba or db.

If $pa = cb$, $pc = pd = 0$ then we can assume that $pb = 0$, ba, db or ωdb.

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb, pc, pd,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - ba, pc, pd,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - db, pc, pd,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - \omega db, pc, pd,\ \text{class } 2\rangle.$$

If $pa = cb$, $pc = 0$, $pd = ba$ then we can assume that $pb = 0$, ba, db or ωdb.

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb, pc, pd - ba,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - ba, pc, pd - ba,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - db, pc, pd - ba,\ \text{class } 2\rangle,$$

$$\langle a, b, c, d \,|\, ca, da, dc - ba, pa - cb, pb - \omega db, pc, pd - ba,\ \text{class } 2\rangle.$$

If $pa = cb$, $pc = 0$, $pd = db$ then we can assume that $pb = \rho ba + \sigma db$, but scaling a and c by the same scale factor we can assume that $\rho = 0, 1$, giving

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - \rho ba - \sigma db, pc, pd - db, \text{ class } 2\rangle \; (\rho = 0, 1,\ 0 \leq \sigma < p).$$

Next, consider the case when $pa = cb$, $pc = ba$ or ωba. Then we can assume that $pd = \rho db$. But if we let $a' = -a$, $b' = -b$, $c' = c$, $d' = d$, then $pd' = -\rho d'b'$. So we can assume that $0 \leq \rho \leq (p-1)/2$. We can then take $pb = \sigma db$, where $0 \leq \sigma < p$, or $pb = ba + (\rho^2 + 1)db$ when $pc = ba$, or $pb = ba + (\rho^2 + \omega)db$ when $pc = \omega ba$. So we have

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - \sigma db, pc - ba, pd - \rho db, \text{ class } 2\rangle \; (0 \leq \rho \leq (p-1)/2,\ 0 \leq \sigma < p),$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - ba - (\rho^2 + 1)db, pc - ba, pd - \rho db, \text{ class } 2\rangle \; (0 \leq \rho \leq (p-1)/2),$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - \sigma db, pc - \omega ba, pd - \rho db, \text{ class } 2\rangle \; (0 \leq \rho \leq (p-1)/2,\ 0 \leq \sigma < p),$$

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - ba - (\rho^2 + \omega)db, pc - \omega ba, pd - \rho db, \text{ class } 2\rangle \; (0 \leq \rho \leq (p-1)/2).$$

So the total here is $(p+1)^2$.

Finally consider the case when $pa = cb$, $pc = db$. To preserve the relation $pc = db$ we require $\alpha^3 = \xi^4$ so that $\xi^{-1} = (\alpha^{-1}\xi)^3$. So we can only choose ξ to be a cube. But if $\xi = k^3$ then we can take $\alpha = k^4$ or κk^4 or $\kappa^2 k^4$, where κ is any cube root of 1. It follows that we can take $pd = 0$ or ba when $p \neq 1 \bmod 3$, and we can take $pd = 0$, ba, ωba or $\omega^2 ba$ when $p = 1 \bmod 3$.

First consider the case when $pd = 0$. Then we can take $pb = xba + zdb$ and consider a', b', c', d' of the form

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha \xi^{-1} b, \\ c' &= \xi c, \\ d' &= \alpha^2 \xi^{-2} d \end{aligned}$$

with $\xi = k^3$ then we can take $\alpha = k^4$ or κk^4 or $\kappa^2 k^4$, where κ is any cube root of 1. This gives

$$\begin{aligned} b'a' &= \alpha^2 \xi^{-1} ba, \\ c'b' &= \alpha cb, \\ d'b' &= \alpha^3 \xi^{-3} db. \end{aligned}$$

So

$$pb' = \alpha\xi^{-1} pb = \alpha\xi^{-1} xba + \alpha\xi^{-1} zdb = \alpha^{-1} xb'a' + \alpha^{-2}\xi^2 zd'b'.$$

We have $\alpha^{-2}\xi^2 = k^{-2}$ or κk^{-2} or $\kappa^2 k^{-2}$, so if $z = 0$ we can take α to be any fourth power, and if $z \neq 0$ then we can take $z = 1$ or ω, but we then require $\alpha = 1$. So we choose $z = 0, 1, \omega$. We have $0 \leq x < p$ when $z = 1, \omega$; when $z = 0$ we have $x = 0, 1, \omega$, or when $p = 1 \bmod 4$, $x = 0, 1, \omega, \omega^2, \omega^3$. So we have $2p + 3$ algebras when $p \neq 1 \bmod 4$, and $2p + 5$ algebras when $p = 1 \bmod 4$.

$$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - xba - zdb, pc - db, pd, \text{ class } 2\rangle.$$

Now consider the case when $pd \neq 0$. We then require $\xi = 1$, $\alpha^3 = 1$. Again we can take $pb = xba + zdb$, and consider a', b', c', d' of the form

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha b, \\ c' &= \xi c, \\ d' &= \alpha^2 d, \end{aligned}$$

$$\begin{aligned} b'a' &= \alpha^2 ba, \\ c'b' &= \alpha cb, \\ d'b' &= db. \end{aligned}$$

$$pb' = \alpha pb = \alpha xba + \alpha zdb.$$

If $p \neq 1 \bmod 3$ then $\alpha = 1$, and we have p^2 algebras (for this particular pd). If $p = 1 \bmod 3$ then for each value of pd there are $(p-1)^2/3$ algebras when $x, z \neq 0$, $(p-1)/3$ when $x = 0$, $z \neq 0$, another $(p-1)/3$ when $x \neq 0$ and $z = 0$, and one algebra when $x = z = 0$, giving $(p^2+1)/3$ algebras in all. Running over the three non-zero values of pd in this case we have $p^2 + 2$ algebras.

$\langle a, b, c, d \mid ca, da, dc-ba, pa-cb, pb-xba-zdb, pc-db, pd-ba,$ class $2\rangle$ $(0 \leq x, z < p,\ p \neq 1 \bmod 3)$,

$\langle a, b, c, d \mid ca, da, dc - ba, pa - cb, pb - xba - zdb, pc - db, pd - \lambda ba,$ class $2\rangle$ $(p = 1 \bmod 3)$

where $\lambda = 1, \omega, \omega^2$, $0 \leq x, z < p$ with (x, z) giving the same algebra as $(\alpha x, \alpha z)$ whenever $\alpha^3 = 1$.

Case 5 Let L satisfy $da = 0$, $db = ca$, $dc = cb$. It is convenient to replace b by $b + d$, so that L satisøes $da = cb = 0$, $db = ca$. So L^2 is generated by ba, ca and dc, and $pL \leq L^2$. It is fairly easy to see that if a', b', c', d' generate L and satisfy $d'a' = c'b' = 0$, $d'b' = c'a'$, then (modulo L^2)

$$\begin{aligned} a' &= \alpha\lambda a + \beta\lambda b + \beta\mu c - \alpha\mu d, \\ b' &= \gamma\lambda a + \delta\lambda b + \delta\mu c - \gamma\mu d, \\ c' &= \gamma\nu a + \delta\nu b + \delta\xi c - \gamma\xi d, \\ d' &= -\alpha\nu a - \beta\nu b - \beta\xi c + \alpha\xi d \end{aligned}$$

with (α, β) and (γ, δ) linearly independant, and with (λ, μ) and (ν, ξ) linearly independant. Furthermore

$$\begin{pmatrix} b'a' \\ c'a' \\ d'c' \end{pmatrix} = (\alpha\delta - \beta\gamma) \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ dc \end{pmatrix}.$$

So we consider orbits of 4×3 matrices A (representing pa, pb, pc, pd) under transformations of the form

$$A \longmapsto (\alpha\delta - \beta\gamma)^{-1} \begin{pmatrix} \alpha\lambda & \beta\lambda & \beta\mu & -\alpha\mu \\ \gamma\lambda & \delta\lambda & \delta\mu & -\gamma\mu \\ \gamma\nu & \delta\nu & \delta\xi & -\gamma\xi \\ -\alpha\nu & -\beta\nu & -\beta\xi & \alpha\xi \end{pmatrix} A \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}^{-1}.$$

We note that if we multiply $\alpha, \beta, \gamma, \delta$ through by a factor k (in the expression above), and multiply λ, μ, ν, ξ through by a factor l, then the image of A is multiplied by a factor $k^{-1}l^{-1}$. So we can ignore the factor $(\alpha\delta - \beta\gamma)^{-1}$ and still get the same orbits. We work out the dimension of the space of matrices A øxed by any given choice of $\alpha, \beta, \gamma, \delta$ and λ, μ, ν, ξ. It turns out that it only depends on the conjugacy classes of then matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$. See Appendix A for details.

The number of algebras is 550 when $p = 3$ and

$$\begin{aligned} p^5 + p^4 + 4p^3 + 6p^2 + 18p + 19 \text{ if } p &= 1 \bmod 3, \\ p^5 + p^4 + 4p^3 + 6p^2 + 16p + 17 \text{ if } p &= 2 \bmod 3. \end{aligned}$$

See orb4.1j and orb4.1j2.

Case 6 The total for case 6 is $\frac{9}{2}p + \frac{13}{2} + 3p^2 + \frac{1}{2}p^4 + \frac{1}{2}p^3$. This gives 101 when $p = 3$, 479 when $p = 5$, and 1557 when $p = 7$. (See orb4.1k ˇ orb4.1o.)

Let L satisfy $da = 0$, $db = \omega ca$, $dc = ba$. Then L^2 is generated by ba, ca, cb and $pL \leq L^2$. It is straightforward to show that all elements in the linear span of a, b, c, d have breadth 3, except for those of the form $\alpha a + \delta d$. Using this we can show that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then (modulo L^2)

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We let

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

where A is a 4×3 matrix over $\mathbb{Z}_p$. Then under a change of generating set of the form described above we see that

$$A \mapsto \begin{pmatrix} \alpha & 0 & 0 & \delta \\ \pm\lambda & \pm\gamma & \pm\omega\beta & \pm\mu \\ \nu & \beta & \gamma & \xi \\ \pm\omega\delta & 0 & 0 & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}.$$

We note that $\langle a, d\rangle + L^2$ is a characteristic subalgebra, and ørst investigate the orbits of pa, pd. We consider three separate cases: $pa = pd = 0$, pa and pd span a one dimensional subspace, and pa, pd are linearly independent. For details see Appendix A. For each choice of pa, pd we compute the orbits of pb, pc. We list the possibilities for pa, pd below, and for each possibility we give the name of the magma program which computes the orbits of pc, pd.

- $pa = pd = 0$ — orb4.1l. Use this when $pb, pc \in \langle ba, ca\rangle$. If $pb, pc \notin \langle ba, ca\rangle$ then we can take $pb \in \langle ba, ca\rangle$ and $pc \notin \langle ba, ca\rangle$, which mean we need to take $\beta = 0$. We can then take $pc = cb$, which means we need to take $\gamma = 1$ in the $+$ matrices and $\gamma = -1$ in the $-$ matrices. We can then take $pc = 0$ or ca.
- $pa = 0$, $pd = ca$ — orb4.1m — also see below.
- $pa = 0$, $pd = cb$ — orb4.1n and orb4.1n2
- $pa = ca$, $pd = cb$ — orb4.1o — also see below.
- $pa = ca$, $pd = rba + sca$ ($r \neq 0$, p orbits) — orb4.1k and orb4.1k2. Orb4.1k gives p values for pa, pd which can be taken to go with $pb = pc = 0$. Junk the second half of orb4.1k and use orb4.1k2 to generate orbits pa, pb, pc, pd where $pL = L^3$ and pa, pd span $\langle ba, ca\rangle$. If pa, pd span $\langle ba, ca\rangle$ and pa, pb, pc, pd generate L^2 then we can assume that $pb = 0$.

These are all put together in orb4.16.

Suppose $pa = 0$, $pd = ca$.

We need $\beta = 0$, $\delta = 0$ and $\gamma = 1$ in the $+$ matrix and $\gamma = -1$ in the $-$ matrix, which gives

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ \lambda & 1 & 0 & \mu \\ \nu & 0 & 1 & \xi \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ -\nu-\mu & \lambda+\omega\xi & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ -\lambda & 1 & 0 & -\mu \\ \nu & 0 & -1 & \xi \\ 0 & 0 & 0 & -\alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ -\nu-\mu & \lambda+\omega\xi & -1 \end{pmatrix}$$

Represent pa, pb, pc, pd by

$$\begin{pmatrix} 0 & 0 & 0 \\ x & y & z \\ u & v & w \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ \lambda & 1 & 0 & \mu \\ \nu & 0 & 1 & \xi \\ 0 & 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ x & y & z \\ u & v & w \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ -\nu-\mu & \lambda+\omega\xi & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{x+z\nu+z\mu}{\alpha} & \frac{y+\mu-z\lambda-z\omega\xi}{\alpha} & z \\ \frac{u+w\nu+w\mu}{\alpha} & \frac{v+\xi-w\lambda-w\omega\xi}{\alpha} & w \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ -\lambda & 1 & 0 & -\mu \\ \nu & 0 & -1 & \xi \\ 0 & 0 & 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ x & y & z \\ u & v & w \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ -\nu-\mu & \lambda+\omega\xi & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{x-z\nu-z\mu}{\alpha} & -\frac{y-\mu+z\lambda+z\omega\xi}{\alpha} & -z \\ \frac{-u+w\nu+w\mu}{\alpha} & \frac{v-\xi+w\lambda+w\omega\xi}{\alpha} & w \\ 0 & 1 & 0 \end{pmatrix}$$

So we can take $y = v = 0$. If $z \neq 0$ we can take $0 < z \leq (p-1)/2$, $x = 0$, $u = 0$ or 1, and w arbitrary ($p(p-1)$). If $z = 0$, $w \neq 0$ we can take $0 < w < p$, $x = 0$ or 1 and $u = 0$ ($2(p-1)$). If $w = z = 0$ then we can take $x = 0$ or 1; if $x = 0$ we can take $u = 0$ or 1, and if $x = 1$ we can take $0 \leq u \leq (p-1)/2$ ($(p+5)/2$). So there are a total of $p^2 + (3p+1)/2$ algebras here.

Now suppose $pa = ca$, $pd = cb$. We need $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$, $(-\nu - \mu) = (\lambda + \omega\xi) = 0$, giving

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\omega\xi & 1 & 0 & -\nu \\ \nu & 0 & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ x & -\omega\xi + y & z - \nu \\ u & \nu + v & w + \xi \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega\xi & -1 & 0 & \nu \\ \nu & 0 & 1 & \xi \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ x & \omega\xi - y & z - \nu \\ -u & \nu + v & -w - \xi \\ 0 & 0 & 1 \end{pmatrix}$$

So we can take $y = z = 0$ and then we have $(x, u, v, w) \sim (x, -u, v, -w)$ so we can take x, v arbitrary, $0 \leq u \leq (p-1)/2$, w arbitrary if $u \neq 0$ and $0 \leq w \leq (p-1)/2$ if $u = 0$. So there are $p^2(p^2+1)/2$ algebras here.

17 Immediate descendants of algebra 41 (5.3)

Algebra 5.3 has $p^4 + 5p^3 + 19p^2 + 64p + 140 + (p+6)\gcd(p-1,3) + (p+7)\gcd(p-1,4) + \gcd(p-1,5)$ immediate descendants of order p^7 and p-class 3.

Algebra 5.3 has presentation

$$\langle a, b, c, d \mid ca, da, cb, db, dc, pa, pb, pc, pd, \text{ class } 2 \rangle.$$

So it has characteristic p and derived algebra of order p generated by ba, with all other commutators trivial. So if L is an immediate descendant of 5.3 then L has class 3, L_3 is generated by baa, bab, and the elements $ca, da, cb, db, dc, pa, pb, pc, pd$ are all linear combinations of baa, bab. The commutator structure of L must correspond to one of the algebras 7.21 ˘ 7.28 in the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of order p^7. So we can assume that one of the following sets of commutator relations holds. For any given set of commutator relations, pa, pb, pc, pd are linear combinations of baa, bab.

$$\begin{aligned}
ca &= cb = da = db = dc = 0, \\
cb &= da = db = dc = 0,\ ca = bab, \\
cb &= da = db = dc = 0,\ ca = baa, \\
da &= db = dc = 0,\ ca = bab,\ cb = \omega baa, \\
ca &= da = dc = 0,\ cb = baa,\ db = bab, \\
da &= dc = 0,\ ca = db = bab,\ cb = baa, \\
da &= dc = 0,\ ca = db = bab,\ cb = \omega baa, \\
ca &= cb = da = db = 0,\ dc = baa, \\
cb &= da = db = 0,\ ca = bab,\ dc = baa.
\end{aligned}$$

17.1 Case 1

There are $p+12$ algebras in this case.

Let L satisfy $ca = cb = da = db = dc = 0$. Note that the centre of L is $\langle c, d\rangle + L^3$. If pc, pd are linearly independent then we may assume that $pc = baa$, $pd = bab$, and that $pa = pb = 0$. This gives

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb, pc - baa, pd - bab, \text{ class } 3\rangle.$$

If pc, pd span a space fof dimension 1 then we may suppose that $pc = bab$, $pd = 0$, and that pa, pb are linear multiples of baa. We can preserve all these relations with transformations of the form

$$\begin{aligned} a &\mapsto \alpha a + \beta b, \\ b &\mapsto \gamma b. \end{aligned}$$

So if $pb \neq 0$ then we may assume that $pa = 0$, $pb = baa$ or ωbaa. And if $pb = 0$ we may assume that $pa = 0$ or baa. This gives

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb, pc - bab, pd \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - baa, pb, pc - bab, pd \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb - baa, pc - bab, pd \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb - \omega baa, pc - bab, pd \text{ class } 3\rangle.$$

Finally, if $pc = pd = 0$ then $\langle a, b\rangle$ has order p^5 and is an immediate descendant of $\langle a, b \mid pa, pb, \text{ class } 2\rangle$, so that $\langle a, b\rangle$ is one of 5.38 ˘5.46, giving

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - bab, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - \omega bab, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - baa, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - baa, pb - \lambda bab, pc, pd, \text{ class } 3\rangle$$

with $\lambda \neq 0$ and λ, λ^{-1} giving isomorphic algebras ($(p+1)/2$ algebras),

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - baa - bab, pb - bab, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - baa - \omega bab, pb - bab, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - \omega bab, pb - baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb, da, db, dc, pa - \alpha bab, pb - baa - bab, pc, pd, \text{ class } 3\rangle$$

where $1 + 4\alpha$ is not a square ($(p-1)/2$ algebras).

17.2 Case 2

The number of algebras in Case 2 is

$p \bmod 12$	algebras
1	$5p+32$
5	$5p+30$
7	$5p+30$
11	$5p+28$

Now let L satisfy $cb = da = db = dc = 0$, $ca = bab$. If a', b', c', d' generate L and satisfy these commutator relations then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \mu b + \nu c + \xi d, \\ c' &= \alpha^2 c + \rho d, \\ d' &= \sigma d \end{aligned}$$

modulo L^2 and

$$\begin{aligned} b'a'a' &= \alpha^2 \mu baa + \alpha\beta\mu bab, \\ b'a'b' &= \alpha\mu^2 bab. \end{aligned}$$

So we may assume that $pd = 0$, baa or bab.

First consider the case when $pd = 0$. Then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is one of 6.131 ˘ 6.149A, so we have

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa,\ pb,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - bab,\ pb,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - \omega bab,\ pb,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - baa,\ pb,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - \alpha baa,\ pb - bab,\ pc, pd,\ \text{class } 3\rangle\ (0 \le \alpha < p),$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - baa - bab,\ pb - bab,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - baa - \omega bab,\ pb - bab,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - baa - \beta bab,\ pb - baa,\ pc, pd,\ \text{class } 3\rangle\ (0 \le \beta < p),$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa,\ pb - baa,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - bab,\ pb - baa,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - \omega bab,\ pb - baa,\ pc, pd,\ \text{class } 3\rangle.$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - baa - \beta bab,\ pb - \omega baa,\ pc, pd,\ \text{class } 3\rangle\ (0 \le \beta < p),$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa,\ pb - \omega baa,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - bab,\ pb - \omega baa,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - bab,\ cb, da, db, dc,\ pa - \omega bab,\ pb - \omega baa,\ pc, pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb-\lambda baa,\ pc-bab,pd,\ \text{class } 3\rangle\ (0\leq\lambda<p),$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-baa,\ pb,\ pc-bab,pd,\ \text{class } 3\rangle.$$

And finally, we have a batch of algebras whose size and composition depends on p.

$p \bmod 12$	algebras
1	$p+7$
5	$p+5$
7	$p+5$
11	$p+3$

When $\beta=0$ and $p=1 \bmod 4$ we have five algebras

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-\alpha bab,\ pb,\ pc-baa,pd,\ \text{class } 3\rangle\ (\alpha=0,1,\omega,\omega^2,\omega^3),$$

and when $\beta=0$ and $p=3 \bmod 4$ we have three algebras

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-\alpha bab,\ pb,\ pc-baa,pd,\ \text{class } 3\rangle\ (\alpha=0,1,\omega),$$

When $\beta\neq 0$ and $p=1 \bmod 3$ we have $p+2$ algebras

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-\alpha bab,\ pb-\beta bab,\ pc-baa,pd,\ \text{class } 3\rangle\ (\beta=1,\omega,\omega^2),$$

where for a fixed value of β, two values α_1,α_2 of α give isomorphic algebras if and only if $\alpha_1^3=\alpha_2^3$. And finally, when $\beta\neq 0$ and $p=2 \bmod 3$ we have p algebras

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-\alpha bab,\ pb-bab,\ pc-baa,pd,\ \text{class } 3\rangle\ (0\leq\alpha<p).$$

Next consider the case when $pd=baa$. Then we are restricted to a',b',c',d' as above with $\beta=0$, and we may assume that pa,pb,pc are all scalar multiples of bab. If $pc\neq 0$ then we can assumbe that $pa=pb=0$, $pc=bab$. And if $pc=0$ we can assume that $pa=0,bab$ or ωbab and that $pb=0$ or bab. So we have seven algebras

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb,\ pc-bab,pd-baa,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-xbab,\ pb-ybab,\ pc,pd-baa,\ \text{class } 3\rangle$$

where $x\in\{0,1,\omega\}$ and $y\in\{0,1\}$.

Finally consider the case when $pd=bab$. Then we may assume that pa,pb,pc are scalar multiples of baa. If $pc\neq 0$ we may assume that $pa=pb=0$ and that $pc=baa$. If $pc=0$ and $pb\neq 0$ then we can assume that $pa=0$ and $pb=baa$ or ωbaa. And if $pb=pc=0$ then we can assume that $pa=0$ or baa. So we have

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb,\ pc-baa,pd-bab,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb-baa,\ pc,pd-bab,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb-\omega baa,\ pc,pd-bab,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa,\ pb,\ pc,pd-bab,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-bab,\ cb,da,db,dc,\ pa-baa,\ pb,\ pc,pd-bab,\ \text{class } 3\rangle.$$

The number of algebras in Case 3 is $\frac{43}{2}+6p+\frac{3}{2}p^2$, but we need to add 1 if $p=1 \bmod 4$.

Now consider the case when $cb=da=db=dc=0$, $ca=baa$. If a',b',c',d' generate L and satisfy the same commutator relations as a,b,c,d then

$$\begin{aligned} a' &= \alpha a+\gamma c+\delta d, \\ b' &= \mu b+\nu c+\xi d, \\ c' &= \alpha\mu c+\rho d, \\ d' &= \sigma d \end{aligned} \tag{*}$$

modulo L^2 and

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa, \\ b'a'b' &= \alpha\mu^2 bab, \end{aligned}$$

or

$$\begin{aligned} a' &= \beta b+\gamma c+\delta d, \\ b' &= \lambda a+\nu c+\xi d, \\ c' &= \beta\lambda c+\rho d, \\ d' &= \sigma d \end{aligned} \tag{**}$$

modulo L^2 and

$$\begin{aligned} b'a'a' &= -\beta^2\lambda bab, \\ b'a'b' &= -\beta\lambda^2 baa. \end{aligned}$$

So we may assume that $pd=0, baa$ or $baa+bab$.

First consider the case when $pd=0$. Then $L=\langle a,b,c\rangle\oplus\langle d\rangle$, and $\langle a,b,c\rangle$ is one of 6.150 ˘ 6.172, giving

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa,\ pb,\ pc,pd,\ \text{class } 3\rangle.$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-baa,\ pb,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-\omega bab,\ pb,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-baa-\omega bab,\ pb,\ pc,pd,\ \text{class } 3\rangle.$$

So if $p=1 \bmod 4$ we have three algebras

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb-baa,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb-\omega baa,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-\omega bab,\ pb-\omega baa,\ pc,pd,\ \text{class } 3\rangle.$$

And if $p=3 \bmod 4$ we have three algebras

$$\langle a,b,c,d \mid ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb-baa,\ pc,pd,\ \text{class } 3\rangle,$$

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb-\omega baa,\ pc,pd,\ \text{class } 3\rangle,$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-\omega bab,\ pb-baa,\ pc,pd,\ \text{class } 3\rangle.$$

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\mu bab,\ pb-baa,\ pc,pd,\ \text{class } 3\rangle\ (0\le\mu<p),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\mu bab,\ pb-\omega baa,\ pc,pd,\ \text{class } 3\rangle\ (0\le\mu<p).$$

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa,\ pb-\xi bab,\ pc,pd,\ \text{class } 3\rangle\ (\xi\ne 0,\ \xi\sim\xi^{-1}).$$

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\xi bab,\ pc,pd,\ \text{class } 3\rangle\ (\xi\ne 0),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\omega bab,\ pb-\xi bab,\ pc,pd,\ \text{class } 3\rangle\ (\xi\ne 0).$$

Take $\lambda=1$ and let ξ take a value in S, where S is a set of representatives for the classes $\{\xi,\xi^{-1}\}$ of non-zero elements in $\mathbb{Z}_p$. We can also take $\mu=1$ or ω. If $\xi\in S$ and $\xi\ne\pm1$ then a change of generating set of the form (**) with $\alpha\beta=-\xi$ keeps $\lambda=1$ but changes ξ to ξ^{-1} which does not lie in S. So for each $\xi\in S\backslash\{\pm1\}$ we have $2(p-1)$ algebras with $\lambda=1$, $\mu=1$ or ω, and $\nu\ne0$. This gives $(p-1)(p-3)$ algebras

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\nu baa-\xi bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0,\ \xi\in S\backslash\{\pm1\}),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\omega bab,\ pb-\nu baa-\xi bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0,\ \xi\in S\backslash\{\pm1\}),$$

If $p=1 \bmod 4$ we obtain $3(p-1)$ algebras

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\nu^2baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\nu^2\omega baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\omega bab,\ pb-\nu^2\omega baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0).$$

Finally, if $p=3 \bmod 4$ we obtain $3(p-1)$ algebras

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\nu^2baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-bab,\ pb-\nu^2\omega baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0),$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-baa-\omega bab,\ pb-\nu^2 baa\pm bab,\ pc,pd,\ \text{class } 3\rangle\ (\nu\ne 0).$$

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa,\ pb,\ pc-baa,pd,\ \text{class } 3\rangle,$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-bab,\ pb,\ pc-baa,pd,\ \text{class } 3\rangle,$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-\omega bab,\ pb,\ pc-baa,pd,\ \text{class } 3\rangle.$$
$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-\nu bab,\ pb-bab,\ pc-baa,pd,\ \text{class } 3\rangle\ (0\le\nu<p).$$

And ønally we have $p(p+1)/2$ algebras

$$\langle a,b,c,d \,|\, ca-baa,\ cb,da,db,dc,\ pa-\nu bab,\ pb-\xi bab,\ pc-baa-bab,pd,\ \text{class } 3\rangle\ (0\le\nu\le\xi<p).$$

Next consider the case when $pd=baa$. Then we can assume that pa,pb,pc are scalar multiples of bab, though we are restricted to a',b',c',d' of form (*). If $pc\ne0$ we may assume that $pa=pb=0$ and $pc=bab$. And if $pc=0$ then we can assume that $pa=0,bab$ or ωbab and that $pb=0$ or bab. Thus we have seven algebras

$$\langle a, b, c, d \mid ca - baa,\ cb, da, db, dc,\ pa,\ pb,\ pc - bab, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca - baa,\ cb, da, db, dc,\ pa - xbab,\ pb - ybab,\ pc, pd - baa, \text{ class } 3\rangle$$

with $x \in \{0, 1, \omega\}$ and $y \in \{0, 1\}$.

Finally consider the case when $pd = baa + bab$. Then we can assume that pa, pb, pc are linear combinations of baa. And we can consider a', b', c', d' as above with $\alpha = \mu$ in (*) and $\beta = \lambda$ in (**). If $pc \neq 0$ we may assume that $pa = pb = 0$, $pc = baa$ giving

$$\langle a, b, c, d \mid ca - baa,\ cb, da, db, dc,\ pa,\ pb,\ pc - baa, pd - baa - bab, \text{ class } 3\rangle.$$

If $pc = 0$ then we let $pa = xbaa$, $pb = ybaa$. Then a transformation of the form (*) with $\alpha = \mu$ gives $pa' = \alpha^{-2}xb'a'a'$ and $pb' = \alpha^{-2}yb'a'a'$, and a transformation of the form (**) with $\beta = \lambda$ gives $pa' = \beta^{-2}yb'a'a'$ and $pb' = \beta^{-2}xb'a'a'$ (after adding a multiple of d to a and b). One possibility is $x = y = 0$. If one of x, y is zero and the other non-zero then we may suppose that $x = 1$ or ω and that $y = 0$. Thus we obtain three algebras

$$\langle a, b, c, d \mid ca - baa,\ cb, da, db, dc,\ pa - xbaa,\ pb,\ pc, pd - baa - bab, \text{ class } 3\rangle\ (x = 0, 1, \omega).$$

If x, y are both non-zero then we may suppose that $x = 1$ or ω and that $0 < y < p$. We write the possible pairs (x, y) as $(1, k^2)$, $(1, \omega k^2)$, (ω, k^2), $(\omega, \omega k^2)$. then $(1, \omega k^2) \sim (\omega, k^{-2})$, $(1, k^2) \sim (1, k^{-2})$, $(\omega, \omega k^2) \sim (\omega, \omega k^{-2})$. So the number of algebras depends on the value of $p \bmod 4$. We obtain $(p-1)/2$ algebras

$$\langle a, b, c, d \mid ca-baa,\ cb, da, db, dc,\ pa-baa,\ pb-\omega k^2 baa,\ pc, pd-baa-bab, \text{ class } 3\rangle\ (0 < k \leq (p-1)/2).$$

We also have

$$\langle a, b, c, d \mid ca-baa,\ cb, da, db, dc,\ pa-baa,\ pb-k^2 baa,\ pc, pd-baa-bab, \text{ class } 3\rangle\ (k \neq 0,\ k^2 \sim k^{-2}),$$

$$\langle a, b, c, d \mid ca-baa,\ cb, da, db, dc,\ pa-\omega baa,\ pb-\omega k^2 baa,\ pc, pd-baa-bab, \text{ class } 3\rangle\ (k \neq 0,\ k^2 \sim k^{-2})$$

where the number of algebras of these two types is $(p+3)/2$ if $p = 1 \bmod 4$ and $(p+1)/2$ if $p = 3 \bmod 4$.

17.4 Case 4

The total number of algebras in this case is $\frac{3}{2}p^2 + 3p + \frac{13}{2}$, but we have to add 1 to this øgure if $p = 1 \bmod 4$.

Let L satisfy $da = db = dc = 0$, $ca = bab$, $cb = \omega baa$. Note that $D = \langle d\rangle + L_2$ and $C = \langle c, d\rangle + L_3$ are characteristic subalgebras. It is straightforward to show that if a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } D, \\ b' &= \pm(\omega\beta a + \alpha b) + \delta c \text{ modulo } D, \\ c' &= (\alpha^2 - \omega\beta^2)c \text{ modulo } D \end{aligned}$$

for some α, β which are not both zero. (In establishing this fact, we ørst show that if $a' = a$ modulo C then $b' = \pm b$ modulo C and $c' = c$ modulo L^3. It follows from this that b' and c' are determined by a'.) We then have

$$\begin{aligned} b'a'a' &= \pm\alpha(\alpha^2 - \omega\beta^2)baa \pm \beta(\alpha^2 - \omega\beta^2)bab, \\ b'a'b' &= \omega\beta(\alpha^2 - \omega\beta^2)baa + \alpha(\alpha^2 - \omega\beta^2)bab. \end{aligned}$$

So we can assume that $pd = 0$ or baa.

If $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$ and so $\langle a, b, c\rangle$ is one of 6.173 ˇ 6.179 giving

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa,\ pb,\ pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca-bab,\ cb-\omega baa, da, db, dc,\ pa-\lambda baa-bab,\ pb,\ pc, pd, \text{ class } 3\rangle\ (0 \le \lambda \le (p-1)/2),$$

$$\langle a, b, c, d \,|\, ca-bab,\ cb-\omega baa, da, db, dc,\ pa-\lambda baa-\omega bab,\ pb,\ pc, pd, \text{ class } 3\rangle\ (0 \le \lambda \le (p-1)/2),$$

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa - baa,\ pb,\ pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa - \omega baa,\ pb,\ pc, pd, \text{ class } 3\rangle\ (p = 1 \bmod 4).$$

Next consider the case when $pC = \{0\}$ and pa, pb span a space of dimension two. We write

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = A \begin{pmatrix} baa \\ bab \end{pmatrix}$$

for some non-singular matrix A. Then we consider the set of matrices

$$P = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix}$$

which form a group G of order $2(p^2 - 1)$. The isomorphism classes of algebras L with $pC = \{0\}$ and pa, pb linearly independant correspond to the orbits of the set of all non-singular 2×2 matrices A under the action of G deøned by setting

$$A \to \frac{1}{\det P} PAP^{-1}.$$

We will show that there are $p^2 + (p-3)/2$ such orbits when $p = 1 \bmod 4$, and $p^2 + (p-1)/2$ such orbits when $p = 3 \bmod 4$. (See dec5.34.tex and orb5.34.) So we obtain

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa - \lambda baa - \mu bab,\ pb - \nu baa - \xi bab,\ pc, pd, \text{ class } 3\rangle,$$

where $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ runs over a set of representatives for these orbits. [It may help in computing these orbit representatives to note that every orbit contains a representative with $\lambda = 0$ or $\lambda = 1$.]

We have $p(p + 1)/2$ algebras

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa - \lambda baa,\ pb - \mu baa,\ pc - bab, pd, \text{ class } 3\rangle,$$

where λ and $-\lambda$ give isomorphic algebras for any given μ, so that we get distinct algebras if we let $0 \le \lambda \le (p - 1)/2$, $0 \le \mu < p$.

Next consider the case when $pd = baa$. Then we need to consider a', b', c' as above with $\beta = 0$ giving

$$\begin{aligned} a' &= \alpha a + \gamma c \text{ modulo } D, \\ b' &= \pm\alpha b + \delta c \text{ modulo } D, \\ c' &= \alpha^2 c \text{ modulo } D \end{aligned}$$

$$\begin{aligned} b'a'a' &= \pm\alpha^3 baa, \\ b'a'b' &= \alpha^3 bab. \end{aligned}$$

We can assume that pa, pb, pc are scalar multiples of bab. If $pc \neq 0$ we can assume that $pa = pb = 0$ and that $pc = bab$, giving

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa,\ pb,\ pc - bab, pd - baa, \text{ class } 3\rangle.$$

And if $pc = 0$ we can assume that $pa = xbab$, $pb = ybab$, with $pa' = \alpha^{-2}xb'a'b'$, $pb' = \pm\alpha^{-2}yb'a'b'$. So if $pa \neq 0$ we can take $x = 1, \omega$ and $0 \leq y \leq (p-1)/2$, and if $pa = 0$ we can take $y = 0$ or 1 when $p = 3 \bmod 4$, or $0, 1, \omega$ when $p = 1 \bmod 4$. So we have $p + 1$ algebras

$$\langle a, b, c, d \,|\, ca{-}bab,\ cb{-}\omega baa, da, db, dc,\ pa{-}bab,\ pb{-}ybab,\ pc, pd{-}baa, \text{ class } 3\rangle\ (0 \leq y \leq (p{-}1)/2),$$

$$\langle a, b, c, d \,|\, ca{-}bab,\ cb{-}\omega baa, da, db, dc,\ pa{-}\omega bab,\ pb{-}ybab,\ pc, pd{-}baa, \text{ class } 3\rangle\ (0 \leq y \leq (p{-}1)/2),$$

and

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa,\ pb,\ pc, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa,\ pb - bab,\ pc, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca - bab,\ cb - \omega baa, da, db, dc,\ pa,\ pb - \omega bab,\ pc, pd - baa, \text{ class } 3\rangle\ (p = 1 \bmod 4).$$

17.5 Case 5

The number of algebras is $p^2 + 9p + 20 + \gcd(p-1, 3) + \gcd(p-1, 4)$.

$p = 5$ gives 95, $p = 7$ gives 137.

Next let L satisfy $ca = da = dc = 0$, $cb = baa$, $db = bab$. Then if a', b', c', d' generate L and satisfy these commutator relations we have

$$\begin{aligned} a' &= \alpha a + \gamma c + \delta d, \\ b' &= \lambda a + \mu b + \nu c + \xi d, \\ c' &= \alpha^2 c, \\ d' &= \alpha\lambda c + \alpha\mu d \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \alpha^2\mu baa, \\ b'a'b' &= \alpha\lambda\mu baa + \alpha\mu^2 bab. \end{aligned}$$

We can assume that $pc = 0$, baa or bab.

$pc = 0$ If $pc = 0$ we can assume that $pd = 0$, baa or bab.

So consider the case $pc = pd = 0$. Then we can assume that $pa = 0$, baa, bab or ωbab. If $pa = pc = pd = 0$ we have $pb = 0$, baa, ωbaa or bab, giving

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - \omega baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - bab, pc, pd, \text{ class } 3\rangle.$$

If $pa = baa$, $pc = pd = 0$ we let $pb = xbaa + ybab$. We then require $\alpha\mu = 1$

$$pb' = \lambda baa + \mu xbaa + \mu ybab = \alpha^{-1}\lambda b'a'a' + \alpha^{-2}xb'a'a' + yb'a'b' - \alpha^{-1}\lambda yb'a'a'.$$

So if $y \neq 1$ then we can always take $x = 0$, and if $y = 1$ we can take $x = 0, 1, \omega$. So we have p algebras

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - baa, pb - ybab, pc, pd, \text{ class } 3\rangle \; (0 \leq y < p),$$

and two more algebras when $y = 1$:

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - baa, pb - baa - bab, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - baa, pb - \omega baa - bab, pc, pd, \text{ class } 3\rangle.$$

If $pa = bab$ or ωbab, $pc = pd = 0$ then we require $\lambda = 0$, $\mu = \pm 1$, giving

$$\begin{aligned} a' &= \alpha a + \gamma c + \delta d, \\ b' &= \pm b + \nu c + \xi d, \\ c' &= \alpha^2 c, \\ d' &= \pm \alpha d \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \pm \alpha^2 baa, \\ b'a'b' &= \alpha bab. \end{aligned}$$

Let $pb = xbaa + ybab$ then

$$pb' = \pm xbaa \pm ybab = \alpha^{-2} xb'a'a' \pm \alpha^{-1} yb'a'b'$$

so that we can take $y = 0$ and $x = 0, 1, \omega$ or $y = 1$, x arbitrary. This gives

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - bab, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - bab, pb - baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - bab, pb - \omega baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - bab, pb - xbaa - bab, pc, pd, \text{ class } 3\rangle \; (0 \leq x < p),$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - \omega bab, pb, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - \omega bab, pb - baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - \omega bab, pb - \omega baa, pc, pd, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - \omega bab, pb - xbaa - bab, pc, pd, \text{ class } 3\rangle \; (0 \leq x < p).$$

Next consider the case $pc = 0$, $pd = baa$. We need $\alpha = 1$.

$$\begin{aligned} a' &= a + \gamma c + \delta d, \\ b' &= \lambda a + \mu b + \nu c + \xi d, \\ c' &= c, \\ d' &= \alpha \lambda c + \mu d, \end{aligned}$$

$$\begin{aligned} b'a'a' &= \mu baa, \\ b'a'b' &= \lambda \mu baa + \mu^2 bab. \end{aligned}$$

We can assume that pa, pb are scalar multiples of bab, and if $pa \neq 0$ we can assume that $pb = 0$. So we have

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb, pc, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - bab, pc, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - bab, pb, pc, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - \omega bab, pb, pc, pd - baa, \text{ class } 3\rangle.$$

Now consider the case $pc = 0$, $pd = bab$. We need $\lambda = 0$ and $\mu = 1$.

$$\begin{aligned} a' &= \alpha a + \gamma c + \delta d, \\ b' &= b + \nu c + \xi d, \\ c' &= \alpha^2 c, \\ d' &= \alpha d \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a' &= \alpha^2 baa, \\ b'a'b' &= \alpha bab. \end{aligned}$$

We may assume that $pa = xbaa$, $pb = ybaa$. This gives

$$\begin{aligned} pa' &= \alpha xbaa = \alpha^{-1} xb'a'a', \\ pb' &= ybaa = \alpha^{-2} yb'a'a'. \end{aligned}$$

So we have

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb, pc, pd - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - baa, pc, pd - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa, pb - \omega baa, pc, pd - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - baa, pb - ybaa, pc, pd - bab, \text{ class } 3\rangle \; (0 \leq y < p).$$

And now the case $pc = baa$. We need to consider a', b', c', d' as above with $\mu = 1$.

$$\begin{aligned} a' &= \alpha a + \gamma c + \delta d, \\ b' &= \lambda a + b + \nu c + \xi d, \\ c' &= \alpha^2 c, \\ d' &= \alpha \lambda c + \alpha d, \end{aligned}$$

$$\begin{aligned} b'a'a' &= \alpha^2 baa, \\ b'a'b' &= \alpha \lambda baa + \alpha bab. \end{aligned}$$

Let $pd = xbaa + ybab$. Then

$$pd' = (\alpha\lambda + \alpha x)baa + \alpha ybab = \alpha^{-1}(\lambda + x - \lambda y)b'a'a' + yb'a'b'.$$

So if $y \neq 1$ we can take $pd = ybab$, and if $y = 1$ we can take $pd = bab$ or $baa + bab$. If $pd \neq 0$ we can assume that $pa = pb = 0$. So consider the case when $pd = 0$. This restricts us to $\lambda = 0$. We can suppose that $pa = zbab$, $pb = tbab$ (restricting us to $\gamma = \nu = 0$). Then

$$\begin{aligned} pa' &= \alpha zbab = zb'a'b', \\ pb' &= tbab = \alpha^{-1}tb'a'b. \end{aligned}$$

So we can take $t = 0$ or 1. So we have

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa, pb, pc - baa, pd - ybab, \text{ class } 3\rangle \ (0 < y < p),$$

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa, pb, pc - baa, pd - baa - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa - zbab, pb, pc - baa, pd, \text{ class } 3\rangle \ (0 \leq z < p),$$

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa - zbab, pb - bab, pc - baa, pd, \text{ class } 3\rangle \ (0 \leq z < p).$$

Finally consider the case when $pc = bab$. We need to consider a', b', c', d' as above with $\alpha = \mu^2$ and $\lambda = 0$.

$$\begin{aligned} a' &= \mu^2 a + \gamma c + \delta d, \\ b' &= \mu b + \nu c + \xi d, \\ c' &= \mu^4 c, \\ d' &= \mu^3 d, \end{aligned}$$

$$\begin{aligned} b'a'a' &= \mu^5 baa, \\ b'a'b' &= \mu^4 bab. \end{aligned}$$

Let $pd = xbaa + ybab$. Then

$$pd' = \mu^3 xbaa + \mu^3 ybab = \mu^{-2} xb'a'a' + \mu^{-1} yb'a'b'.$$

So if $y \neq 0$ we can take $y = 1$, and if $y = 0$ we can take $x = 0, 1, \omega$. In the cases when $x \neq 0$ we can take $pa = pb = 0$, giving

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa, pb, pc - bab, pd - xbaa - bab, \text{ class } 3\rangle \ (0 < x < p),$$

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa, pb, pc - bab, pd - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c, d \mid ca, cb - baa, da, db - bab, dc, pa, pb, pc - bab, pd - \omega baa, \text{ class } 3\rangle.$$

So consider the cases $pd = 0$ when $x = 0$. Then we can assume that $pa = zbaa$, $pb = tbaa$, giving

$$\begin{aligned} pa' &= \mu^{-3} zb'a'a', \\ pb' &= \mu^{-4} tb'a'a'. \end{aligned}$$

One possibility is $z = t = 0$.

If $z \neq 0$, $t = 0$ then we can take $z = 1$ when $p = 2 \bmod 3$ and $z = 1, \omega$ or ω^2 when $p = 1 \bmod 3$.

If $z = 0$, $t \neq 0$ then we can take $t = 1, \omega$ when $p = 3 \bmod 4$ and $t = 1, \omega, \omega^2, \omega^3$ when $p = 1 \bmod 4$.

Now consider the case when z, t are both non-zero. If $p = 2 \bmod 3$ then we can take $z = 1$ and we get $p - 1$ values for t. But if $p = 1 \bmod 3$ then we can take $z = 1, \omega, \omega^2$ and take t from a set of representatives for the equivalence classes of the equivalence relation

$t \sim t'$ if $t^3 = t'^3$. Again we have $p-1$ diøerent cases. So we have a total of $p+3$ algebras, though we have to add 2 if $p = 1 \bmod 3$ and add 2 if $p = 1 \bmod 4$.

$$\langle a, b, c, d \,|\, ca, cb - baa, da, db - bab, dc, pa - zbaa, pb - tbaa, pc - bab, pd, \text{ class } 3\rangle,$$

with z, t as speciøed.

And ønally consider the case $pc = pd = bab$. We now need $\mu = 1$, so that if $pa = zbaa$, $pb = tbaa$ then z, t can take all possible values, giving p^2 algebras

$$\langle a, b, c, d \,|\, ca, cb-baa, da, db-bab, dc, pa-zbaa, pb-tbaa, pc-bab, pd-bab, \text{ class } 3\rangle \ (0 \le z, t < p).$$

17.6 Case 6

Now let L satisfy $da = dc = 0$, $ca = db = bab$, $cb = baa$. If a', b', c', d' satisfy the same commutator relations then

$$\begin{aligned} a' &= \alpha a - \beta b + \gamma c + \delta d, \\ b' &= \pm(\beta a + \alpha b + \lambda c + \mu d), \\ c' &= (\alpha^2 - \beta^2)c - 4\alpha\beta d, \\ d' &= \pm(\alpha\beta c + (\alpha^2 - \beta^2)d) \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm(\alpha^2 + \beta^2)(\alpha baa - \beta bab), \\ b'a'b' &= (\alpha^2 + \beta^2)(\beta baa + \alpha bab). \end{aligned}$$

We show in Appendix B that the number of algebras is as follows:

$p = 1 \bmod 12$

$(p^2 + 3p + 14)/2 - (p + 11)/2 - 1 = \frac{1}{2}p^2 + p + \frac{1}{2}$ with pc, pd rank 2.

$p^2 + (7p + 15)/2$ with $pc = pd = 0$.

$p(p + 1)/2 + ((p + 3)/4 - 2)p^2 + p(p + 1) + p^2(p + 3)/4 + p(p + 1)/2 + 2p + p + 7 = \frac{3}{2}p^2 + 5p + \frac{1}{2}p^3 + 7$ with pc, pd rank 1.

So the grand total is $\frac{1}{2}p^2+p+\frac{1}{2}+p^2+(7p+15)/2+\frac{3}{2}p^2+5p+\frac{1}{2}p^3+7 = 3p^2+\frac{19}{2}p+15+\frac{1}{2}p^3$

$p = 5 \bmod 12$

$(p^2 + 3p + 12)/2 - (p + 11)/2 - 1 = \frac{1}{2}p^2 + p - \frac{1}{2}$ with pc, pd rank 2.

$p^2 + (7p + 15)/2$ with $pc = pd = 0$.

$p(p + 1)/2 + ((p + 3)/4 - 2)p^2 + p(p + 1) + p^2(p + 3)/4 + p(p + 1)/2 + 2p + p + 5 = \frac{3}{2}p^2 + 5p + \frac{1}{2}p^3 + 5$ with pc, pd rank 1.

So the grand total is $\frac{1}{2}p^2+p-\frac{1}{2}+p^2+(7p+15)/2+\frac{3}{2}p^2+5p+\frac{1}{2}p^3+5 = 3p^2+\frac{19}{2}p+12+\frac{1}{2}p^3$

$p = 7 \bmod 12$

$(p^2 + p + 4)/2 - (p + 3)/2 - 1 = \frac{1}{2}p^2 - \frac{1}{2}$ with pc, pd of rank 2.

$p^2 + (3p + 5)/2$ with $pc = pd = 0$.

$p(p + 1)/2 + p^2((p + 1)/4 - 1) + p(p + 1)/2 + p^2(p + 1)/4 = \frac{1}{2}p^2 + p + \frac{1}{2}p^3$ with pc, pd of rank 1.

So the grand total is $\frac{1}{2}p^2 - \frac{1}{2} + p^2 + (3p + 5)/2 + \frac{1}{2}p^2 + p + \frac{1}{2}p^3 = 2p^2 + 2 + \frac{5}{2}p + \frac{1}{2}p^3$.

$p = 11 \bmod 12$

$(p^2 + p + 6)/2 - (p + 3)/2 - 1 = \frac{1}{2}p^2 + \frac{1}{2}$ with pc, pd of rank 2.

$p^2 + (3p + 5)/2$ with $pc = pd = 0$.

$p(p + 1)/2 + p^2((p + 1)/4 - 1) + p(p + 1)/2 + p^2(p + 1)/4 = \frac{1}{2}p^2 + p + \frac{1}{2}p^3$ with pc, pd of rank 1.

So the grand total is $\frac{1}{2}p^2 + \frac{1}{2} + p^2 + (3p + 5)/2 + \frac{1}{2}p^2 + p + \frac{1}{2}p^3 = 2p^2 + 3 + \frac{5}{2}p + \frac{1}{2}p^3$

See orb5.36 for a program to generate the orbits.

17.7 Case 7

Now let L satisfy $da = dc = 0$, $ca = db = bab$, $cb = \omega baa$. If a', b', c', d' satisfy the same commutator relations then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \pm(-\omega\beta a + \alpha b + \lambda c + \mu d), \\ c' &= (\alpha^2 - \omega\beta^2)c + 4\omega\alpha\beta d), \\ d' &= \pm(-\alpha\beta c + (\alpha^2 - \omega\beta^2)d) \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm(\alpha^2 + \omega\beta^2)(\alpha baa + \beta bab), \\ b'a'b' &= (\alpha^2 + \omega\beta^2)(-\omega\beta baa + \alpha bab). \end{aligned}$$

We show in Appendix C that the total number of algebras is
$2p^2 + \frac{5}{2}p + 2 + \frac{1}{2}p^3$ if $p = 1 \bmod 12$,
$2p^2 + \frac{5}{2}p + 3 + \frac{1}{2}p^3$ if $p = 5 \bmod 12$,
$3p^2 + \frac{19}{2}p + 13 + \frac{1}{2}p^3$ if $p = 7 \bmod 12$,
$3p^2 + \frac{19}{2}p + 10 + \frac{1}{2}p^3$ if $p = 11 \bmod 12$.
See orb5.37.

17.8 Case 8

Now let L satisfy $ca = cb = da = db = 0$, $dc = baa$. If a', b', c', d' satisfy the same commutator relations then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \alpha^{-2}(\lambda\xi - \mu\nu)b, \\ c' &= \lambda c + \mu d, \\ d' &= \nu c + \xi d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= (\lambda\xi - \mu\nu)baa, \\ b'a'b' &= \alpha^{-1}\beta(\lambda\xi - \mu\nu)baa + \alpha^{-3}(\lambda\xi - \mu\nu)^2 bab. \end{aligned}$$

We ørst consider the possibilities for pc, pd.

One possibility is that $pc = pd = 0$.

If pc, pd span a space of dimension 1 then we can assume that $pd = 0$, and that $pc = baa$ or bab.

If pc, pd span a space of dimension 2 then we may suppose that $pc = baa$, $pd = bab$.

Consider the case when $pc = pd = 0$. Then we can suppose that $pa = 0$, baa, bab or ωbab.

Suppose that $pa = pc = pd = 0$. Then we can take $pb = 0$, baa, ωbaa, or bab. (4)xxxxxx

Suppose that $pa = baa$, $pc = pd = 0$. Then we require $\alpha = (\lambda\xi - \mu\nu)$. Let $pb = xbaa + ybab$. Then

$$pb' = (\beta + \alpha^{-1}x)baa + \alpha^{-1}ybab = \alpha^{-1}(\beta + \alpha^{-1}x - \beta y)b'a'a' + yb'a'b'.$$

So if $y \neq 1$ we can take $x = 0$, and if $y = 1$ we can take $x = 0, 1, \omega$, giving $p + 2$ algebras. $(p + 2)$xxxxxxxxx

Suppose that $pa = bab$ or ωbab, $pc = pd = 0$. Then we require $\alpha^4 = (\lambda\xi - \mu\nu)^2$, $\beta = 0$. Let $pb = xbaa + ybab$. Then

$$\begin{aligned} pb' &= \alpha^{-2}(\lambda\xi - \mu\nu)xbaa + \alpha^{-2}(\lambda\xi - \mu\nu)ybab \\ &= \alpha^{-2}xb'a'a' \pm \alpha^{-1}yb'a'b'. \end{aligned}$$

So we can take $y = 0$ or 1, and if $y = 0$ we can take $x = 0, 1, \omega$. $(2(p+3))$xxxxxx

Next consider the case when $pc = baa$, $pd = 0$. We require $\nu = 0$, $\xi = 1$, so we have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \beta a + \alpha^{-2}\lambda b, \\ c' &= \lambda c + \mu d, \\ d' &= d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \lambda baa, \\ b'a'b' &= \alpha^{-1}\beta\lambda baa + \alpha^{-3}\lambda^2 bab. \end{aligned}$$

We can assume that $pa = 0$, baa, bab or ωbab.

If $pa = 0$ we can take $pb = 0$, baa, ωbaa or bab. (4)xxxxx

If $pa = baa$ we need $\alpha = \lambda$. Suppose that $pb = xbaa + ybab$. Then we have

$$pb' = (\beta + \alpha^{-1}x)baa + \alpha^{-1}ybab = \alpha^{-1}(\beta + \alpha^{-1}x - \beta y)b'a'a' + yb'a'b'.$$

So if $y \neq 1$ we can take $x = 0$, and if $y = 1$ we can take $x = 0, 1, \omega$, giving $p + 2$ algebras. $(p+2)$xxxxxxxxx

If $pa = bab$ or ωbab then we require $\beta = 0$, $\alpha^4 = \lambda^2$. If $pb = xbaa + ybab$ then

$$pb' = \alpha^{-2}\lambda xbaa + \alpha^{-2}\lambda ybab = \alpha^{-2}xb'a'a' + \alpha\lambda^{-1}yb'a'b'.$$

Now $\lambda = \pm\alpha^2$ and so $\alpha\lambda^{-1} = \pm\alpha^{-1}$. So we can take $y = 0, 1$, and if $y = 0$ we can take $x = 0, 1, \omega$. $(2(p+3))$xxxxxx

Next consider the case when $pc = bab$, $pd = 0$. We require $\beta = \nu = 0$, $\lambda = \alpha^3\xi^{-2}$ which gives

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha\xi^{-1}b, \\ c' &= \alpha^3\xi^{-2}c + \mu d, \\ d' &= \xi d, \end{aligned}$$

$$\begin{aligned} b'a'a' &= \alpha^3\xi^{-1}baa, \\ b'a'b' &= \alpha^3\xi^{-2}bab. \end{aligned}$$

Let

$$\begin{pmatrix} pa \\ pb \end{pmatrix} = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} baa \\ bab \end{pmatrix}.$$

Then

$$\begin{pmatrix} pa' \\ pb' \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha\xi^{-1} \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha^3\xi^{-1} & 0 \\ 0 & \alpha^3\xi^{-2} \end{pmatrix}^{-1} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix} = \begin{pmatrix} \alpha^{-2}\xi u & \alpha^{-2}\xi^2 v \\ \alpha^{-2}w & \alpha^{-2}\xi x \end{pmatrix} \begin{pmatrix} b'a'a' \\ b'a'b' \end{pmatrix}$$

So the total number of algebras here is $2p^2+5p+12$.

Finally consider the case when $pc = baa$, $pd = bab$. We need $\mu = 0$, $\xi = 1$, $\alpha^3 = \lambda^2$, $\nu = \alpha^{-1}\beta\lambda$. The condition $\alpha^3 = \lambda^2$ implies that $\alpha = \delta^2$ for some δ, so that $\lambda = \pm\delta^3$, so that

$$\begin{aligned} a' &= \delta^2 a, \\ b' &= \beta a \pm \delta^{-1} b, \\ c' &= \pm\delta^3 c, \\ d' &= \pm\beta\delta c + d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm\delta^3 baa, \\ b'a'b' &= \pm\beta\delta baa + bab. \end{aligned}$$

We can choose $pa = 0$, baa, bab or ωbab.

If $pa = 0$ then we can take $pb = 0$, baa, ωbaa or bab, or (when $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. (4+2 if $p = 1 \bmod 4$)xxxx

If $pa = baa$ then we require $\delta = \pm 1$, giving

$$\begin{aligned} a' &= a, \\ b' &= \beta a + b, \\ c' &= c, \\ d' &= \beta c + d, \end{aligned}$$

$$\begin{aligned} b'a'a' &= baa, \\ b'a'b' &= \beta baa + bab. \end{aligned}$$

Let $pb = xbaa + ybab$. Then

$$pb' = (\beta + x)baa + ybab = (\beta + x - \beta y)b'a'a' + yb'a'b'.$$

So if $y \neq 1$ we can take $x = 0$, giving $2p - 1$ algebras

If $pa = bab$ or ωbab then we need $\beta = 0$, $\delta^2 = 1$.

$$\begin{aligned} a' &= a, \\ b' &= \pm b, \\ c' &= \pm c, \\ d' &= d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm baa, \\ b'a'b' &= bab. \end{aligned}$$

Let $pb = xbaa + ybab$. Then

$$pb' = \pm xbaa \pm ybab = xb'a'a' \pm yb'a'b'.$$

So we have $p(p+1)$ algebras.

So the total number of algebras is $3p^2 + 14p + 37 + \gcd(p-1, 4)$. See orb5.38.

Finally let L satisfy $cb = da = db = 0$, $ca = bab$, $dc = baa$. If a', b', c', d' satisfy the same commutator relations then

$$\begin{aligned} a' &= \alpha a - 2\alpha\beta\gamma^{-1}d, \\ b' &= \beta a + \gamma b - \beta^2\gamma^{-1}d, \\ c' &= \gamma^2 c + \delta d, \\ d' &= \alpha^2\gamma^{-1}d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \alpha^2\gamma baa, \\ b'a'b' &= \alpha\beta\gamma baa + \alpha\gamma^2 bab. \end{aligned}$$

The number of algebras is $p^4 + 4p^3 + 7p^2 + 14p + 10 + (p+3)\gcd(p-1,3) + (p+2)\gcd(p-1,4) + \gcd(p-1,5)$.

$p = 5$ get 1417.

$p = 7$ get 4273.

$p = 11$ get 21021.

See orb5.39.

We can assume that $pd = 0$, baa, ωbaa or bab.

If $pd = 0$ we can assume that $pc = 0$, baa or bab.

If $pd = baa$ or ωbaa we can assume that $pc = 0$ or bab.

If $pd = bab$ then we can assume that $pc = 0$ or baa or (in the case when $p = 1 \bmod 5$) ωbaa, $\omega^2 baa$, $\omega^3 baa$ or $\omega^4 baa$.

First consider the case when $pc = pd = 0$. Then we can assume that $pa = 0$, baa, bab or ωbab. If $pa = 0$ we can assume that $pb = 0$, baa, ωbaa or bab (xxx 4 algebras). If $pa = baa$ then we need $\gamma = \alpha^{-1}$, so that we can take

$$\begin{aligned} a' &= \alpha a - 2\alpha^2\beta d, \\ b' &= \beta a + \alpha^{-1}b - \alpha\beta^2 d, \\ c' &= \alpha^{-2}c + \delta d, \\ d' &= \alpha^3 d \end{aligned}$$

which gives

$$\begin{aligned} b'a'a' &= \alpha baa, \\ b'a'b' &= \beta baa + \alpha^{-1}bab. \end{aligned}$$

If we let $pb = \lambda baa + \mu bab$ then we have

$$pb' = (\beta + \alpha^{-1}\lambda)baa + \alpha^{-1}\mu bab = \alpha^{-1}(\beta + \alpha^{-1}\lambda - \beta\mu)b'a'a' + \mu b'a'b'.$$

So if $\mu \neq 1$ we can take $pb = \mu bab$, and if $\mu = 1$ we can take $pb = bab$, $baa + bab$ or $\omega baa + bab$ (xxx $p+2$ algebras). If $pa = bab$ or ωbab then we need $\beta = 0$ and $\gamma = \pm 1$ so we can take

$$\begin{aligned} a' &= \alpha a, \\ b' &= \pm b, \\ c' &= c + \delta d, \\ d' &= \pm\alpha^2 d \end{aligned}$$

which gives

$$\begin{aligned}
b'a'a' &= \pm\alpha^2 baa,\\
b'a'b' &= \alpha bab.
\end{aligned}$$

So if $pb = \lambda baa + \mu bab$ then we have

$$pb' = \pm\lambda baa \pm \mu bab = \alpha^{-2}\lambda b'a'a' \pm \alpha^{-1}\mu bab.$$

So we can take $\mu = 0$ or 1, and if $\mu = 0$ we can take $\lambda = 0$, 1 or ω and if $\mu = 1$ we can take $0 \leq \lambda < p$ (xxx $2(p+3)$ algebras).

Next consider the case when $pc = baa$, $pd = 0$. Then we need $\gamma = \alpha^2$ and so we can take

$$\begin{aligned}
a' &= \alpha a - 2\alpha^{-1}\beta d,\\
b' &= \beta a + \alpha^2 b - \alpha^{-2}\beta^2 d,\\
c' &= \alpha^4 c + \delta d,\\
d' &= d
\end{aligned}$$

$$\begin{aligned}
b'a'a' &= \alpha^4 baa,\\
b'a'b' &= \alpha^3\beta baa + \alpha^5 bab.
\end{aligned}$$

We can take $pa = 0$, baa or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$, or bab, ωbab or (if $p = 1 \bmod 4$) $\omega^2 bab$ or $\omega^3 bab$. If $pa = 0$ we can take $pb = 0$, baa, ωbaa, bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$ (xxx $3 + \gcd(p-1,3)$ algebras). If $pa = kbaa$ where $k = 1$, ω or ω^2 then we need $\alpha^3 = 1$. If $pb = \lambda baa + \mu bab$ then we have

$$pb' = (\beta k + \alpha^2\lambda)baa + \alpha^2\mu bab = \alpha^{-1}(\beta k + \alpha^2\lambda - \mu\beta)baa + \mu b'a'b'$$

so if $\mu \neq k$ we can take $pb = \mu bab$, and if $\mu = k$ we can take $pb = \lambda baa + kbab$ where $\lambda = 0$, or λ lies in a transversal for the cube roots of unity (xxx $p\gcd(p-1,3) + p - 1$ algebras). If $pa = kbab$ where $k = 1$, ω, ω^2 or ω^3 then we need $\beta = 0$ and $\alpha^4 = 1$. If $pb = \lambda baa + \mu bab$ then we have

$$pb' = \alpha^2\lambda baa + \alpha^2\mu bab = \alpha^2\lambda b'a'a' + \alpha\mu b'a'b'$$

so if $p = 3 \bmod 4$ we can take $0 \leq \lambda < p$, $0 \leq \mu \leq (p-1)/2$ and if $p = 1 \bmod 4$ we can take $\mu = 0$ and $0 \leq \lambda \leq (p-1)/2$ or μ in a transversal for the fourth roots of unity and $0 \leq \lambda < p$ (xxx $p^2 + p - 2 + \gcd(p-1,4)$ algebras).

Now consider the case when $pc = bab$, $pd = 0$. Then we need $\alpha = 1$ and $\beta = 0$ so we can take

$$\begin{aligned}
a' &= a,\\
b' &= \gamma b,\\
c' &= \gamma^2 c + \delta d,\\
d' &= \gamma^{-1} d
\end{aligned}$$

which gives

$$\begin{aligned}
b'a'a' &= \gamma baa,\\
b'a'b' &= \gamma^2 bab.
\end{aligned}$$

If we let $pa = \zeta baa + \eta bab$ and $pb = \lambda baa + \mu bab$ then

$$\begin{aligned} pa' &= \zeta baa + \eta bab = \gamma^{-1}\zeta b'a'a' + \gamma^{-2}\eta b'a'b', \\ pb' &= \gamma\lambda baa + \gamma\mu bab = \lambda b'a'a' + \gamma^{-1}\mu b'a'b'. \end{aligned}$$

So if $\zeta \neq 0$ we can take $\zeta = 1$ and η, λ, μ arbitrary, if $\zeta = 0$ and $\mu \neq 0$ we can take $\mu = 1$ and η, λ arbitary, and if $\zeta = \mu = 0$ we can take $\eta = 0, 1, \omega$ and λ arbitrary (xxx $p^3 + p^2 + 3p$ algebras).

And now consider the case when $pc = 0$ and $pd = baa$ or ωbaa. Then we need $\gamma = \pm 1$ and $\delta = 0$, so we can take

$$\begin{aligned} a' &= \alpha a \mp 2\alpha\beta d, \\ b' &= \beta a \pm b \mp \beta^2 d, \\ c' &= c, \\ d' &= \pm\alpha^2 d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm\alpha^2 baa, \\ b'a'b' &= \pm\alpha\beta baa + \alpha bab. \end{aligned}$$

If we let $pa = \rho baa + \sigma bab$ then

$$pa' = \alpha\rho baa + \alpha\sigma bab \mp 2\alpha\beta pd = \sigma b'a'b' + \alpha\rho baa \mp \alpha\beta\sigma baa \mp 2\alpha\beta pd.$$

We can take $pa = \sigma bab$ with $0 \leq \sigma < p$ though we then need $\beta = 0$, unless $\sigma = -2k$ where $k = 1$ when $pd = baa$, and $k = \omega$ when $pd = \omega baa$.

First consider the case when $\sigma \neq -2k$. If we let $pb = \lambda baa + \mu bab$ then

$$pb' = \pm\lambda baa \pm \mu bab = \alpha^{-2}\lambda b'a'a' \pm \alpha^{-1}\mu b'a'b'.$$

So if $\mu \neq 0$ we can take $\mu = 1$ and λ arbitrary, and if $\mu = 0$ we can take $\lambda = 0, 1, \omega$ (xxx $2(p-1)(p+3)$ algebras).

Next, consider the case when $\sigma = -2k$. Then we have

$$pa' = \pm\alpha^{-1}\rho ba'a' + \sigma b'a'b'$$

so we can take $\rho = 0$ or 1. If $\rho = 0$ then

$$pb' = \pm pb \mp k\beta^2 baa - 2k\beta bab,$$

so we can assume that $pb = 0$, baa or ωbaa. If $\rho = 1$ then we need $\alpha = \gamma$ (where $\gamma = \pm 1$) and

$$pb' = \pm pb + (\beta \mp k\beta^2)baa - 2k\beta bab,$$

so we can assume that $pb = \lambda baa$ with $0 \leq \lambda < p$ (xxx $2(p+3)$ algebras).

Suppose that $pc = bab$ and $pd = baa$ or ωbaa. Then we need $\alpha = 1$ and $\gamma = \pm 1$ (with suitable δ) so we can take

$$\begin{aligned} a' &= a \mp 2\beta d, \\ b' &= \beta a \pm b \mp \beta^2 d, \\ c' &= c + \delta d, \\ d' &= \pm d \end{aligned}$$

which gives

$$\begin{aligned} b'a'a' &= \pm baa, \\ b'a'b' &= \pm\beta baa + bab. \end{aligned}$$

If we let $pa = \rho baa + \sigma bab$ then

$$pa' = \rho baa + \sigma bab \mp 2\alpha\beta pd = \sigma b'a'b' + \rho baa \mp \beta\sigma baa \mp 2\beta pd.$$

As above we can take $pa = \sigma bab$ with $0 \leq \sigma < p$ though we then need $\beta = 0$, unless $\sigma = -2k$.

First consider the case when $\sigma \neq -2k$. If we let $pb = \lambda baa + \mu bab$ then

$$pb' = \pm\lambda baa \pm \mu bab = \lambda b'a'a' \pm \mu b'a'b'$$

so we can take $0 \leq \lambda < p$, $0 \leq \mu \leq (p-1)/2$ (xxx $p(p^2-1)$ algebras).

Next consider the case when $\sigma = -2k$. We have

$$pa' = \sigma b'a'b' + \rho baa$$

so we can take $0 \leq \rho \leq (p-1)/2$ though if $\rho \neq 0$ we need $\gamma = 1$. We then have

$$pb' = \gamma pb - k\beta^2\gamma baa - 2k\beta bab,$$

and so we can take $pb = \lambda baa$ where $0 \leq \lambda < p$ (xxx $p(p+1)$ algebras).

Next consider the case when $pc = 0$, $pd = bab$. Then we need $\beta = 0$, $\alpha = \gamma^3$ and $\delta = 0$ so we can take

$$\begin{aligned} a' &= \gamma^3 a, \\ b' &= \gamma b, \\ c' &= \gamma^2 c, \\ d' &= \gamma^5 d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \gamma^7 baa, \\ b'a'b' &= \gamma^5 bab. \end{aligned}$$

If we let $pa = \zeta baa + \eta bab$ and $pb = \lambda baa + \mu bab$ then

$$\begin{aligned} pa' &= \gamma^3\zeta baa + \gamma^3\eta bab = \gamma^{-4}\zeta baa + \gamma^{-2}\eta bab \\ pb' &= \gamma\lambda baa + \gamma\mu bab = \gamma^{-6}\lambda baa + \gamma^{-4}\mu bab. \end{aligned}$$

So if $\eta \neq 0$ we can take $\eta = 1, \omega$ and ζ, λ, μ arbitrary; if $\eta = 0$ and $\zeta \neq 0$ and $p = 1 \bmod 4$ then we can take $\zeta = 1, \omega, \omega^2, \omega^3$, μ arbitry and $0 \leq \lambda \leq (p-1)/2$; if $\eta = 0$ and $\zeta \neq 0$ and $p = 3 \bmod 4$ then we can take $\zeta = 1, \omega$, λ, μ arbitry; if $\eta = \zeta = 0$ and $\mu \neq 0$ and $p = 1 \bmod 4$ then we can take $\mu = 1, \omega, \omega^2, \omega^3$, and $0 \leq \lambda \leq (p-1)/2$; if $\eta = \zeta = 0$ and $\mu \neq 0$ and $p = 3 \bmod 4$ then we can take $\mu = 1, \omega$, and λ arbitary; and if $\zeta = \eta = \mu = 0$ then we can take $\lambda = 0, 1, \omega$ or (if $p = 1 \bmod 3$) $\omega^2, \omega^3, \omega^4, \omega^5$; (xxx $2p^3 + 2p^2 - 1 + 2\gcd(p-1,3) + (p+1)\gcd(p-1,4)$ algebras).

Finally consider the case when $pc = kbaa$ and $pd = bab$ with $k = 1$ or (if $p = 1 \bmod 5$) ω, ω^2, ω^3 or ω^4.Then we need $\beta = 0$, $\alpha = \gamma^3$, $\gamma^5 = 1$ and $\delta = 0$ so we can take

$$\begin{aligned} a' &= \gamma^3 a, \\ b' &= \gamma b, \\ c' &= \gamma^2 c, \\ d' &= d \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \gamma^2 baa, \\ b'a'b' &= bab. \end{aligned}$$

If we let $pa = \zeta baa + \eta bab$ and $pb = \lambda baa + \mu bab$ then

$$\begin{aligned} pa' &= \gamma^3\zeta baa + \gamma^3\eta bab = \gamma\zeta baa + \gamma^{-2}\eta bab \\ pb' &= \gamma\lambda baa + \gamma\mu bab = \gamma^{-1}\lambda baa + \gamma\mu bab. \end{aligned}$$

If $p \neq 1 \bmod 5$ then we need $\gamma = 1$ and so $\zeta, \eta, \lambda, \mu$ are arbitrary. And if $p = 1 \bmod 5$ then we can take any one non-zero parameter among $\zeta, \eta, \lambda, \mu$ to lie in a tansversal for the øfth roots of unity, and the others arbitrary (xxx $p^4 - 1 + \gcd(p-1, 5)$ algebras).

18 Immediate descendants of algebra 42 (5.1)

Let L be an immediate descendant of 5.1 of order p^7. Then L is generated by a, b, c, d, e. We divide the various cases up according to their commutator structure.

18.1 L abelian

$$\langle a, b, c, d, e \mid ba, ca, da, ea, cb, db, eb, dc, ec, ed, pc, pd, pe, \text{ class } 2\rangle \tag{7.11}$$

18.2 L^2 has order p

We may assume that L^2 is generated by ba, and that the commutator structure is given by one of the following two sets of relations:

$$\begin{aligned} ca &= da = ea = cb = db = eb = dc = ec = ed = 0, \\ ca &= da = ea = cb = db = eb = ec = ed = 0, \ dc = ba. \end{aligned}$$

Case 1 First consider the case when L satisøes the relations

$$ca = da = ea = cb = db = eb = dc = ec = ed = 0.$$

If pL has order p then $L_2 = L^2 \oplus pL$, and we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.12}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa, pb, pd, pe, \text{ class } 2\rangle. \tag{7.13}$$

So consider the case when pL has order p^2. If $p\langle c, d, e\rangle \not\leq L^2$ then we may assume that L_2 is generated by ba, pc, and we may assume that pa, pb, pd, pe are all linear multiples of ba. If pd or pe is non-zero then we may assume that $pd = ba$, $pa = pb = pe = 0$. And if $pd = pe = 0$ then we may assume that $pa = ba$, $pb = 0$. So we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa, pb, pd - ba, pe, \text{ class } 2\rangle, \tag{7.14}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pa - ba, pb, pd, pe, \text{ class } 2\rangle. \tag{7.15}$$

On the other hand, if $p\langle c, d, e\rangle \leq L^2$ then we may assume that L_2 is generated by ba, pa, and we may assume that pb, pc, pd, pe are all linear multiples of ba. This gives

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pb, pc - ba, pd, pe, \text{ class } 2\rangle, \tag{7.16}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc, ec, ed, pb - ba, pc, pd, pe, \text{ class } 2\rangle. \tag{7.17}$$

Case 2 Now consider the case when L satisøes the relations

$$ca = da = ea = cb = db = eb = ec = ed = 0, \ dc = ba.$$

If pL has order p then $L_2 = L^2 \oplus pL$, and we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.18}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pa, pb, pc, pd, \text{ class } 2\rangle. \tag{7.19}$$

So consider the case when pL has order p^2. We can assume either that L_2 is generated by ba and pe or that L_2 is generated by ba and pa. If L_2 is generated by ba and pe then we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pa - ba, pb, pc, pd, \text{ class } 2\rangle, \tag{7.20}$$

and if L_2 is generated by ba and pa then we have

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pb, pc, pd, pe - ba, \text{ class } 2\rangle, \tag{7.21}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \tag{7.22}$$

$$\langle a, b, c, d, e \mid ca, da, ea, cb, db, eb, dc - ba, ec, ed, pb, pc - ba, pd, pe, \text{ class } 2\rangle. \tag{7.22A}$$

18.3 L^2 has order p^2

If L^2 has order p^2, then we can assume that L has the same commutator structure as one of the class 2, øve generator, seven dimensional Lie algebras over $\mathbb{Z}_p$. So we may assume that one of the following sets of commutator relations holds:

$$\begin{aligned}
da &= ea = cb = db = eb = dc = ec = ed = 0, \\
da &= ea = cb = eb = dc = ec = ed = 0, \ db = ba, \\
da &= ea = cb = eb = dc = ec = ed = 0, \ db = ca, \\
da &= ea = cb = eb = ec = ed = 0, \ db = ca, \ dc = \omega ba, \\
da &= ea = cb = db = eb = dc = ec = 0, \ ed = ba, \\
da &= ea = cb = eb = dc = ec = 0, \ db = ca, \ ed = ba.
\end{aligned}$$

If $pL = \{0\}$ then we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.23}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ba, eb, dc, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.24}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.25}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc - \omega ba, ec, ed, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.26}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.27}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc, pd, pe, \text{ class } 2\rangle. \tag{7.28}$$

Next, consider the case when pL has order p.

Case 1 If

$$da = ea = cb = db = eb = dc = ec = ed = 0$$

then we may suppose that $pe = 0$ and that the subalgebras generated by a, b, c, d is isomorphic to one of 6.20, 6.21, 6.23 or 6.29. so we have

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.29}$$

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \tag{7.30}$$

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa, pb - ca, pc, pd, pe, \text{ class } 2\rangle, \tag{7.31}$$

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa, pb, pc, pd - ca, pe, \text{ class } 2\rangle. \tag{7.32}$$

Case 2 Next, consider the case when

$$da = ea = cb = eb = dc = ec = ed = 0, \; db = ba.$$

If $pe = 0$ then the subalgebra generated by a, b, c, d is isomorphic to one of 6.34 ˘ 6.38. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.33}$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \tag{7.34}$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ba, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \tag{7.35}$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ba - ca, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.36}$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ba - ca, pb - ba - ca, pc, pd, pe, \text{ class } 2\rangle. \tag{7.37}$$

On the other hand, if $pe \neq 0$ then we can assume that $pa = pb = pc = pd = 0$, and (using the same argument as in the calculation of 6.34 ˘ 6.38) we may assume that $pe = ba$ or $ba + ca$. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa, pb, pc, pd, pe - ba, \text{ class } 2\rangle, \tag{7.38}$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa, pb, pc, pd, pe - ba - ca, \text{ class } 2\rangle. \tag{7.39}$$

Case 3 Now suppose that

$$da = ea = cb = eb = dc = ec = ed = 0, \; db = ca.$$

Again, e is central, and if $pe = 0$ then the subalgebra generated by a, b, c, d is isomorphic to one of 6.49 ˘ 6.52. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.40}$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc - ba, pd, pe, \text{ class } 2\rangle, \tag{7.41}$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ca, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.42}$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc - ca, pd, pe, \text{ class } 2\rangle. \tag{7.43}$$

And if $pe \neq 0$ then we can assume that $pa = pb = pc = pd = 0$, and (using the same argument as in the computation of 6.49 ˘ 6.52) we may assume that $pe = ba$ or ca. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc, pd, pe - ba, \text{ class } 2\rangle, \tag{7.44}$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc, pd, pe - ca, \text{ class } 2\rangle. \tag{7.45}$$

Case 4 Now consider the case when

$$da = ea = cb = eb = ec = ed = 0, \; db = ca, \; dc = \omega ba.$$

Once again, e is central. If $pe = 0$ then the subalgebra generated by a, b, c, d is isomorphic to 6.60B, and so we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc - \omega ba, ea, eb, ec, ed, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle. \quad (7.46)$$

And if $pe \neq 0$ then we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc - \omega ba, ea, eb, ec, ed, pa, pb, pc, pd, pe - ba, \text{ class } 2\rangle. \quad (7.47)$$

Case 5 Next consider the case when

$$da = ea = cb = db = eb = dc = ec = 0, \; ed = ba.$$

It is straightforward to show that if a', b', c', d', e' generate L and satisfy these same commutator relations, then (modulo L^2)

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d + \varepsilon e, \\ b' &= \alpha^{-1}(\lambda\xi - \mu\nu)b, \\ c' &= \zeta b + \eta c, \\ d' &= -\alpha^{-1}(\delta\mu - \varepsilon\lambda)b + \lambda d + \mu e, \\ e' &= -\alpha^{-1}(\delta\xi - \varepsilon\nu)b + \nu d + \xi e, \end{aligned}$$

with $\alpha, \eta, \lambda\xi - \mu\nu \neq 0$. (Furthermore any a', b', c', d', e' of this form satisfy these same relations.) It is clear that we may assume that pL is generated by ba or by ca. First consider the case when $pb = \rho ba$ for some $\rho \neq 0$. Replacing a, b, c, d, e by a', b', c', d', e' where

$$\begin{aligned} a' &= \rho a + \beta b + \delta d + \varepsilon e, \\ b' &= \rho^{-1} b, \\ c' &= \zeta b + c, \\ d' &= \varepsilon b + d, \\ e' &= -\delta b + e \end{aligned}$$

for suitable $\beta, \delta, \varepsilon, \zeta$, we may assume that $pa = pc = pd = pe = 0$ and that $pb = ba$. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ba, pc, pd, pe, \text{ class } 2\rangle. \quad (7.48)$$

Next let $pb = \rho ca$ for some $\rho \neq 0$ and let $pc = \sigma ca$. Replacing a, b, c, d, e by a', b', c', d', e' where

$$\begin{aligned} a' &= a + \beta b + \delta d + \varepsilon e, \\ b' &= b, \\ c' &= c, \\ d' &= \varepsilon b + d, \\ e' &= -\delta b + e \end{aligned}$$

for suitable $\beta, \delta, \varepsilon$, we may assume that $pa = pd = pe = 0$. Then, if we let $a' = \alpha a$, $b' = \alpha^{-1} b$, $c' = \eta c$ we have

$$\begin{aligned} b'a' &= ba, \\ c'a' &= \alpha\eta ca, \\ pb' &= \alpha^{-1} pb = \alpha^{-1} \rho ca, \\ pc' &= \eta pc = \eta\sigma ca. \end{aligned}$$

Setting $\eta = \alpha^{-2}\rho$ we have

$$\begin{aligned} pb' &= c'a', \\ pc' &= \alpha^{-1}\sigma c'a'. \end{aligned}$$

If $\sigma \neq 0$ we can take $\alpha = \sigma$, and so we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ca, pc, pd, pe, \text{ class } 2\rangle, \tag{7.49}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ca, pc - ca, pd, pe, \text{ class } 2\rangle. \tag{7.50}$$

So assume that $pb = 0$. If one of pd, pe is non-zero then we may assume that $pe = 0$ and that $pd = ba$ or ca. Subtracting a suitable scalar multiple of d from a we may assume that $pa = 0$. If $pd = ba$ then scaling c we may assume that $pc = 0$ or ba. And if $pd = ca$ then scaling a, b, d, e we may suppose that $pc = 0$ or ca. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc, pd - ba, pe, \text{ class } 2\rangle, \tag{7.51}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd - ba, pe, \text{ class } 2\rangle, \tag{7.52}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc, pd - ca, pe, \text{ class } 2\rangle, \tag{7.53}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ca, pd - ca, pe, \text{ class } 2\rangle. \tag{7.54}$$

Now assume that $pb = pd = pe = 0$. If $pc \neq 0$ then subtracting a suitable scalar multiple of c from a we may suppose that $pa = 0$. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd, pe, \text{ class } 2\rangle, \tag{7.55}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ca, pd, pe, \text{ class } 2\rangle, \tag{7.56}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle, \tag{7.57}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa - ca, pb, pc, pd, pe, \text{ class } 2\rangle. \tag{7.58}$$

Case 6 Finally consider the case when L satisøes

$$da = ea = cb = eb = dc = ec = 0,\ db = ca,\ ed = ba.$$

It is straightforward to show that the only elements of breadth 1 in the span of a, b, c, d, e are elements of the form

$$\lambda(b + \mu c - \mu^{-1} e),\ \lambda c,\ \lambda e.$$

In particular, the elements of breadth 1 all lie in the linear span of b, c, e. It is straightforward to show that we may assume that pL is generated by ba. And then, if we consider possible generating sets a', b', c', d', e' for L satisfying the same commutator relations as

a, b, c, d, e and such that $b'a'$ is a linear multiple of ba, it is straightforward to show that (up to scale factors, and modulo L^2)

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c + \varepsilon e, \\
b' &= \lambda b - 2\lambda\mu e, \\
c' &= \lambda\mu b + \lambda c - \lambda\mu^2 e, \\
d' &= \alpha\mu a + (\beta\mu + \varepsilon)b + (-\beta + \gamma\mu)c + \alpha d + \nu e, \\
e' &= \lambda e,
\end{aligned}$$

with α, λ both non-zero. We then have $b'a' = \alpha\lambda ba$.

First consider the case when $pe \neq 0$. Scaling a and d we may assume that $pe = ba$. Then, if we let

$$\begin{aligned}
a' &= a + \varepsilon e, \\
b' &= b - 2\mu e, \\
c' &= \mu b + c - \mu^2 e, \\
d' &= \mu a + \varepsilon b + d + \nu e, \\
e' &= e,
\end{aligned}$$

we can choose ε, μ, ν so that $pa' = pb' = pd' = 0$. Let $pc = kba$. Then if we let $a' = \alpha a$, $b' = \beta b$, $c' = \alpha^{-1}\beta c$, $d' = d$, $e' = \alpha\beta e$ then a', b', c', d', e' satisfy the same commutator relations as a, b, c, d, e and $pe' = b'a'$. Furthermore

$$pc' = \alpha^{-1}\beta kba = \alpha^{-2}kb'a'.$$

So we can take $k = 0, 1, \omega$ giving

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc, pd, pe - ba, \text{ class } 2\rangle, \quad (7.59)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd, pe - ba, \text{ class } 2\rangle, \quad (7.60)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc - \omega ba, pd, pe - ba, \text{ class } 2\rangle. \quad (7.60B)$$

Next consider the case when $pe = 0$, $pb \neq 0$. Scaling a and d we may assume that $pb = ba$. Then, if we let

$$\begin{aligned}
a' &= a, \\
b' &= \lambda b - 2\lambda\mu e, \\
c' &= \lambda\mu b + \lambda c - \lambda\mu^2 e, \\
d' &= \mu a + d, \\
e' &= \lambda e,
\end{aligned}$$

we can choose μ so that $pc' = 0$. So we suppose that $pc = pe = 0$, $pb = ba$. Then, if we let

$$\begin{aligned}
a' &= a + \beta b + \varepsilon e, \\
b' &= \lambda b, \\
c' &= \lambda c, \\
d' &= \varepsilon b + -\beta c + d, \\
e' &= \lambda e,
\end{aligned}$$

we can choose β and ε so that $pa' = pd' = 0$. This gives

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ba, pc, pd, pe, \text{ class } 2\rangle. \tag{7.61}$$

Now let $pb = pe = 0$, and suppose that $pc \neq 0$. As above, we may assume that $pc = ba$. If we let

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \lambda b, \\ c' &= \lambda c, \\ d' &= -\beta c + d, \\ e' &= \lambda e, \end{aligned}$$

we can choose β, γ so that $pa' = pd' = 0$. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd, pe, \text{ class } 2\rangle. \tag{7.62}$$

Now assume that $pb = pc = pe = 0$, and that $pa = ba$. Then letting

$$\begin{aligned} a' &= a, \\ b' &= b - 2\mu e, \\ c' &= \mu b + c - \mu^2 e, \\ d' &= \mu a + d, \\ e' &= e, \end{aligned}$$

we can choose μ so that $pd' = 0$. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ba, pb, pc, pd, pe, \text{ class } 2\rangle. \tag{7.63}$$

Finally, let $pa = pb = pc = pe = 0$, and let $pd = ba$. This gives

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc, pd - ba, pe, \text{ class } 2\rangle. \tag{7.64}$$

Now consider the case when pL has order p^2.

Case 1 Let L satisfy

$$da = ea = cb = db = eb = dc = ec = ed = 0.$$

If pd and pe are linearly independant then we may suppose that $pd = ba$, $pe = ca$, and that $pa = pb = pc = 0$. This gives

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed, pa, pb, pc, pd - ba, pe - ca, \text{ class } 2\rangle. \tag{7.65}$$

On the other hand, if pd and pe are linearly dependant then we may suppose that $pe = 0$. So the subalgebras generated by a, b, c, d is isomorphic to one of 6.22, 6.24 ˘ 6.28, 6.30 ˘ 6.32. So we have

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa - ca, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \tag{7.66}$$

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa - ba, pb - ca, pc, pd, pe, \text{ class } 2\rangle, \tag{7.67}$$

$$\langle a, b, c, d, e \mid cb, da, db, dc, ea, eb, ec, ed, pa, pb - ba, pc - \lambda ca, pd, pe, \text{ class } 2\rangle \tag{7.68}$$

with $\lambda \neq 0$, where λ and λ^{-1} give isomorphic algegbras ($(p+1)/2$ algebras),

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa,pb-ba-ca,pc-ca,pd,pe,\text{ class } 2\rangle, \tag{7.69}$$

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa,pb-\omega ca,pc-ba,pd,pe,\text{ class } 2\rangle, \tag{7.70}$$

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa,pb-\alpha ca,pc-ba-ca,pd,pe,\text{ class } 2\rangle \tag{7.71}$$

where $1+4\alpha$ is not a square ($(p-1)/2$ algebras),

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa-ba,pb,pc,pd-ca,pe,\text{ class } 2\rangle, \tag{7.72}$$

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa,pb-ba,pc,pd-ca,pe,\text{ class } 2\rangle, \tag{7.73}$$

$$\langle a,b,c,d,e \mid cb,da,db,dc,ea,eb,ec,ed,pa,pb,pc-ba,pd-ca,pe,\text{ class } 2\rangle. \tag{7.74}$$

Case 2 Next let L satisfy

$$da = ea = cb = eb = dc = ec = ed = 0,\ db = ba.$$

The generator e lies in the centre of L, and if $pe = 0$ then the subalgebra generated by a,b,c,d is isomorphic to one of 6.39 ˘ 6.47. This gives

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba,pb-ba,pc-ca,pd-ca,pe,\text{ class } 2\rangle, \tag{7.75}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba,pb-ba,pc,pd-ca,pe,\text{ class } 2\rangle, \tag{7.76}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba+ca,pb-ba,pc,pd-ca,pe,\text{ class } 2\rangle, \tag{7.77}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa+ca,pb-ba,pc,pd-ca,pe,\text{ class } 2\rangle, \tag{7.78}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa,pb-ba,pc,pd-ca,pe,\text{ class } 2\rangle. \tag{7.79}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba-ca,pb-ba-uca,pc,pd,pe,\text{ class } 2\rangle \tag{7.80}$$

($u \neq 0,1$, $p-2$ diœerent algebras),

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba-ca,pb-ba,pc,pd,pe,\text{ class } 2\rangle, \tag{7.81}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba-ca,pb-ca,pc,pd,pe,\text{ class } 2\rangle, \tag{7.82}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ba,pb-ca,pc,pd,pe,\text{ class } 2\rangle, \tag{7.83}$$

$$\langle a,b,c,d,e \mid cb,da,db-ba,dc,ea,eb,ec,ed,pa-ca,pb-ba,pc,pd,pe,\text{ class } 2\rangle. \tag{7.84}$$

Now consider the case when $pe \neq 0$. As in the calculation of 6.34 ˘ 6.38, we may assume that $pe = ba$ or $ba + ca$. From the computation of 6.34 ˘ 6.47 we see that if a', b', c', d', e' is any other set of generators of L satisfying these commutator relations then we either have

$$\begin{aligned}
a' &= \alpha a - \nu b + \beta c + (\alpha - \xi)d, \\
b' &= \gamma b + \delta d, \\
c' &= \lambda a + \mu c + \lambda d, \\
d' &= \nu b + \xi d
\end{aligned} \tag{(*)}$$

modulo $\langle e\rangle + L^2$, for some $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \xi$ with $\alpha\mu - \beta\lambda \neq 0$ and $\gamma\xi - \delta\nu \neq 0$, or we have

$$\begin{aligned} a' &= -\nu a + \alpha b - \xi c + \beta d, \\ b' &= \gamma a + \delta c + \gamma d, \\ c' &= \lambda b + \mu d, \\ d' &= \nu a + \xi c + \nu d \end{aligned} \tag{(**)}$$

modulo $\langle e\rangle + L^2$, for some $\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \xi$ with $\alpha\mu - \beta\lambda - \lambda\nu \neq 0$ and $\gamma\xi - \delta\nu \neq 0$. In the ørst of these

$$\begin{aligned} b'a' &= (\gamma\xi - \delta\nu)ba, \\ c'a' &= (\alpha\mu - \beta\lambda)ca, \end{aligned}$$

and in the second

$$\begin{aligned} b'a' &= (\gamma\xi - \delta\nu)ca, \\ c'a' &= (\alpha\mu - \beta\lambda - \lambda\nu)ba. \end{aligned}$$

Subtracting suitable scalar multiples of e from a, b, c, d we may suppose that pa, pb, pc, pd are all scalar multiples of ca.

First consider the case when $pe = ba$. To preserve the relation $pe = ba$ we can only consider a change of generating set of the form (*). If $pb = pd = 0$, then we may assume that $pa = \rho ca$, $pc = 0$ for some $\rho \neq 0$. Then replacing c by ρc we have $pa = ca$. This gives

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ca, pb, pc, pd, pe - ba, \text{ class } 2\rangle. \tag{7.85}$$

On the other hand, if one of pb, pd is non-zero then we may assume that $pb = 0$, $pd = \rho ca$ for some $\rho \neq 0$. Let $pa = \sigma ca$, and let $pc = \tau ca$, and let

$$\begin{aligned} a' &= \alpha a + \beta c + (\alpha - \xi)d, \\ b' &= \gamma b, \\ c' &= \lambda a + \mu c + \lambda d, \\ d' &= \xi d. \end{aligned}$$

Then

$$\begin{aligned} b'a' &= \gamma\xi ba, \\ c'a' &= (\alpha\mu - \beta\lambda)ca, \end{aligned}$$

and

$$\begin{aligned} pa' &= (\alpha\sigma + \beta\tau + (\alpha - \xi)\rho)ca, \\ pb' &= 0, \\ pc' &= (\lambda\sigma + \mu\tau + \lambda\rho)ca, \\ pd' &= \xi\rho ca. \end{aligned}$$

We can pick λ, μ so that $pc' = 0$, and so we can assume that $\tau = 0$. First consider the case when $\sigma = -\rho$. Then

$$pa' = -\xi\rho ca = -pd'$$

and by scaling we can assume that

$$pa' = -c'a' = -pd',$$

giving

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa + ca, pb, pc, pd - ca, pe - ba, \text{ class } 2\rangle. \quad (7.86)$$

On the other hand, of $\sigma \neq -\rho$ then we must have $\lambda = 0$ to ensure the relation $pc' = 0$, and then we require $\alpha\mu \neq 0$. Now

$$pa' = (\alpha(\rho + \sigma) - \xi\rho)ca$$

and we can choose ξ so that $pa' = 0$, and choose α, μ so that $pd' = c'a'$. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa, pb, pc, pd - ca, pe - ba, \text{ class } 2\rangle. \quad (7.87)$$

Now consider the case when $pe = ba = ca$. Using transformations of the form (*), as above, we see that L is isomorphic to one of the following algebras.

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa - ca, pb, pc, pd, pe - ba - ca, \text{ class } 2\rangle, \quad (7.88)$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa + ca, pb, pc, pd - ca, pe - ba - ca, \text{ class } 2\rangle,$$

$$\langle a, b, c, d, e \mid cb, da, db - ba, dc, ea, eb, ec, ed, pa, pb, pc, pd - ca, pe - ba - ca, \text{ class } 2\rangle. \quad (7.89)$$

However we must consider the eœect of transformations of the form (**) on these algebras. Let

$$\begin{aligned} a' &= -\nu a + \alpha b - \xi c + \beta d, \\ b' &= \gamma a + \delta c + \gamma d, \\ c' &= \lambda b + \mu d, \\ d' &= \nu a + \xi c + \nu d. \end{aligned}$$

Note that

$$\begin{aligned} b'a' &= (\gamma\xi - \delta\nu)ca, \\ c'a' &= (\alpha\mu - \beta\lambda - \lambda\nu)ba. \end{aligned}$$

If $pa = pb = pc = 0$, $pd = ca$ then to ensure that $pb' = pc' = 0$ we need $\gamma = \mu = 0$. Then

$$\begin{aligned} pa' &= \beta ca, \\ pd' &= \nu ca. \end{aligned}$$

But if $\gamma = 0$ then we need $\nu \neq 0$, so we cannot choose ν so that $pd' = 0$. Next consider the case when $pb = pc = 0$, $pa = -ca$, $pd = ca$. Then

$$\begin{aligned} pa' &= (\nu + \beta)ca, \\ pb' &= 0, \\ pc' &= \mu ca, \\ pd' &= 0. \end{aligned}$$

So we can assume that $pa \neq 0$, $pb = pc = pd = 0$, so that this algebra is isomorphic to 7.88.

Case 3 Now we consider the situation when

$$da = ea = cb = eb = dc = ec = ed = 0,\ db = ca.$$

Once again e is central, and so if $pe = 0$ then L is isomorphic to one of 6.53 ˘ 6.59. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ba, pb - ca, pc, pd, pe, \text{ class } 2\rangle, \quad (7.90)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ca, pb, pc - ba, pd, pe, \text{ class } 2\rangle, \quad (7.91)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb - ca, pc - ba, pd, pe, \text{ class } 2\rangle, \quad (7.92)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb - \omega ca, pc - ba, pd, pe, \text{ class } 2\rangle, \quad (7.93)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ba, pb, pc - ca, pd, pe, \text{ class } 2\rangle, \quad (7.94)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb - kba, pc - ca, pd, pe, \text{ class } 2\rangle \quad (7.95)$$

for $k = 1, 2, \ldots, p - 1$ ($p - 1$ algebras),

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc - ba, pd - ca, pe, \text{ class } 2\rangle. \quad (7.96)$$

So suppose that $pe \neq 0$. As we saw in the computation of 6.53 ˘ 6.59, if a', b', c', d', e' generate L and satisfy the same commutator relations as a, b, c, d, e then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \varepsilon a + \zeta b + \eta c + \theta d, \\ c' &= \lambda \zeta c - \lambda \varepsilon d, \\ d' &= -\lambda \beta c + \lambda \alpha d \end{aligned}$$

modulo $\langle e\rangle + L^2$ for some $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \lambda$ with $\alpha\zeta - \beta\varepsilon \neq 0$, $\lambda \neq 0$. If a', b', c', d' are as above then

$$\begin{aligned} b'a' &= (\alpha\zeta - \beta\varepsilon)ba + (\alpha\eta + \beta\theta - \gamma\varepsilon - \delta\zeta)ca, \\ c'a' &= \lambda(\alpha\zeta - \beta\varepsilon)ca. \end{aligned}$$

So we may assume that $pe = ba$ or ca. First consider the case when $pe = ba$. Subtracting suitable multiples of e from a, b, c, d we may suppose that pa, pb, pc, pd are scalar multiples of ca. If $pc = pd = 0$ then we can choose $\alpha, \beta, \varepsilon, \zeta$ so that $pa' \neq 0$, $pb' = 0$, and we can choose $\gamma, \delta, \eta, \theta$ so that $\alpha\eta + \beta\theta - \gamma\varepsilon - \delta\zeta = 0$. Then we can choose λ so that $pa' = c'a'$. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ca, pb, pc, pd, pe - ba, \text{ class } 2\rangle. \quad (7.97)$$

If pc, pd are not both zero then we can assume that $pc = \rho ca$ for some $\rho \neq 0$, $pd = 0$. We now consider generators a', b', c', d', e' for L where

$$\begin{aligned} a' &= \rho a + \gamma c + \rho \eta d, \\ b' &= b + \eta c, \\ c' &= c, \\ d' &= \rho d \end{aligned}$$

We can choose γ, η so that $pa' = pb' = 0$ so that we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc - ca, pd, pe - ba, \text{ class } 2\rangle. \quad (7.98)$$

Now consider the case when $pe = ca$. Subtracting suitable multiples of e from a, b, c, d we can assume that pa, pb, pc, pd are scalar multiples of ba. If $pc = pd = 0$ then we can assume that $pa = ba$, $pb = 0$. And if pc, pd are not both zero then we may assume that $pc = ba$, $pd = 0$, and we can assume that $pa = pb = 0$. So we have

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa - ba, pb, pc, pd, pe - ca, \text{ class } 2\rangle, \quad (7.99)$$

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc, ea, eb, ec, ed, pa, pb, pc - ba, pd, pe - ca, \text{ class } 2\rangle. \quad (7.100)$$

Case 4 Next let L satisfy

$$da = ea = cb = eb = ec = ed = 0, \ db = ca, \ dc = \omega ba.$$

As above, if $pe = 0$ then the subalgebra generated by a, b, c, d is isomorphic to one of 6.61 or 6.62.

$$\langle a, b, c, d, e \mid cb, da, db{-}ca, dc{-}\omega ba, ea, eb, ec, ed, pa, pb, pc{-}ba, pd{-}ca, pe, \text{ class } 2\rangle, \ (7.101)$$

and p algebras corresponding to the p equivalence classes of non-singular matrices $\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix}$ under the equivalence relation

$$\begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \sim \begin{pmatrix} \mu & \nu \\ \omega\nu & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \begin{pmatrix} \mu + \nu x & \nu y \\ \omega\nu y & \mu + \nu x \end{pmatrix}^{-1}.$$

Thus we have the following algebras (with x, y to correspond to representives of these equivalence classes)

$$\langle a, b, c, d, e \mid cb, da, db - ca, dc - \omega ba, ea, eb, ec, ed, pa, pb - ba, pc - xba - yca, pd, pe, \text{ class } 2\rangle. \quad (7.102)$$

Now let $pe \neq 0$. By subtracting suitable multiples of e from a, b, c, d we may assume that pa, pb, pc, pd span a space of dimension 1. As in the computation of 6.60B we can assume that $pa \neq 0$, $pb = pc = pd = 0$. From the computation of 6.60 ˘ 6.62, we see that if we let

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d, \\ b' &= \lambda a + \mu b + \nu c + \xi d, \\ c' &= -\omega\xi a + \omega\nu b + \mu c - \lambda d, \\ d' &= \omega\delta a - \omega\gamma b - \beta c + \alpha d \end{aligned}$$

then a', b', c', d' satisfy the same commutator relations as a, b, c, d. Furthermore, we can take a' to be any non-zero element in the span of a, b, c, d, and (having øxed a') we can take b' to be any element outside the centralizer of a'. This øxes c', d' up to change of sign. If a', b', c', d' have this form and if $pb' = pc' = pd' = 0$ then we have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \mu b + \nu c, \\ c' &= \omega\nu b + \mu c, \\ d' &= -\omega\gamma b - \beta c + \alpha d. \end{aligned}$$

We can choose μ, ν so that $pe = b'a'$, and then subtracting a suitable scalar multiple of e from a' we can assume that $pa' = c'a'$. So we have

$$\langle a, b, c, d, e \mid cb, da, db{-}ca, dc{-}\omega ba, ea, eb, ec, ed, pa{-}ca, pb, pc, pd, pe{-}ba, \text{ class } 2\rangle. \ (7.103)$$

Case 5 Now let L satisfy

$$da = ea = cb = db = eb = dc = ec = 0,\ ed = ba.$$

As we showed above, if a', b', c', d', e' generate L and satisfy the same commutator relations as a, b, c, d, e then

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c + \delta d + \varepsilon e, \\
b' &= \alpha^{-1}(\lambda\xi - \mu\nu)b, \\
c' &= \zeta b + \eta c, \\
d' &= -\alpha^{-1}(\delta\mu - \varepsilon\lambda)b + \lambda d + \mu e, \\
e' &= -\alpha^{-1}(\delta\xi - \varepsilon\nu)b + \nu d + \xi e,
\end{aligned}$$

and

$$\begin{aligned}
b'a' &= (\lambda\xi - \mu\nu)ba, \\
c'a' &= \alpha\zeta ba + \alpha\eta ca.
\end{aligned}$$

First consider the case when $pb = 0$. We can assume that $pc = 0$ or ba or ca. If pd, pe are linearly independant then we may suppose that $pd = \rho ba$, $pe = \sigma ca$ for some $\rho, \sigma \neq 0$, and we may suppose that $pa = 0$. Now let

$$\begin{aligned}
a' &= \alpha a, \\
b' &= \alpha^{-1}\lambda\xi b, \\
c' &= \eta c, \\
d' &= \lambda d, \\
e' &= \xi e.
\end{aligned}$$

Then

$$\begin{aligned}
b'a' &= \lambda\xi ba, \\
c'a' &= \alpha\eta ca,
\end{aligned}$$

and

$$\begin{aligned}
pc' &= 0 \text{ or } \eta\lambda^{-1}\xi^{-1}b'a' \text{ or } \alpha^{-1}c'a' \\
pd' &= \lambda pd = \lambda\rho ba = \rho\xi^{-1}b'a', \\
pe' &= \xi pe = \xi\sigma ca = \alpha^{-1}\eta^{-1}\xi\sigma c'a'.
\end{aligned}$$

We can choose $\alpha, \eta, \lambda, \xi$ so that $pd' = b'a'$, $pe' = c'a'$ and $pc' = 0$ or $b'a'$ or $c'a'$. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc, pd - ba, pe - ca, \text{ class } 2\rangle, \quad (7.104)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd - ba, pe - ca, \text{ class } 2\rangle, \quad (7.105)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb, pc - ca, pd - ba, pe - ca, \text{ class } 2\rangle. \quad (7.106)$$

Next suppose that $pb = 0$ and that pd, pe generate a subalgebra of order p. As above we may suppose that $pc = 0$ or ba or ca. We may assume that $pe = 0$. If $pc = 0$ then we may assume that $pd = \rho ba$ and $pa = \sigma ca$ or that $pd = \rho ca$ and $pa = \sigma ba$. By scaling we can take $\rho = \sigma = 1$ in both cases, and so we have

$$\langle a, b, c, d, e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa - ca, pb, pc, pd - ba, pe, \text{ class } 2\rangle, \quad (7.107)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb,pc,pd-ca,pe,\text{ class }2\rangle. \quad (7.108)$$

If $pc = ba$ then we may assume that $pd = \rho ba$ or ρca. Let $pc = ba$, $pd = \rho ba$. Then we may assume that $pa = \sigma ca$. By scaling we can take $\rho = \sigma = 1$ and so we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ca,pb,pc-ba,pd-ba,pe,\text{ class }2\rangle. \quad (7.109)$$

On the other hand if $pc = ba$, $pd = \rho ca$ then we can assume that $pa = 0$, and by scaling we can take $\rho = \sigma = 1$. So we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb,pc-ba,pd-ca,pe,\text{ class }2\rangle. \quad (7.110)$$

Now let $pc = ca$. If $pd = \rho ca$ then we can assume that $pa = \sigma ba$, and by scaling we can take $\rho = \sigma = 1$ and so we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb,pc-ca,pd-ca,pe,\text{ class }2\rangle. \quad (7.111)$$

And if $pc = ca$, and pc, pd are linearly independant then we may assume that $pa = 0$. Let $pd = rba + sca$ (with $r \neq 0$). If we let

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^{-1}\lambda\xi b, \\ c' &= \eta c, \\ d' &= \lambda d, \\ e' &= \xi e, \end{aligned}$$

then

$$\begin{aligned} b'a' &= \lambda\xi ba, \\ c'a' &= \alpha\eta ca. \end{aligned}$$

To ensure that $pc' = c'a'$ we require $\alpha = 1$. Then

$$pd' = \xi rba + \xi sca = \lambda^{-1} rb'a' + \eta^{-1}\xi sc'a'.$$

So we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb,pc-ca,pd-ba,pe,\text{ class }2\rangle, \quad (7.112)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb,pc-ca,pd-ba-ca,pe,\text{ class }2\rangle. \quad (7.113)$$

Next let $pb = pd = pe = 0$. Then we must have $pc \neq 0$ and we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ca,pb,pc-ba,pd,pe,\text{ class }2\rangle, \quad (7.113B)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb,pc-ca,pd,pe,\text{ class }2\rangle. \quad (7.114)$$

We now consider the situation when $pb \neq 0$. We can assume that $pb = ba$ or ca. First let $pb = ba$. If we let

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + \delta d + \varepsilon e, \\ b' &= \alpha^{-1}(\lambda\xi - \mu\nu)b, \\ c' &= \zeta b + \eta c, \\ d' &= -\alpha^{-1}(\delta\mu - \varepsilon\lambda)b + \lambda d + \mu e, \\ e' &= -\alpha^{-1}(\delta\xi - \varepsilon\nu)b + \nu d + \xi e, \end{aligned}$$

then

$$\begin{aligned} b'a' &= (\lambda\xi - \mu\nu)ba, \\ c'a' &= \alpha\zeta ba + \alpha\eta ca. \end{aligned}$$

To ensure that $pb' = b'a'$ we require $\alpha = 1$. Taking $\lambda = \xi = 1$, and $\mu = \nu = 0$ we have

$$\begin{aligned} a' &= a + \beta b + \gamma c + \delta d + \varepsilon e, \\ b' &= b, \\ c' &= \zeta b + \eta c, \\ d' &= \varepsilon b + d, \\ e' &= -\delta b + e, \end{aligned}$$

and

$$\begin{aligned} b'a' &= ba, \\ c'a' &= \zeta ba + \eta ca. \end{aligned}$$

We can assume that pa', pd', pe' are all scalar multiples of ca. Let $pc = rba + sca$, and consider pc'.

$$pc' = \zeta ba + \eta rba + \eta sca = (\zeta + \eta r - \zeta s)b'a' + sc'a',$$

so we can assume that $pc' = sc'a'$ or $b'a' + c'a'$. So we may assume that $pb = ba$ and that pa, pc, pd, pe are all scalar multiples of ca, or we can assume that $pa = 0$, $pb = ba$, $pc = ba + ca$, and that pd, pe are scalar multiples of ca.

Let $pb = ba$, and let pa, pc, pd, pe all be scalar multiples of ca. Replacing a, b, c, d, e by a', b', c', d', e' of the form

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= (\lambda\xi - \mu\nu)b, \\ c' &= \eta c, \\ d' &= \lambda d + \mu e, \\ e' &= \nu d + \xi e, \end{aligned}$$

we have

$$\begin{aligned} b'a' &= (\lambda\xi - \mu\nu)ba, \\ c'a' &= \eta ca, \end{aligned}$$

and so we can suppose that $pe = 0$, $pd = 0$ or ca, $pc = sca$, $pb = ba$, $pa = 0$ or ca. Furthermore, if $s \neq 0$ we can assume that $pa = 0$. Also, one of pa, pc, pd must be non-zero. So we have $2(p-1)$ algebras

$$\langle a,b,c,d,e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ba, pc - sca, pd, pe, \text{ class } 2\rangle, \quad (7.115)$$

$$\langle a,b,c,d,e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ba, pc - sca, pd - ca, pe, \text{ class } 2\rangle, \quad (7.116)$$

with $0 < s < p$, and

$$\langle a,b,c,d,e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa, pb - ba, pc, pd - ca, pe, \text{ class } 2\rangle, \quad (7.117)$$

$$\langle a,b,c,d,e \mid da, ea, cb, db, eb, dc, ec, ed - ba, pa - ca, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \quad (7.118)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ca,pb-ba,pc,pd-ca,pe,\text{ class }2\rangle. \quad (7.119)$$

Finally, let $pa=0$, $pb=ba$, $pc=ba+ca$, and let pd,pe be scalar multiples of ca. As above we can assume that $pe=0$ and that $pd=0$ or ca. So we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ba,pc-ba-ca,pd,pe,\text{ class }2\rangle, \quad (7.120)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ba,pc-ba-ca,pd-ca,pe,\text{ class }2\rangle. \quad (7.121)$$

Now let $pb=ca$ and consider replacing a,b,c,d,e by a',b',c',d',e' where

$$\begin{aligned}
a' &= \alpha a+\beta b+\gamma c+\delta d+\varepsilon e,\\
b' &= \alpha^{-1}(\lambda\xi-\mu\nu)b,\\
c' &= \zeta b+\eta c,\\
d' &= -\alpha^{-1}(\delta\mu-\varepsilon\lambda)b+\lambda d+\mu e,\\
e' &= -\alpha^{-1}(\delta\xi-\varepsilon\nu)b+\nu d+\xi e.
\end{aligned}$$

To preserve the relation $pb=ca$ we need $\zeta=0$ and $\alpha^{-1}(\lambda\xi-\mu\nu)=\alpha\eta$. Using a similar argument to that given above, we may assume that $pe=0$ and that $pd=0$ or ba. Let $pc=rba+sca$. If we let

$$\begin{aligned}
a' &= \alpha a+\beta b+\gamma c,\\
b' &= \alpha^{-1}\lambda\xi b,\\
c' &= \alpha^{-2}\lambda\xi c,\\
d' &= \lambda d,\\
e' &= \xi e,
\end{aligned}$$

then

$$pc'=\alpha^{-2}\lambda\xi rba+\alpha^{-2}\lambda\xi sca=\alpha^{-2}rb'a'+\alpha^{-1}sc'a'.$$

We also have $pd'=0$ or $\xi^{-1}b'a'$. So we can take $pc=0, ba$ or ωba, or $rba+ca$ $(0\le r<p)$, while preserving the relation $pd=0$ or ba. Furthermore, we can take $pa=0$ unless $pc=0$ or ca, and in those to cases we can take $pa=0$ or ba. So we have

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb-ca,pc,pd,pe,\text{ class }2\rangle, \quad (7.122)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc,pd-ba,pe,\text{ class }2\rangle, \quad (7.123)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb-ca,pc,pd-ba,pe,\text{ class }2\rangle, \quad (7.124)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb-ca,pc-ca,pd,pe,\text{ class }2\rangle, \quad (7.125)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-ca,pd-ba,pe,\text{ class }2\rangle, \quad (7.126)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa-ba,pb-ca,pc-ca,pd-ba,pe,\text{ class }2\rangle, \quad (7.127)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-ba,pd,pe,\text{ class }2\rangle, \quad (7.128)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-ba,pd-ba,pe,\text{ class }2\rangle, \quad (7.129)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-\omega ba,pd,pe,\text{ class }2\rangle, \quad (7.130)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-\omega ba,pd-ba,pe,\text{ class }2\rangle, \quad (7.131)$$

and $2(p-1)$ algebras

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-rba-ca,pd,pe,\text{ class }2\rangle, \quad (7.132)$$

$$\langle a,b,c,d,e \mid da,ea,cb,db,eb,dc,ec,ed-ba,pa,pb-ca,pc-rba-ca,pd-ba,pe,\text{ class }2\rangle, \quad (7.133)$$

with $0<r<p$.

Case 6 Finally consider the case when L satisøes

$$da = ea = cb = eb = dc = ec = 0,\ db = ca,\ ed = ba.$$

As above, if a', b', c', d', e' generate L and satisfy these commutator relations then (modulo L^2) b', c', e' span the same space as b, c, e. Using this fact, we can show that

$$\begin{aligned}
a' &= \alpha a + (\alpha\eta - \gamma\varepsilon)^{-1}(\alpha\beta\varepsilon + \gamma\delta\eta - \alpha^2\zeta - \gamma^2\theta)b + \beta c + \gamma d + \delta e, \\
b' &= \lambda((\alpha\eta + \gamma\varepsilon)b + 2\gamma\eta c - 2\alpha\varepsilon e), \\
c' &= \lambda(\varepsilon\eta b + \eta^2 c - \varepsilon^2 e), \\
d' &= \varepsilon a + (\alpha\eta - \gamma\varepsilon)^{-1}(-\alpha\varepsilon\zeta - \gamma\eta\theta + \beta\varepsilon^2 + \delta\eta^2)b + \zeta c + \eta d + \theta e, \\
e' &= \lambda(-\alpha\gamma b - \gamma^2 c + \alpha^2 e)
\end{aligned}$$

for some (arbitrary) $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \lambda$ with $\alpha\eta - \gamma\varepsilon \neq 0$ and $\lambda \neq 0$. Note that, up to the scalar multiple λ, b', c', e' are determined by $\alpha, \gamma, \varepsilon, \eta$. We then have

$$\begin{aligned}
b'a' &= \lambda\alpha(\alpha\eta - \gamma\varepsilon)ba + \lambda\gamma(\alpha\eta - \gamma\varepsilon)ca, \\
c'a' &= \lambda\varepsilon(\alpha\eta - \gamma\varepsilon)ba + \lambda\eta(\alpha\eta - \gamma\varepsilon)ca.
\end{aligned}$$

First consider the case when $pb = pc = pe = 0$. Then pa, pd must be linearly independant. Let

$$\begin{pmatrix} pa \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix},$$

where A is a non-singular matrix over $\mathbb{Z}_p$. Then

$$\begin{pmatrix} pa' \\ pd' \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \lambda^{-1}(\alpha\eta - \gamma\varepsilon)^{-1} \begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix} A \begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix}.$$

By Theorem 6, we can take A to be one of the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} (\lambda \neq 0),\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},$$

or to a matrix of the form

$$\begin{pmatrix} 0 & \lambda \\ 1 & 1 \end{pmatrix},$$

where $x^2 - x - \lambda$ is irreducible. Furthermore none of these matrices give isomorphic algebras, except that if $\lambda \neq 0$ then the algebras determined by $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ are isomorphic. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa-ba, pb, pc, pd-\lambda ca, pe, \text{ class } 2\rangle, \quad (7.134)$$

($\lambda \neq 0$, λ and λ^{-1} give isomorphic algebras),

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ba - ca, pb, pc, pd - ca, pe, \text{ class } 2\rangle, \quad (7.135)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa-\omega ca, pb, pc, pd-ba, pe, \text{ class } 2\rangle, \quad (7.136)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - \lambda ca, pb, pc, pd - ba - ca, pe, \text{ class } 2\rangle, \quad (7.137)$$

where $1 + 4\lambda$ is not a square.

Next suppose that pb, pc, pe span a space of dimension 1. We can suppose that this subspace is spanned by ba. If we let a', b', c', d', e' generate L and satisfy the same commutator relations as a, b, c, d, e, and if we let $b'a'$ be a scalar multiple of ba then

$$\begin{aligned}
a' &= \alpha a + (\alpha\eta)^{-1}(\alpha\beta\varepsilon - \alpha^2\zeta)b + \beta c + \delta e, \\
b' &= \lambda(\alpha\eta b - 2\alpha\varepsilon e), \\
c' &= \lambda(\varepsilon\eta b + \eta^2 c - \varepsilon^2 e), \\
d' &= \varepsilon a + (\alpha\eta)^{-1}(-\alpha\varepsilon\zeta + \beta\varepsilon^2 + \delta\eta^2)b + \zeta c + \eta d + \theta e, \\
e' &= \lambda\alpha^2 e
\end{aligned}$$

where $\alpha\eta\lambda \neq 0$. We then have

$$\begin{aligned}
b'a' &= \lambda\alpha^2\eta ba, \\
c'a' &= \lambda\varepsilon\alpha\eta ba + \lambda\alpha\eta^2 ca.
\end{aligned}$$

First consider the case when $pe \neq 0$. We can assume that $pe = ba$, and $pb = 0$, and then to ensure that $pe' = b'a'$, $pb' = 0$, we require

$$\begin{aligned}
a' &= \alpha a + -\alpha\zeta b + \beta c + \delta e, \\
b' &= \lambda\alpha b, \\
c' &= \lambda c, \\
d' &= \alpha^{-1}\delta b + \zeta c + d + \theta e, \\
e' &= \lambda\alpha^2 e,
\end{aligned}$$

which gives

$$\begin{aligned}
b'a' &= \lambda\alpha^2 ba, \\
c'a' &= \lambda\alpha ca.
\end{aligned}$$

Clearly we can assume that $pc = 0$, ba or ωba. (If $pc \neq 0$ then we require $\alpha = \pm 1$.) We can then assume that pa and pd are scalar multiples of ca. If $pd \neq 0$ we can take $pd = ca$ (which ıøxes $\lambda\alpha$ȷ). And if $pd = 0$ we can take $pa = ca$. If $pd = ca$ and $pc = 0$ then we can take $pa = 0$ or ca. And if $pd = ca$ and $pc \neq 0$ then we can take $pa = 0$ or $pa = kca$ where $1 \leq k \leq (p-1)/2$ (k and $-k$ give isomorphic algebras. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb, pc, pd, pe - ba, \text{ class } 2\rangle, \quad (7.138)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc, pd - ca, pe - ba, \text{ class } 2\rangle, \quad (7.139)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb, pc, pd - ca, pe - ba, \text{ class } 2\rangle, \quad (7.140)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb, pc - ba, pd, pe - ba, \text{ class } 2\rangle, \quad (7.141)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - kca, pb, pc - ba, pd - ca, pe - ba, \text{ class } 2\rangle, \quad (7.142)$$

where $0 \leq k \leq (p-1)/2$ with k and $-k$ giving isomorphic algebras ($(p+1)/2$ algebras),

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb, pc - \omega ba, pd, pe - ba, \text{ class } 2\rangle, \quad (7.143)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - kca, pb, pc - \omega ba, pd - ca, pe - ba, \text{ class } 2\rangle, \quad (7.144)$$

where $0 \leq k \leq (p-1)/2$ with k and $-k$ giving isomorphic algebras ($(p+1)/2$ algebras).

Next, consider the case when $pe = 0$ and $pb \neq 0$. We may assume that $pb = ba$ and $pc = 0$. If $pb' = b'a'$ and $pc' = 0$ then

$$\begin{aligned}
a' &= a - \eta^{-1}\zeta b + \beta c + \delta e, \\
b' &= \lambda\eta b, \\
c' &= \lambda\eta^2 c, \\
d' &= \delta\eta b + \zeta c + \eta d + \theta e, \\
e' &= \lambda e
\end{aligned}$$

where $\eta\lambda \neq 0$. We then have

$$\begin{aligned}
b'a' &= \lambda\eta ba, \\
c'a' &= \lambda\eta^2 ca.
\end{aligned}$$

We can then assume that pa and pd are both equal to 0 or ca. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ba, pc, pd - ca, pe, \text{ class } 2\rangle, \quad (7.145)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb - ba, pc, pd, pe, \text{ class } 2\rangle, \quad (7.146)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb - ba, pc, pd - ca, pe, \text{ class } 2\rangle. \quad (7.147)$$

Next, suppose that $pb = pd = 0$ and that $pc \neq 0$. We can assume that $pc = ba$ (ıwhich forces $\eta = \alpha^2$ȷ) and that pa and pd are both scalar multiples of ca. We now let

$$\begin{aligned}
a' &= \alpha a, \\
b' &= \lambda(\alpha^3 b - 2\alpha\varepsilon e), \\
c' &= \lambda(\varepsilon\alpha^2 b + \alpha^4 c - \varepsilon^2 e), \\
d' &= \varepsilon a + \alpha^2 d, \\
e' &= \lambda\alpha^2 e
\end{aligned}$$

which gives

$$\begin{aligned}
b'a' &= \lambda\alpha^4 ba, \\
c'a' &= \lambda\varepsilon\alpha^3 ba + \lambda\alpha^5 ca.
\end{aligned}$$

Replacing a, b, c, d, e by suitable a', b', c', d', e' of this form we may assume that one of pa, pd is zero, and the other equals ca. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb, pc - ba, pd - ca, pe, \text{ class } 2\rangle, \quad (7.148)$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa - ca, pb, pc - ba, pd, pe, \text{ class } 2\rangle. \quad (7.149)$$

Finally, consider the case when pb, pc, pe span a space of dimension 2. Once again we consider replacing a, b, c, d, e by a', b', c', d', e' where

$$\begin{aligned}
a' &= \alpha a + (\alpha\eta - \gamma\varepsilon)^{-1}(\alpha\beta\varepsilon + \gamma\delta\eta - \alpha^2\zeta - \gamma^2\theta)b + \beta c + \gamma d + \delta e, \\
b' &= \lambda((\alpha\eta + \gamma\varepsilon)b + 2\gamma\eta c - 2\alpha\varepsilon e), \\
c' &= \lambda(\varepsilon\eta b + \eta^2 c - \varepsilon^2 e), \\
d' &= \varepsilon a + (\alpha\eta - \gamma\varepsilon)^{-1}(-\alpha\varepsilon\zeta - \gamma\eta\theta + \beta\varepsilon^2 + \delta\eta^2)b + \zeta c + \eta d + \theta e, \\
e' &= \lambda(-\alpha\gamma b - \gamma^2 c + \alpha^2 e).
\end{aligned}$$

If $pb = 0$ then pc, pe must be linearly independant, and we can suppose that $pa = pd = 0$. If $pa' = pb' = pd' = 0$ then $\gamma\eta = \alpha\varepsilon = 0$. So

$$\begin{array}{rcl} a' & = & \alpha a, \\ b' & = & \lambda\alpha\eta b, \\ c' & = & \lambda\eta^2 c, \\ d' & = & \eta d, \\ e' & = & \lambda\alpha^2 e, \end{array}$$

and

$$\begin{array}{rcl} b'a' & = & \lambda\alpha^2\eta ba, \\ c'a' & = & \lambda\alpha\eta^2, \end{array}$$

or

$$\begin{array}{rcl} a' & = & \gamma d, \\ b' & = & \lambda\gamma\varepsilon b, \\ c' & = & -\lambda\varepsilon^2 e, \\ d' & = & \varepsilon a, \\ e' & = & -\lambda\gamma^2 c, \end{array}$$

and

$$\begin{array}{rcl} b'a' & = & -\lambda\gamma^2\varepsilon ca, \\ c'a' & = & -\lambda\gamma\varepsilon^2 ba. \end{array}$$

Let

$$\begin{pmatrix} pc \\ pe \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix},$$

where A is a non-singular 2×2 matrix. Then

$$\begin{pmatrix} pc' \\ pe' \end{pmatrix} = \begin{pmatrix} \lambda\eta^2 & 0 \\ 0 & \lambda\alpha^2 \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \begin{pmatrix} \lambda\eta^2 & 0 \\ 0 & \lambda\alpha^2 \end{pmatrix} A \begin{pmatrix} \lambda\alpha^2\eta & 0 \\ 0 & \lambda\alpha\eta^2 \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix},$$

or

$$\begin{pmatrix} pc' \\ pe' \end{pmatrix} = \begin{pmatrix} 0 & -\lambda\varepsilon^2 \\ -\lambda\gamma^2 & 0 \end{pmatrix} A \begin{pmatrix} ba \\ ca \end{pmatrix} = \begin{pmatrix} 0 & -\lambda\varepsilon^2 \\ -\lambda\gamma^2 & 0 \end{pmatrix} A \begin{pmatrix} 0 & -\lambda\gamma^2\varepsilon \\ -\lambda\gamma\varepsilon^2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} b'a' \\ c'a' \end{pmatrix},$$

So the matrix A can be transformed to

$$\begin{pmatrix} \alpha^{-1}\eta & 0 \\ 0 & \alpha\eta^{-1} \end{pmatrix} A \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \eta^{-1} \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & \varepsilon^2 \\ \gamma^2 & 0 \end{pmatrix} A \begin{pmatrix} 0 & \gamma^2\varepsilon \\ \gamma\varepsilon^2 & 0 \end{pmatrix}^{-1}.$$

If $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ then

$$\begin{pmatrix} \alpha^{-1}\eta & 0 \\ 0 & \alpha\eta^{-1} \end{pmatrix} A \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \eta^{-1} \end{pmatrix} = \begin{pmatrix} \eta\frac{r}{\alpha^2} & \frac{s}{\alpha} \\ \frac{t}{\eta} & \alpha\frac{u}{\eta^2} \end{pmatrix}$$

and

$$\left(\begin{array}{cc} 0 & \varepsilon^2 \\ \gamma^2 & 0 \end{array}\right) A \left(\begin{array}{cc} 0 & \gamma^2\varepsilon \\ \gamma\varepsilon^2 & 0 \end{array}\right)^{-1} = \left(\begin{array}{cc} \varepsilon\frac{u}{\gamma^2} & \frac{t}{\gamma} \\ \frac{s}{\varepsilon} & \gamma\frac{r}{\varepsilon^2} \end{array}\right)$$

So we can take A to be one of the following matrices

$$\left(\begin{array}{cc} r & 1 \\ 1 & s \end{array}\right) (rs \neq 1,\, r \leq s),\ \left(\begin{array}{cc} r & 0 \\ 1 & 1 \end{array}\right) (r \neq 0),\ \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),$$

or (when $p = 1 \bmod 3$)

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & \omega \end{array}\right).$$

So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb, pc-rba-ca, pd, pe-ba-sca, \text{class } 2\rangle \tag{7.150}$$

($0 \leq r \leq s < p$, $rs \neq 1 \bmod p$, $(p^2-1)/2$ algebras),

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb, pc-rba, pd, pe-ba-ca, \text{class } 2\rangle \tag{7.151}$$

($r \neq 0$, $p-1$ algebras),

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb, pc-ba, pd, pe-ca, \text{class } 2\rangle, \tag{7.152}$$

and when $p = 1 \bmod 3$ we have one extra algebra

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb, pc-ba, pd, pe-\omega ca, \text{class } 2\rangle. \tag{7.153}$$

Now suppose that $pb \neq 0$, and that $pb' \neq 0$ for all b' of the form above.

Let

$$\left(\begin{array}{c} pb \\ pc \\ pe \end{array}\right) = A \left(\begin{array}{c} ba \\ ca \end{array}\right)$$

where A is a 3×2 matrix of rank 2. Then if a', b', c', d', e' are as above, then

$$\left(\begin{array}{c} pb' \\ pc' \\ pe' \end{array}\right) = (\alpha\eta - \gamma\varepsilon)^{-2} \left(\begin{array}{lll} (\alpha\eta+\gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{array}\right) A \left(\begin{array}{cc} \eta & -\gamma \\ -\varepsilon & \alpha \end{array}\right) \left(\begin{array}{c} b'a' \\ c'a' \end{array}\right).$$

We can take $pb = ca$, and so we have

$$(\alpha\eta - \gamma\varepsilon)^{-2} \left(\begin{array}{lll} (\alpha\eta+\gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{array}\right) \left(\begin{array}{cc} 0 & 1 \\ w & x \\ y & z \end{array}\right) \left(\begin{array}{cc} \eta & -\gamma \\ -\varepsilon & \alpha \end{array}\right)$$

If $(wz-xy)^2 - wy$ is a non-zero square then we can take $pb = 0$, and we are back in a previous case. And if $(wz-xy)^2 - wy \neq 0$ we can take $pa = pd = 0$.

So consider the case when $(wz-xy)^2 - wy$ is a not a square. We consider the orbits of matrices

$$\left(\begin{array}{cc} u & v \\ w & x \\ y & z \end{array}\right)$$

where $(wz - xy)^2 - (ux - vw)(uz - vy)$ is not a square under the transformations given above. Each such orbit contains a matrix with $u = 0$ and $v = 1$, and we pick one matrix of this form out of each orbit, giving k algebras

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc-wba-xca, pd, pe-yba-zca, \text{ class } 2\rangle, \tag{7.154}$$

where $k = 4$ when $p = 3$, $k = (p^2 - 1)/2$ when $p = 1 \bmod 3$, and $k = (p^2 + 1)/2$ when $p = 2 \bmod 3$. (See Appendix I.)

Finally we consider the orbits of matrices

$$\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix}$$

where $(wz - xy)^2 - (ux - vw)(uz - vy) = 0$. There are $2p + 1$ orbits with representatives

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ y & 0 \end{pmatrix} (0 < y < p), \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \omega & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ y & 1 \\ 0 & 0 \end{pmatrix} (0 < y < p).$$

In the case when $pc = 0$ we can take $pa = 0$, $pd = 0$ or ca, and in the case when $pe = 0$ we can take $pa = 0$, $pd = 0$ or ba. So we have

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc, pd, pe-yba, \text{ class } 2\rangle \tag{7.155}$$

$(0 < y < p)$,

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ca, pc, pd - ca, pe - yba, \text{ class } 2\rangle \tag{7.156}$$

$(0 < y < p)$,

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ca, pc, pd, pe + ba - ca, \text{ class } 2\rangle, \tag{7.157}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc, pd-ca, pe+ba-ca, \text{ class } 2\rangle, \tag{7.158}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc-ba, pd, pe, \text{ class } 2\rangle, \tag{7.159}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ca, pc - ba, pd - ba, pe, \text{ class } 2\rangle, \tag{7.160}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc-\omega ba, pd, pe, \text{ class } 2\rangle, \tag{7.161}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ca, pc - \omega ba, pd - ba, pe, \text{ class } 2\rangle, \tag{7.162}$$

$$\langle a, b, c, d, e \mid da, ea, cb, db - ca, eb, dc, ec, ed - ba, pa, pb - ca, pc - yba - ca, pd, pe, \text{ class } 2\rangle \tag{7.163}$$

$(0 < y < p)$,

$$\langle a, b, c, d, e \mid da, ea, cb, db-ca, eb, dc, ec, ed-ba, pa, pb-ca, pc-yba-ca, pd-ba, pe, \text{ class } 2\rangle \tag{7.164}$$

$(0 < y < p)$.

19 Grandchildren of algebra 1

Algebra 1 is the cyclic Lie ring of order p^5, and its only grandchild of order p^7 is the cyclic one.

20 Grandchildren of algebra 2 (4.6)

All the descendants of 4.6 of order p^6 are terminal, except for 6.366 and 6.367.

20.1 Descendants of 6.366

Algebra 6.366 has presentation

$$\langle a, b \mid ba, \text{ class } 3 \rangle,$$

and so if L is a descendant of 6.366 of order p^7 then L_2 is generated modulo L_3 by pa, pb, L_3 is generated modulo L_4 by p^2a, p^2b and L_4 has order p and is generated by p^3a, p^3b. Furthermore $ba \in L_4$. We can assume that $p^3b = 0$ and that L_4 is generated by p^3a, and so we have

$$\langle a, b \mid ba, p^3b, \text{ class } 4 \rangle,$$

$$\langle a, b \mid ba - p^3a, p^3b, \text{ class } 4 \rangle.$$

20.2 Descendants of 6.367

Algebra 6.367 has presentation

$$\langle a, b \mid ba - p^2a, \text{ class } 3 \rangle,$$

and so if L is a descendant of 6.367 of order p^7 then L_2 is generated modulo L_3 by pa, pb, L_3 is generated modulo L_4 by p^2a, p^2b and L_4 has order p and is generated by p^3a, p^3b. Furthermore $ba - p^2a \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_2 then $a' = \alpha a + c$, $b' = \beta a + b + d$ for some α, β and some (arbitrary) c, d in L_2. So we have $p^3a' = \alpha p^3a$ and $p^3b' = \beta p^3a + p^3b$. So we can either assume that L_4 is generated by p^3a and that $p^3b = 0$, or we can assume that L_4 is generated by p^3b and that $p^3a = 0$.

Consider the case when $p^3a = 0$. Then if we let $a' = a + \lambda pb$, $b' = b$ we have

$$b'a' = ba,\ p^2a' = p^2a + \lambda p^3b$$

so that $b'a' - p^2a' = ba - p^2a - \lambda p^3b$.So we can assume that $ba = p^2a$, giving

$$\langle a, b \mid ba - p^2a, p^3a, \text{ class } 4 \rangle.$$

Now consider the case when $p^3b = 0$. We have $pba = p^3a$ and so adding a suitable scalar multiple of pb to b we can assume that $ba = p^2a$, giving

$$\langle a, b \mid ba - p^2a, p^3b, \text{ class } 4 \rangle.$$

21 Grandchildren of algebra 3 (4.7)

Algebra 4.7 has 16 capable descendants of order p^6: 6.368 ˘ 6.383.

21.1 Descendants of 6.368

Algebra 6.368 has $9 + 6\gcd(p-1, 3)$ descendants of order p^7.

Algebra 6.368 has presentation

$$\langle a, b \mid p^2a,\ pb, \text{ class } 3 \rangle,$$

and so if L is a descendant of 6.368 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$, $baab$ and $babb$. Furthermore p^2a and $pb \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_2 then $a' = \alpha a + \beta b + c$, $b' = \gamma b + d$ for some α, β, γ and some (arbitrary) c, d in L_2, and

$$\begin{aligned} b'a'a'a' &= \alpha^3\gamma baaa + 2\alpha^2\beta\gamma baab + \alpha\beta^2\gamma babb, \\ b'a'a'b' &= \alpha^2\gamma^2 baab + \alpha\beta\gamma^2 babb, \\ b'a'b'b' &= \alpha\gamma^3 babb. \end{aligned}$$

So if $babb \neq 0$ we can assume that $baab = 0$ (though we then need $\beta = 0$), and we can either assume that $baaa = 0$ or that $baaa \neq 0$ and that $babb = baaa$ or $\omega baaa$. If $babb = 0$ and $baab \neq 0$ then we can assume that $baaa = 0$ (though we then again need $\beta = 0$). The ønal possibility is that $baab = babb = 0$ and that L_4 is generated by $baaa$.

21.1.1 $baaa = baab = 0$

If $baaa = babb = 0$ then L_4 is generated by $babb$. If a', b' are as above (with $\beta = 0$) then we have

$$b'a'b'b' = \alpha\gamma^3 babb,\ p^2a' = \alpha p^2 a$$

and so we can assume that $p^2a = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $p^2a \neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb = 0$. And if $p^2a = 0$ then $pb' = \gamma pb$ so we may assume that $pb = 0$ or $babb$. This gives

$$\langle a, b \mid baaa, baab, p^2a, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baaa, baab, p^2a, pb - babb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baaa, baab, p^2a - babb, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baaa, baab, p^2a - \omega babb, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \mid baaa, baab, p^2a - \omega^2 babb, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3).$$

21.1.2 $baab = 0,\ babb = baaa$

If $baab = 0$ and $babb = baaa$ then we need $\beta = 0$ and $\gamma = \pm\alpha$, so that

$$b'a'a'a' = \pm\alpha^4 baaa,\ p^2a' = \alpha p^2 a.$$

So we can assume that $p^2a = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $p^2a \neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb = 0$. And if $p^2a = 0$ then $pb' = \pm\alpha pb$ so we may assume that $pb = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. This gives

$$\langle a, b \mid baab, babb - baaa, p^2a, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baab, babb - baaa, p^2a, pb - baaa, \text{ class } 4\rangle,$$

$$\langle a, b \mid baab, babb - baaa, p^2a, pb - \omega baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \mid baab, babb - baaa, p^2a, pb - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \mid baab, babb - baaa, p^2a - baaa, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baab, babb - baaa, p^2a - \omega baaa, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \mid baab, babb - baaa, p^2a - \omega^2 baaa, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3).$$

21.1.3 $baab = 0,\ babb = \omega baaa$

This case is similar to the one above, and gives

$$\langle a, b \mid baab, babb - \omega baaa, p^2a, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baab, babb - \omega baaa, p^2a, pb - baaa, \text{ class } 4\rangle,$$
$$\langle a, b \mid baab, babb - \omega baaa, p^2a, pb - \omega baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baab, babb - \omega baaa, p^2a, pb - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baab, babb - \omega baaa, p^2a - baaa, pb, \text{ class } 4\rangle,$$
$$\langle a, b \mid baab, babb - \omega baaa, p^2a - \omega baaa, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baab, babb - \omega baaa, p^2a - \omega^2 baaa, pb, \text{ class } 4\rangle\ (p = 1 \bmod 3).$$

21.1.4 $baaa = babb = 0$

If $baaa = babb = 0$ then L_4 is generated by $baab$, and we require $\beta = 0$. We then have

$$b'a'a'b' = \alpha^2\gamma^2 baab,\ p^2a' = \alpha p^2 a,$$

and so we can assume that $p^2a = 0$ or $baab$. As above, if $p^2a \neq 0$ we can assume that $pb = 0$, and if $p^2a = 0$ then we can assume that $pb = 0$ or $baab$. so we have

$$\langle a, b \mid baaa, babb, p^2a, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baaa, babb, p^2a, pb - baab, \text{ class } 4\rangle,$$
$$\langle a, b \mid baaa, babb, p^2a - baab, pb, \text{ class } 4\rangle.$$

21.1.5 $baab = babb = 0$

If $baab = babb = 0$ then L_4 is generated by $baaa$, and

$$b'a'a'a' = \alpha^3\gamma baaa,\ p^2a' = \alpha p^2 a.$$

So we can assume that $p^2a = 0$ or $baaa$, and if $p^2a = baaa$ we can (as above) assume that $pb = 0$. If $p^2a = 0$ then $pb' = \gamma pb$ and so we can assume that $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, giving

$$\langle a, b \mid baab, babb, p^2a, pb, \text{ class } 4\rangle,$$

$$\langle a, b \mid baab, babb, p^2a, pb - baaa, \text{ class } 4\rangle,$$
$$\langle a, b \mid baab, babb, p^2a, pb - \omega baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baab, babb, p^2a, pb - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baab, babb, p^2a - baaa, pb, \text{ class } 4\rangle.$$

21.2 Descendants of 6.369

Algebra 6.369 has $5 + \gcd(p-1, 3)$ descendants of order p^7 and class 4.

Algebra 6.369 has presentation

$$\langle a, b \,|\, p^2 a,\ pb - bab,\ \text{class } 3\rangle,$$

and so if L is a descendant of 6.369 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa and bab modulo L_4, and L_4 has order p and is generated by $baaa$, $baab$. Furthermore $babb = 0$ and $p^2 a,\ pb - bab \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_2 then $a' = \alpha a + \beta b + c$, $b' = \alpha^{-1} b + d$ for some α, β, γ and some (arbitrary) c, d in L_2, and

$$\begin{aligned} b'a'a'a' &= \alpha^2 baaa + 2\alpha\beta baab, \\ b'a'a'b' &= baab. \end{aligned}$$

So we can assume that $baaa = 0$ and that L_4 is generated by $baab$ (though we then need $\beta = 0$), or we can assume that $baab = 0$ and that L_4 is generated by $baaa$.

21.2.1 $baaa = 0$

Consider the case when $baaa = 0$ and L_4 is generated by $baab$. Then we can assume that $p^2 a = 0$ or $baab$. If $p^2 a = baab$ then adding a suitable scalar multiple of pa to b we may assume that $pb - bab = 0$. And if $p^2 a = 0$ then $pb' - b'a'b' = \alpha^{-1}(pb - bab)$ so we may assume that $pb - bab = 0$ or $baab$. So we have

$$\langle a, b \,|\, baaa, p^2 a, pb - bab,\ \text{class } 4\rangle,$$

$$\langle a, b \,|\, baaa, p^2 a, pb - bab - baab,\ \text{class } 4\rangle,$$

$$\langle a, b \,|\, baaa, p^2 a - baab, pb - bab,\ \text{class } 4\rangle.$$

21.2.2 $baab = 0$

Now consider the case when $baab = 0$ and L_4 is generated by $baaa$. Then

$$b'a'a'a' = \alpha^2 baaa,\ p^2 a' = \alpha p^2 a$$

so we can assume that $p^2 a = 0$ or $baaa$. As above, if $p^2 a \neq 0$ we can assume that $pb - bab = 0$. If $p^2 a = 0$ then

$$pb' - b'a'b' = \alpha^{-1}(pb - bab)$$

and so we can assume that $pb - bab = 0$, $baaa$ or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baaa$. This gives

$$\langle a, b \,|\, baab, p^2 a, pb - bab,\ \text{class } 4\rangle,$$

$$\langle a, b \,|\, baab, p^2 a, pb - bab - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b \,|\, baab, p^2 a, pb - bab - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \,|\, baab, p^2 a, pb - bab - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \,|\, baab, p^2 a - baaa, pb - bab,\ \text{class } 4\rangle.$$

21.3 Descendants of 6.370

Algebra 6.370 has $2p+2+(p+1)/2+\gcd(p-1,3)+\gcd(p-1,4)/2$ descendants of order p^7 and class 4.

Algebra 6.370 has presentation

$$\langle a,b\,|\,p^2a,\ pb-baa,\ \text{class }3\rangle,$$

and so if L is a descendant of 6.370 of order p^7 then L_2 is generated by ba,pa modulo L_3, L_3 is generated by baa and bab modulo L_4, and L_4 is generated by $baaa$ and $babb$. Furthermore L_4 has order p and $p^2a, pb-baa\in L_4$. If a',b' generate L and satisfy the same relations as a,b modulo L_2 then $a'=\pm a+c$, $b'=\beta b+d$ for some β and some (arbitrary) c,d in L_2, and

$$\begin{aligned} b'a'a'a' &= \pm\beta baaa,\\ b'a'b'b' &= \pm\beta^3 babb. \end{aligned}$$

So we can assume that $baaa=0$ or $babb=0$ or $babb=baaa$ or $babb=\omega baaa$.

21.3.1 $baaa=0$

Consider the case when $baaa=0$ and L_4 is generated by $babb$. Then

$$b'a'b'b'=\pm\beta^3 babb,\ p^2a'=\pm p^2a$$

and so we may assume that $p^2a=0$, $babb$ or (if $p=1\bmod 3$) $\omega babb$ or $\omega^2 babb$. If $p^2a\neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb-baa=0$. If $p^2a=0$ then we have

$$pb'-b'a'a=\beta(pb-baa)$$

and so we may assume that $pb-baa=0$, $babb$ or (if $p=1\bmod 4$) $\omega babb$. This gives

$$\langle a,b\,|\,baaa,p^2a,pb-baa,\ \text{class }4\rangle,$$

$$\langle a,b\,|\,baaa,p^2a,pb-baa-babb,\ \text{class }4\rangle,$$

$$\langle a,b\,|\,baaa,p^2a,pb-baa-\omega babb,\ \text{class }4\rangle\ (p=1\bmod 4),$$

$$\langle a,b\,|\,baaa,p^2a-babb,pb-baa,\ \text{class }4\rangle,$$

$$\langle a,b\,|\,baaa,p^2a-\omega babb,pb-baa,\ \text{class }4\rangle\ (p=1\bmod 3),$$

$$\langle a,b\,|\,baaa,p^2a-\omega^2 babb,pb-baa,\ \text{class }4\rangle\ (p=1\bmod 3).$$

21.3.2 $babb=0$

Now consider the case when $babb=0$ and L_4 is spanned by $baaa$. Then

$$b'a'a'a'=\pm\beta baaa,\ p^2a'=\pm p^2a$$

and so we can assume that $p^2a=0$ or $baaa$. As above, if $p^2a\neq 0$ then we can assume that $pb-baa=0$. If $p^2a=0$ then

$$pb'-b'a'a'=\beta(pb-baa)$$

and so we can assume that $pb-baa=xbaaa$ with $0\leq x\leq (p-1)/2$. This gives

$$\langle a,b\,|\,babb,p^2a,pb-baa-xbaaa,\ \text{class }4\rangle\ (0\leq x\leq (p-1)/2),$$

$$\langle a,b\,|\,babb,p^2a-baaa,pb-baa,\ \text{class }4\rangle.$$

21.3.3 $babb = baaa$

If $babb = baaa$ then we need $\beta = \pm 1$ and so we have $a' = \pm a$ modulo L_2 and we independently have $b' = \pm b$ modulo L_2. So we can assume that $p^2a = xbaaa$ where $0 \leq x \leq (p-1)/2$. As above, if $p^2a \neq 0$ we can assume that $pb - baa = 0$, and if $p^2a = 0$ then we can assume that $pb - baa = xbaaa$ with $0 \leq x \leq (p-1)/2$. This gives

$$\langle a, b \mid babb - baaa, p^2a, pb - baa - xbaaa, \text{ class } 4\rangle \; (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb - baaa, p^2a - xbaaa, pb - baa, \text{ class } 4\rangle \; (1 \leq x \leq (p-1)/2).$$

21.3.4 $babb = \omega baaa$

This case is similar to the one above, and gives

$$\langle a, b \mid babb - \omega baaa, p^2a, pb - baa - xbaaa, \text{ class } 4\rangle \; (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb - \omega baaa, p^2a - xbaaa, pb - baa, \text{ class } 4\rangle \; (1 \leq x \leq (p-1)/2).$$

21.4 Descendants of 6.371

Algebra 6.371 has $2p + 2 + (p+1)/2 + \gcd(p-1, 3) + \gcd(p-1, 4)/2$ descendants of order p^7 and class 4.

Algebra 6.371 has presentation

$$\langle a, b \mid p^2a, \; pb - \omega baa, \text{ class } 3\rangle,$$

so this case is almost identical to 6.370, and its decendants of order p^7 are

$$\langle a, b \mid baaa, p^2a, pb - \omega baa, \text{ class } 4\rangle,$$

$$\langle a, b \mid baaa, p^2a, pb - \omega baa - babb, \text{ class } 4\rangle,$$
$$\langle a, b \mid baaa, p^2a, pb - \omega baa - \omega babb, \text{ class } 4\rangle \; (p = 1 \bmod 4),$$
$$\langle a, b \mid baaa, p^2a - babb, pb - \omega baa, \text{ class } 4\rangle,$$
$$\langle a, b \mid baaa, p^2a - \omega babb, pb - \omega baa, \text{ class } 4\rangle \; (p = 1 \bmod 3),$$
$$\langle a, b \mid baaa, p^2a - \omega^2 babb, pb - \omega baa, \text{ class } 4\rangle \; (p = 1 \bmod 3),$$

$$\langle a, b \mid babb, p^2a, pb - \omega baa - xbaaa, \text{ class } 4\rangle \; (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb, p^2a - baaa, pb - \omega baa, \text{ class } 4\rangle,$$

$$\langle a, b \mid babb - baaa, p^2a, pb - \omega baa - xbaaa, \text{ class } 4\rangle \; (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb - baaa, p^2a - xbaaa, pb - \omega baa, \text{ class } 4\rangle \; (1 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb - \omega baaa, p^2a, pb - \omega baa - xbaaa, \text{ class } 4\rangle \; (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \mid babb - \omega baaa, p^2a - xbaaa, pb - \omega baa, \text{ class } 4\rangle \; (1 \leq x \leq (p-1)/2).$$

Algebras 6.372 has $(p+1)/2+\gcd(p-1,3)+\gcd(p-1,4)/2$ descendants of order p^7.

Algebra 6.372 has presentation

$$\langle a,b \,|\, p^2a-bab,\, pb,\, \text{class } 3\rangle,$$

so if L is a descendant of 6.372 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$. In addition, $baab=babb=0$ and p^2a-bab and pb are scalar mutiples of $baaa$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a'=\alpha a+\beta b$ modulo L_2 and $b'=\pm b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a' &= \pm\alpha^3 baaa, \\ p^2a'-b'a'b' &= \alpha(p^2a-bab), \\ pb' &= \pm pb. \end{aligned}$$

So we can assume that $p^2a-bab=0$, $baaa$ or (if $p=1 \bmod 4$) $\omega baaa$. If $p^2a-bab=0$ we can assume that $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Consider the case when $p^2a-bab\neq 0$, and $pb=\lambda baaa$ If $p=1 \bmod 4$ then we either have

$$\begin{aligned} b'a'a'a' &= \alpha^3 baaa, \\ p^2a'-b'a'b' &= \alpha(p^2a-bab), \\ pb' &= pb=\lambda baaa \end{aligned}$$

with $\alpha^2=1$, which gives

$$pb'=\pm\lambda b'a'a'a',$$

or we have

$$\begin{aligned} b'a'a'a' &= -\alpha^3 baaa, \\ p^2a'-b'a'b' &= \alpha(p^2a-bab), \\ pb' &= -pb=-\lambda baaa \end{aligned}$$

with $\alpha^2=-1$, which gives

$$pb'=\pm i\lambda b'a'a'a'$$

where $i^2=-1$. So we can take $\lambda=0$, or λ in a transversal for the fourth roots of unity. On the other hand, if $p=1 \bmod 4$ then we have

$$\begin{aligned} b'a'a'a' &= \alpha^3 baaa, \\ p^2a'-b'a'b' &= \alpha(p^2a-bab), \\ pb' &= pb=\lambda baaa \end{aligned}$$

with $\alpha^2=1$, which gives

$$pb'=\pm\lambda b'a'a'a',$$

so we can take $0\leq\lambda\leq(p-1)/2$. So we have

$$\langle a,b \,|\, p^2a-bab,\, pb,\, \text{class } 4\rangle,$$

$$\langle a,b \,|\, p^2a-bab,\, pb-baaa,\, \text{class } 4\rangle,$$

$$\langle a,b \,|\, p^2a-bab,\, pb-\omega baaa,\, \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a, b \,|\, p^2a - bab,\, pb - \omega^2 baaa,\, \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \,|\, p^2a - bab - baaa,\, pb - xbaaa,\, \text{class } 4\rangle\ (p = 1 \bmod 4,\ \text{all } x, x^4 \sim x'^4),$$

$$\langle a, b \,|\, p^2a - bab - \omega baaa,\, pb - xbaaa,\, \text{class } 4\rangle\ (p = 1 \bmod 4,\ \text{all } x, x^4 \sim x'^4),$$

$$\langle a, b \,|\, p^2a - bab - baaa,\, pb - xbaaa,\, \text{class } 4\rangle\ (p = 3 \bmod 4, 0 \leq x \leq (p-1)/2).$$

21.6 Descendants of 6.373

Algebras 6.373 has p descendants of order p^7 and class 4.

Algebra 6.373 has presentation

$$\langle a, b \,|\, p^2a - bab,\, pb - baa,\, \text{class } 3\rangle,$$

and so if L is a descendant of 6.373 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $baaa$. In addition, $baab = babb = 0$ and $p^2a - bab$ and $pb - baa$ are scalar mutiples of $baaa$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \pm a + \beta b$ modulo L_2 and $b' = \pm b$ modulo L_2 (the $\pm$ signs are independent of each other). In addition

$$b'a'a'a' = \pm baaa,\ p^2a' - b'a'b' = \pm(p^2a - bab)$$

(with the $\pm$ signs independent of each other). So we can assume that $p^2a - bab = xbaaa$ with $0 \leq x \leq (p-1)/2$. If $p^2a - bab = 0$ then we can similarly assume that $pb - baa = xbaaa$ with $0 \leq x \leq (p-1)/2$. But if $p^2a - bab \neq 0$ then replacing a by $a + \beta b$ for suitable β, and adding a suitable scalar multiple of pa to b, we may assume that $pb - baa = 0$. So we have

$$\langle a, b \,|\, p^2a - bab,\, pb - baa - xbaaa,\, \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$
$$\langle a, b \,|\, p^2a - bab - xbaaa,\, pb - baa,\, \text{class } 4\rangle\ (1 \leq x \leq (p-1)/2).$$

21.7 Descendants of 6.374

Algebras 6.374 has p descendants of order p^7 and class 4.

Algebra 6.374 has presentation

$$\langle a, b \,|\, p^2a - bab,\, pb - \omega baa,\, \text{class } 3\rangle,$$

and so this case is similar to 6.373, and the descendants of order p^7 are

$$\langle a, b \,|\, p^2a - bab,\, pb - \omega baa - xbaaa,\, \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b \,|\, p^2a - bab - xbaaa,\, pb - \omega baa,\, \text{class } 4\rangle\ (1 \leq x \leq (p-1)/2).$$

21.8 Descendants of 6.375

Algebras 6.375 has $(p+1)/2+\gcd(p-1,3)+\gcd(p-1,4)/2$ descendants of order p^7.

Algebra 6.375 has presentation

$$\langle a,b \,|\, p^2a-\omega bab,\, pb,\, \text{class } 3\rangle,$$

and so this case is almost identical to 6.372, and its descendants of order p^7 are

$$\langle a,b \,|\, p^2a-\omega bab,\, pb,\, \text{class } 4\rangle,$$

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-baaa,\, \text{class } 4\rangle,$$

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-\omega baaa,\, \text{class } 4\rangle \; (p=1 \bmod 3),$$

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-\omega^2 baaa,\, \text{class } 4\rangle \; (p=1 \bmod 3),$$

$$\langle a,b \,|\, p^2a-\omega bab-baaa,\, pb-xbaaa,\, \text{class } 4\rangle \; (p=1 \bmod 4,\ \text{all } x, x^4\sim x'^4),$$

$$\langle a,b \,|\, p^2a-\omega bab-\omega baaa,\, pb-xbaaa,\, \text{class } 4\rangle \; (p=1 \bmod 4,\ \text{all } x, x^4\sim x'^4),$$

$$\langle a,b \,|\, p^2a-\omega bab-baaa,\, pb-xbaaa,\, \text{class } 4\rangle \; (p=3 \bmod 4, 0\leq x\leq (p-1)/2).$$

21.9 Descendants of 6.376

Algebra 6.376 has p descendants of order p^7 and class 4.

Algebra 6.376 has presentation

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-baa,\, \text{class } 3\rangle,$$

and so this case is almost identical to 6.373, and its descendants of order p^7 are

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-baa-xbaaa,\, \text{class } 4\rangle \; (0\leq x\leq (p-1)/2),$$

$$\langle a,b \,|\, p^2a-\omega bab-xbaaa,\, pb-baa,\, \text{class } 4\rangle \; (1\leq x\leq (p-1)/2).$$

21.10 Descendants of 6.377

Algebra 6.377 has p descendants of order p^7 and class 4.

Algebra 6.377 has presentation

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-\omega baa,\, \text{class } 3\rangle,$$

and so this case is almost identical to 6.376, and its descendants of order p^7 are

$$\langle a,b \,|\, p^2a-\omega bab,\, pb-\omega baa-xbaaa,\, \text{class } 4\rangle \; (0\leq x\leq (p-1)/2),$$

$$\langle a,b \,|\, p^2a-\omega bab-xbaaa,\, pb-\omega baa,\, \text{class } 4\rangle \; (1\leq x\leq (p-1)/2).$$

21.11 Descendants of 6.378

Algebra 6.378 $p+1+\gcd(p-1,3)$ descendants of order p^7 when $\lambda=0$, and none when $\lambda\neq 0$.

Algebra 6.378 has presentation

$$\langle a,b\,|\,p^2a-baa,\ pb-\lambda bab,\ \text{class }3\rangle\ (0\leq\lambda<p).$$

However this algebra is terminal unless $\lambda=0$. If L is a descendant of 6.378 (with $\lambda=0$) of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, bab modulo L_4, and L_4 has order p and is generated by $babb$. In addition, $baaa=babb=0$ and p^2a-baa and pb are scalar mutiples of $babb$. If a', b' generate L and satisfy the same relations as a,b modulo L_4 then $a'=\alpha a$ modulo L_2 and $b'=\alpha^{-1}b$ modulo L_2. We then have

$$\begin{aligned} b'a'b'b' &= \alpha^{-2}babb, \\ p^2a'-b'a'a' &= \alpha(p^2a-baa), \\ pb' &= \alpha^{-1}pb. \end{aligned}$$

So we can assume that $pb=0$ or $babb$. If $pb=0$ we can assume that $p^2a-baa=0$, $babb$ or (if $p=1\bmod 3$) $\omega babb$ or ω^2babb. And if $pb=babb$ then we have $p^2a-baa=xbabb$ with $0\leq x<p$.

$$\langle a,b\,|\,p^2a-baa,\ pb,\ \text{class }4\rangle,$$

$$\langle a,b\,|\,p^2a-baa-babb,\ pb,\ \text{class }4\rangle,$$

$$\langle a,b\,|\,p^2a-baa-\omega babb,\ pb,\ \text{class }4\rangle\ (p=1\bmod 3),$$

$$\langle a,b\,|\,p^2a-baa-\omega^2babb,\ pb,\ \text{class }4\rangle\ (p=1\bmod 3),$$

$$\langle a,b\,|\,p^2a-baa-xbabb,\ pb-babb,\ \text{class }4\rangle\ (0\leq x<p).$$

21.12 Descendants of 6.379

Algebra 6.379 has $5+2\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Algebra 6.379 has presentation

$$\langle a,b\,|\,baa,\ pb,\ \text{class }3\rangle,$$

and so if L is a descendant of 6.379 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab, p^2a modulo L_4 and L_4 has order p and is generated by $babb, p^3a$. In addition, $baa, pb\in L_4$. If a', b' generate L and satisfy the same relations as a,b modulo L_4 then $a'=\alpha a$ modulo L_2 and $b'=\beta b$ modulo L_2. We then have

$$\begin{aligned} b'a'b'b &= \alpha\beta^3babb, \\ p^3a' &= \alpha p^3a, \end{aligned}$$

so we can assume that $p^3a=0$, or $babb=0$, or $p^3a=babb$, or (if $p=1\bmod 3$) $p^3a=\omega babb$ or ω^2babb.

21.12.1 $\quad p^3a = 0$

If $p^3a = 0$ then L_4 is generated by $babb$, and so $baa = \lambda babb$, $pb = \mu babb$ for some λ, μ. If a', b' are as above then

$$\begin{aligned} b'a'a' &= \alpha^2\beta\lambda babb = \alpha\beta^{-2}\lambda b'a'b'b', \\ pb' &= \beta\mu babb = \alpha^{-1}\beta^{-2}\mu b'a'b'b'. \end{aligned}$$

So we can assume that $\lambda = 0$ or 1. If $\lambda = 0$ we can assume that $\mu = 0$ or 1, and if $\lambda = 1$ we can assume that $\mu = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. This gives

$$\langle a, b \,|\, p^3a, baa, pb, \text{ class } 4\rangle,$$

$$\langle a, b \,|\, p^3a, baa, pb - babb, \text{ class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, baa - babb, pb, \text{ class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, baa - babb, pb - babb, \text{ class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, baa - babb, pb - \omega babb, \text{ class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, baa - babb, pb - \omega^2 babb, \text{ class } 4\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b \,|\, p^3a, baa - babb, pb - \omega^3 babb, \text{ class } 4\rangle \ (p = 1 \bmod 4).$$

21.12.2 $\quad babb = 0$

If $babb = 0$ then L_4 is generated by p^3a. Adding a suitable scalar multiple of p^2a to b we can assume that $pb = 0$. If $baa = \lambda p^3a$ then a', b' are as above then

$$b'a'a' = \alpha^2\beta\lambda p^3a = \alpha\beta\lambda p^3a',$$

So we can assume that $\lambda = 0$ or 1, giving

$$\langle a, b \,|\, babb, baa, pb, \text{ class } 4\rangle,$$

$$\langle a, b \,|\, babb, baa - p^3a, pb, \text{ class } 4\rangle,$$

21.12.3 $\quad p^2a = kbabb$ $(k = 1, \omega, \omega^2)$

If $p^2a = babb$ then L_4 is generated by $babb$, though we then $\beta^3 = 1$. Adding a suitable scalar multiple of p^2a to b we may assume that $pb = 0$. If $baa = \lambda babb$ and if a', b' are as above (with $\beta^3 = 1$) then

$$b'a'a' = \alpha^2\beta\lambda babb = \alpha\beta^{-2}\lambda b'a'b'b',$$

so we can assume that $\lambda = 0$ or 1, giving

$$\langle a, b \,|\, p^3a - babb, baa, pb, \text{ class } 4\rangle,$$

$$\langle a, b \,|\, p^3a - babb, baa - babb, pb, \text{ class } 4\rangle,$$
$$\langle a, b \,|\, p^3a - \omega babb, baa, pb, \text{ class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a - \omega babb, baa - babb, pb, \text{ class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a - \omega^2 babb, baa, pb, \text{ class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a - \omega^2 babb, baa - babb, pb, \text{ class } 4\rangle \ (p = 1 \bmod 3).$$

21.13 Descendants ot 6.380

Algebra 6.380 has p descendants of order p^7 and p-class 4.

Algebra 6.380 has presentation

$$\langle a, b \mid baa,\ pb - bab,\ \text{class } 3\rangle,$$

and so if L is a descendant of 6.380 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab, p^2a modulo L_4 and L_4 has order p and is generated by p^3a. In addition, $baa, pb - bab \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \alpha a$ modulo L_2 and $b' = \alpha^{-1}b$ modulo L_2. Adding a suitable scalar multiple of p^2a to b we may assume that $pb - bab = 0$. If $baa = \lambda p^3a$, $pb - bab = \mu p^3a$ we have

$$b'a'a' = \alpha\lambda p^3a = \lambda p^3a',$$

so we can take $0 \le \lambda < p$ giving

$$\langle a, b \mid baa - xp^3a,\ pb - bab,\ \text{class } 4\rangle\ (0 \le x < p),$$

21.14 Descendants of 6.381

Algebra 6.381 has $6 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Algebra 6.381 has presentation

$$\langle a, b \mid bab,\ pb,\ \text{class } 3\rangle,$$

and so if L is a descendant of 6.381 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, p^2a modulo L_4 and L_4 has order p and is generated by $baaa, p^3a$. In addition, $bab, pb \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a' &= \alpha^3\gamma baaa, \\ p^3a' &= \alpha p^3a. \end{aligned}$$

So we can assume that $baaa = 0$ or $p^3a = 0$ or $p^3a = baaa$.

21.14.1 $baaa = 0$

If $baaa = 0$ then L_4 is generated by p^3a and we have $bab = \lambda p^3a$ for some λ. Adding a suitable scalar multiple of p^2a to b we may assume that $pb = 0$. We have

$$b'a'b' = \alpha\gamma^2\lambda p^3a = \gamma^2\lambda p^3a',$$

and so we can assume that we have $\lambda = 0, 1, \omega$, giving

$$\langle a, b \mid baaa, bab,\ pb,\ \text{class } 4\rangle,$$

$$\langle a, b \mid baaa, bab - p^3a,\ pb,\ \text{class } 4\rangle,$$

$$\langle a, b \mid baaa, bab - \omega p^3a,\ pb,\ \text{class } 4\rangle.$$

21.14.2 $p^3a = 0$

If $p^3a = 0$ then L_4 is generated by $baaa$ and we have $bab = \lambda baaa$, $pb = \mu baaa$ for some λ, μ. We then have

$$\begin{aligned} b'a'b' &= \alpha\gamma^2\lambda baaa = \alpha^{-2}\gamma\lambda b'a'a'a', \\ pb' &= \gamma\mu baaa = \alpha^{-3}\mu b'a'a'a'. \end{aligned}$$

So we can take $\lambda = 0, 1$ and $\mu = 0, 1$ or (if $p = 1 \bmod 3$) ω, ω^2. This gives

$$\langle a, b \,|\, p^3a, bab, pb, \text{class } 4\rangle,$$

$$\langle a, b \,|\, p^3a, bab - baaa, pb, \text{class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, bab, pb - baaa, \text{class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, bab - baaa, pb - baaa, \text{class } 4\rangle,$$
$$\langle a, b \,|\, p^3a, bab, pb - \omega baaa, \text{class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a, bab - baaa, pb - \omega baaa, \text{class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a, bab, pb - \omega^2 baaa, \text{class } 4\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b \,|\, p^3a, bab - baaa, pb - \omega^2 baaa, \text{class } 4\rangle \ (p = 1 \bmod 3).$$

21.14.3 $p^3a = baaa$

If $p^3a = baaa$ then L_4 is generated by $baaa$, though we need $\gamma = \alpha^{-2}$. Subtracting a suitable scalar multiple of p^2a from b we may assume that $pb = 0$. If $bab = \lambda baaa$ then

$$b'a'b' = \alpha^{-3}\lambda baaa = \alpha^{-4}\lambda b'a'a'a'$$

and so we can take $\lambda = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. This gives

$$\langle a, b \,|\, p^3a - baaa, bab, pb, \text{class } 4\rangle,$$

$$\langle a, b \,|\, p^3a - baaa, bab - baaa, pb, \text{class } 4\rangle,$$
$$\langle a, b \,|\, p^3a - baaa, bab - \omega baaa, pb, \text{class } 4\rangle,$$
$$\langle a, b \,|\, p^3a - baaa, bab - \omega^2 baaa, pb, \text{class } 4\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b \,|\, p^3a - baaa, bab - \omega^3 baaa, pb, \text{class } 4\rangle \ (p = 1 \bmod 4).$$

21.15 Descendants of 6.382

Algebra 6.382 has $p + 5 + (p-1)/2$ descendants of order p^7 and p-class 4.

Algebra 6.382 has presentation

$$\langle a, b \,|\, bab, pb - baa, \text{class } 3\rangle,$$

and so if L is a descendant of 6.381 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, p^2a modulo L_4 and L_4 has order p and is generated by $baaa, p^3a$. In addition, $bab, pb - baa \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \pm a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a' &= \pm\gamma baaa, \\ p^3a' &= \pm p^3a, \end{aligned}$$

so we can assume that $baaa = 0$ or $p^3a = 0$ or $p^3a = baaa$.

21.15.1 $baaa = 0$

If $baaa = 0$ then L_4 is generated by p^3a. Adding a suitable scalar multiple of p^2a to b we may assume that $pb = baa$. If $bab = \lambda p^3 a$ then

$$b'a'b' = \pm\gamma^2\lambda p^3 a = \gamma^2\lambda p^3 a'$$

and so we can assume that we have $\lambda = 0, 1, \omega$, giving

$$\langle a, b \mid baaa, bab, pb - baa, \text{class } 4\rangle,$$
$$\langle a, b \mid baaa, bab - p^3a, pb - baa, \text{class } 4\rangle,$$
$$\langle a, b \mid baaa, bab - \omega p^3a, pb - baa, \text{class } 4\rangle.$$

21.15.2 $p^3a = 0$

If $p^3a = 0$ then L_4 is generated by $baaa$ and we have $bab = \lambda baaa$, $pb - baa = \mu baaa$ for some λ, μ. We then have

$$\begin{aligned} b'a'b' &= \pm\gamma^2\lambda baaa = \gamma\lambda b'a'a'a', \\ pb' - b'a'a' &= \mp\beta\gamma bab + \gamma(pb - baa). \end{aligned}$$

So we can take $\lambda = 0, 1$, and if $\lambda = 1$ we can take $\mu = 0$. If $\lambda = 0$ then we have

$$pb' = \pm\mu b'a'a'a'$$

and so we can take $0 \le \mu \le (p-1)/2$ giving

$$\langle a, b \mid p^3a, bab, pb - baa - xbaaa, \text{class } 4\rangle \; (0 \le x \le (p-1)/2),$$
$$\langle a, b \mid p^3a, bab - baaa, pb - baa, \text{class } 4\rangle.$$

21.15.3 $p^3a = baaa$

If $p^3a = baaa$ then L_4 is generated by $baaa$, though we need $\gamma = 1$. Adding a suitable scalar multiple of p^2a to b we may assume that $pb = baa$. If $bab = \lambda baaa$ then

$$b'a'b' = \pm\lambda baaa = \lambda b'a'a'a',$$

so we have

$$\langle a, b \mid p^3a - baaa, bab - xbaaa, pb - baa, \text{class } 4\rangle \; (0 \le x < p).$$

21.16 Descendants of 6.383

Algebra 6.383 has $p + 5 + (p-1)/2$ descendants of order p^7 and p-class 4.

Algebra 6.383 has presentation

$$\langle a, b \mid bab, pb - \omega baa, \text{class } 3\rangle,$$

and so this case is almost identical to 6.382, and we have

$$\langle a, b \mid baaa, bab, pb - \omega baa, \text{class } 4\rangle,$$
$$\langle a, b \mid baaa, bab - p^3a, pb - \omega baa, \text{class } 4\rangle,$$
$$\langle a, b \mid baaa, bab - \omega p^3a, pb - \omega baa, \text{class } 4\rangle,$$
$$\langle a, b \mid p^3a, bab, pb - \omega baa - xbaaa, \text{class } 4\rangle \; (0 \le x \le (p-1)/2),$$
$$\langle a, b \mid p^3a, bab - baaa, pb - \omega baa, \text{class } 4\rangle,$$
$$\langle a, b \mid p^3a - baaa, bab - xbaaa, pb - \omega baa, \text{class } 4\rangle \; (0 \le x < p).$$

22 Grandchildren of algebra 4 (4.8)

All the descendants of 4.8 of order p^6 are terminal, except for 6.384.

22.1 Descendants of 6.384

Algebra 6.384 has 2 descendants of order p^7 and p-class 4.

Algebra 6.384 has presentation

$$\langle a, b \mid pb - ba, \text{ class } 3 \rangle,$$

and so if L is a descendant of 6.381 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa, p^2a modulo L_4 and L_4 has order p and is generated by $baaa, p^3a$. In addition, $pb - ba \in L_4$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a' &= \gamma baaa, \\ p^3a' &= p^3a + \beta baaa. \end{aligned}$$

So we can assume that $baaa = 0$ or that $p^3a = 0$.

22.1.1 $baaa = 0$

If $baaa = 0$ then L_4 is generated by p^3a. Addding a suitable scalar multiple of p^2a to b we may assume that $pb = ba$ and so we have

$$\langle a, b \mid baaa, pb - ba, \text{ class } 4 \rangle.$$

22.1.2 $p^3a = 0$

If $p^3a = 0$ then L_4 is generated by $baaa$, though we then need $\beta = 0$. Adding a suitable scalar multiple of p^2a to a we may assume that $pb = ba$, giving

$$\langle a, b \mid p^3a, pb - ba, \text{ class } 4 \rangle.$$

23 Grandchildren of algebra 5 (5.37)

All the descendants of 5.37 of order p^6 are terminal, except for 6.386 and 6.388.

23.1 Descendants of 6.386

Algebra 6.386 has presentation

$$\langle a, b \mid baa,\ bab,\ pba,\ p^2b, \text{ class } 3 \rangle,$$

so if L is a descendant of 6.386 of order p^7 then L_2 is generated by ba, pa, pb modulo L_3, L_3 is generated by p^2a modulo L_4, and L_4 is generated by p^3a. Furthermore baa, bab, pba and p^2b are all scalar multiples of p^3a. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \alpha a + \beta b + c$, $b' = \gamma b + d$ for some α, β, γ with $\alpha, \gamma \neq 0$ and c, d arbitrary elements in L_2. So adding a suitable multiple of pa to b we can assume that $p^2b = 0$. We let

$$baa = \lambda p^3a,\ bab = \mu p^3a,\ pba = \nu p^3a.$$

Then if a', b' are as above we have

$$\begin{aligned} b'a'a' &= \alpha^2\gamma baa + \alpha\beta\gamma bab = (\alpha^2\gamma\lambda + \alpha\beta\gamma\mu)p^3a = (\alpha\gamma\lambda + \beta\gamma\mu)p^3a', \\ b'a'b' &= \alpha\gamma^2 bab = \alpha\gamma^2\mu p^3a = \gamma^2\mu p^3 a', \\ pb'a' &= \alpha\gamma pba = \alpha\gamma\nu p^3 a = \gamma\nu p^3 a'. \end{aligned}$$

So we can assume that $\nu = 0$ or 1.

If $\nu = 0$ we can assume that $\mu = 0, 1$ or ω, and if $\nu = 1$ we need $\gamma = 1$ and so μ is arbitrary.

If $\mu = \nu = 0$ then we can assume that $\lambda = 0$ or 1.

If $\nu = 0$, $\mu = 1$ or ω then we can assume that $\lambda = 0$.

If $\nu = 1$ and $\mu \neq 0$ then we can assume that $\lambda = 0$.

If $\nu = 1$ and $\mu = 0$ then we can assume that $\lambda = 0$ or 1.

So we have $p+5$ algebras here. I have checked that the recipes above give this number of non-isomorphic groups for $p = 5, 7, 11, 13$, and have also checked that this is the right number of descendants for $p = 5, 7, 11, 13$.

23.2 Descendants of 6.388

Algebra 6.388 has presentation

$$\langle a, b \mid bab, pba,\ p^2a,\ p^2b, \text{ class } 3\rangle,$$

so if L is a descendant of 6.388 of order p^7 then L_2 is generated by ba, pa, pb modulo L_3, L_3 is generated by baa modulo L_4, and L_4 is generated by $baaa$. Furthermore bab, pba, p^2a and p^2b are all scalar multiples of $baaa$. If a', b' generate L and satisfy the same relations as a, b modulo L_4 then $a' = \alpha a + \beta b$, $b' = \gamma b$ modulo L_2 for some α, β, γ with $\alpha, \gamma \neq 0$. We then have

$$\begin{aligned} b'a'a'a' &= \alpha^3\gamma baaa, \\ b'a'b' &= \alpha\gamma^2 bab, \\ pb'a' &= \alpha\gamma pba, \\ p^2a' &= \alpha p^2 a + \beta p^2 b, \\ p^2b' &= \gamma p^2 b. \end{aligned}$$

So we can assume that $p^2b = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

First consider the case when $p^2b = 0$. Then we can assume that $p^2a = 0$ or $baaa$.

Suppose that $p^2a = p^2b = 0$. Then we can assume that $bab = 0$ or $baaa$ and that $pba = 0$, $baaa$ or $\omega baaa$ (6 algebras).

Next suppose that $p^2a = baaa$, $p^2b = 0$. Then we need $\gamma = \alpha^{-2}$ and so we have

$$\begin{aligned} b'a'a'a' &= \alpha baaa, \\ b'a'b' &= \alpha^{-3} bab, \\ pb'a' &= \alpha^{-1} pba. \end{aligned}$$

We can assume that $pba = 0$, $baaa$ or $\omega baaa$. If $pba = 0$ then we can assume that $bab = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$, and if $pba = baaa$ or $\omega baaa$ we can assume that $bab = xbaaa$ with $0 \leq x < p$ ($2p + 1 + \gcd(p-1, 4)$ algebras).

Now suppose that $p^2a = 0$ and $p^2b = baaa$, $\omega baaa$ or $\omega^2 baaa$. Then we need $\alpha^3 = 1$. We can assume that $bab = 0$ or $baaa$ and that $pba = 0$ or $xbaaa$ where $x \neq 0$ lies in a transversal for the cube roots of unity ($2p - 2 + 2\gcd(p-1, 3)$ algebras).

So there are a total of $4p+5+2\gcd(p-1,3)+\gcd(p-1,4)$ algebras here. I have checked that the recipes above give this number of non-isomorphic groups for $p = 5, 7, 11, 13$, and have also checked that this is the right number of descendants for $p = 5, 7, 11, 13$.

24 Grandchildren of algebra 6 (5.38)

All the descendants of 5.38 of order p^6 are terminal, except for 6.394, 6.399 and 6.404.

24.1 Descendants of 6.394

Algebra 6.394 has

$p^2+3p+10+2\gcd(p-1,3)+3\gcd(p-1,4)+2\gcd(p-1,5)+\gcd(p-1,8)+\gcd(p-1,9)$

descendants of order p^7 and p-class 5.

Algebra 6.394 has presentation

$$\langle a,b \mid baab, babb,\ pa,\ pb,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.394 of order p^7 then L/L_4 is free of characteristic p, L_4 is generated by $baaa$ modulo L_5, and L_5 has order p and is generated by $baaaa$ and $baaab$. We also have $baab, babb, pa, pb \in L_5$. If a', b' generate L and satisfy the same realtions as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\gamma baaaa + \alpha^3\beta\gamma baaab, \\ b'a'a'a'b' &= \alpha^3\gamma^2 baaab, \end{aligned}$$

so we can assume that $baaaa = 0$ or that $baaab = 0$.

24.1.1 $baaaa = 0$

If $baaaa = 0$ then L_5 is generated by $baaab$, though we then need $\beta = 0$. Let $baab = \lambda baaab$, $babb = \mu baaab$, $pa = \nu baaab$, $pb = \xi baaab$. Then

$$\begin{aligned} b'a'a'b' &= \alpha^2\gamma^2\lambda baaab = \alpha^{-1}\lambda b'a'a'a'b', \\ b'a'b'b' &= \alpha\gamma^3\mu baaab = \alpha^{-2}\gamma\mu b'a'a'a'b', \\ pa' &= \alpha\nu baaab = \alpha^{-2}\gamma^{-2}\nu b'a'a'a'b', \\ pb' &= \gamma\xi baaab = \alpha^{-3}\gamma^{-1}\xi b'a'a'a'b', \end{aligned}$$

and so we can take $\lambda, \mu = 0, 1$.

If $\lambda = \mu = 0$ then we can assume that $\xi = 0$ or 1. If $\xi = 0$ we can take $\nu = 0, 1, \omega$, and if $\xi = 1$ we can take $\nu = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3.

If $\lambda = 0$ and $\mu = 1$ then we need $\gamma = \alpha^2$ and so we can take $\nu = 0, 1, \omega$ or (if $p = 1 \bmod 3$) $\omega^2, \omega^3, \omega^4, \omega^5$. If $\nu = 0$ then we can take $\xi = 0, 1$ or (if $p = 1 \bmod 5$) $\omega, \omega^2, \omega^3, \omega^4$. And if $\nu \neq 0$ we can take $\xi = 0$, or ξ in a transversal for the sixth roots of unity.

If $\lambda = 1$ and $\mu = 0$ then we need $\alpha = 1$ and so we can take $\xi = 0$ or 1. If $\xi = 0$ we can take $\nu = 0, 1, \omega$, and if $\xi = 1$ then we take $0 \leq \nu < p$.

If $\lambda = \mu = 1$ then we need $\alpha = \gamma = 1$ and so we take $0 \leq \nu, \xi < p$.

24.1.2 $baaab = 0$

If $baaab = 0$ then L_5 is generated by $baaaa$. Let $baab = \lambda baaaa$, $babb = \mu baaaa$, $pa = \nu baaaa$, $pb = \xi baaaa$. Then

$$\begin{aligned} b'a'a'b' &= \alpha^2\gamma^2 baab + \alpha\beta\gamma^2 babb = (\alpha^2\gamma^2\lambda + \alpha\beta\gamma^2\mu)baaaa = (\alpha^{-2}\gamma\lambda + \alpha^{-3}\beta\gamma\mu)b'a'a'a'a', \\ b'a'b'b' &= \alpha\gamma^3\mu baaaa = \alpha^{-3}\gamma^2\mu b'a'a'a'a', \\ pa' &= (\alpha\nu + \beta\xi)baaaa = \alpha^{-4}\gamma^{-1}(\alpha\nu + \beta\xi)b'a'a'a'a', \\ pb' &= \gamma\xi baaaa = \alpha^{-4}\xi b'a'a'a'a'. \end{aligned}$$

So we can take $\mu = 0$ or 1. If $\mu = 0$ we can take $\lambda = 0$ or 1, and if $\mu = 1$ we can take $\lambda = 0$.

If $\lambda = \mu = 0$ then we can take $\xi = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. If $\xi = 0$ we can take $\nu = 0$ or 1, and if $\xi \neq 0$ we can take $\nu = 0$.

If $\lambda = 1$ and $\mu = 0$ then we need $\gamma = \alpha^2$. Again we can take $\xi = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, and if $\xi \neq 0$ we can take $\nu = 0$. But if $\xi = 0$ then we can take $\nu = 0, 1$ or (if $p = 1 \bmod 5$) $\omega, \omega^2, \omega^3, \omega^4$.

If $\lambda = 0$ and $\mu = 1$ then we need $\beta = 0$ and $\gamma^2 = \alpha^3$, so we write $\alpha = \delta^2$, $\gamma = \delta^3$ and we have

$$\begin{aligned} pa' &= \delta^2 \nu baaaa = \delta^{-9} \nu b'a'a'a'a', \\ pb' &= \delta^3 \xi baaaa = \delta^{-8} \xi b'a'a'a'a'. \end{aligned}$$

So we can take $\xi = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3, or (if $p = 1 \bmod 8$) $\omega^4, \omega^5, \omega^6$ or ω^7. If $\xi \neq 0$ we can take $\nu = 0$ or in a transversal for the eighth roots of unity, and if $\xi = 0$ then we can take $\nu = 0, 1$ or (if $p = 1 \bmod 3$) ω or ω^2 or (if $p = 1 \bmod 9$) $\omega^3, \omega^4, \omega^5, \omega^6, \omega^7$ or ω^8.

24.2 Descendants of 6.399

Algebra 6.399 has

$$\frac{1}{2}p^3 + \frac{3}{2}p^2 + 4p + 7 + 3\gcd(p-1,3) + (p+3)\gcd(p-1,4)/2 + \gcd(p-1,5) + \gcd(p-1,8)/2$$

descendants of order p^7 and p-class 5.

Algebra 6.399 has presentation

$$\langle a, b \mid baab, babb + baaa, \ pa, \ pb, \text{ class } 4 \rangle,$$

and so if L is a descendant of 6.399 of order p^7 then L/L_4 is free of characteristic p, L_4 is generated by $baaa$ modulo L_5, and L_5 has order p and is generated by $baaaa, baaab$. We have $baab$, $babb + baaa$, pa, $pd \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \pm(\beta a + \alpha b)$ modulo L_2 for some α, β with $\alpha \neq \pm\beta$. We then have

$$\begin{aligned} b'a'a'a'a' &= \pm(\alpha^2 - \beta^2)^2(\alpha baaaa + \beta baaab), \\ b'a'a'a'b' &= (\alpha^2 - \beta^2)^2(\beta baaaa + \alpha baaab) \end{aligned}$$

so we can assume that $baaab = 0$ or that $baaab = baaaa$.

24.2.1 $baaab = 0$

If $baaab = 0$ then L_5 is generated by $baaaa$ though we need $\beta = 0$. If $baab = \lambda baaaa$, $babb + baaa = \mu baaaa$, $pa = \nu baaaa$, $pb = \xi baaaa$ then

$$\begin{aligned} b'a'a'b' &= \alpha^4 \lambda baaaa = \pm\alpha^{-1}\lambda b'a'a'a'a', \\ b'a'b'b' + b'a'a'a' &= \pm\alpha^4 \mu baaaa = \alpha^{-1}\mu b'a'a'a'a', \\ pa' &= \alpha\nu baaaa = \pm\alpha^{-4}\nu b'a'a'a'a', \\ pb' &= \pm\alpha\xi baaaa = \alpha^{-4}\xi b'a'a'a'a'. \end{aligned}$$

So we can take $\mu = 0$ or 1. If $\mu = 0$ we can take $\lambda = 0$ or 1, and if $\mu = 1$ we can take $0 \leq \lambda \leq (p-1)/2$.

If $\lambda = \mu = 0$ then We can take $\xi = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. If $\xi \neq 0$ we can take $0 \leq \nu \leq (p-1)/2$. If $\xi = 0$ and $p = 3 \bmod 4$ then we can take $\nu = 0, 1$; if $\xi = 0$ and $p = 5 \bmod 8$ then we can take $\nu = 0, 1, \omega$; and if $\xi = 0$ and $p = 1 \bmod 8$ then we can take $\nu = 0, 1, \omega, \omega^2$ or ω^3.

If $\lambda = 1$ and $\mu = 0$ then we need $\alpha = \pm 1$ so we can take $0 \leq \nu \leq (p-1)/2$ and $0 \leq \xi < p$.

If $\lambda = 0$ and $\mu = 1$ then we need $\alpha = 1$, and again we can take $0 \leq \nu \leq (p-1)/2$ and $0 \leq \xi < p$.

If $\lambda \neq 0$ and $\mu = 1$ then we need $\alpha = 1$ and we are restricted to $a' = a$ modulo L_2, $b' = b$ modulo L_2 so that we need to take $0 \leq \nu, \xi < p$.

24.2.2 $baaab = baaaa$

If $baaab = baaaa$ then L_5 is generated by $baaaa$, and we are restricted to $a' = \alpha a + \beta b$ modulo L_2 and $b' = \beta a + \alpha b$ modulo L_2. If $baab = \lambda baaaa$, $babb + baaa = \mu baaaa$, $pa = \nu baaaa$, $pb = \xi baaaa$ then

$$\begin{aligned}
b'a'a'a'a' &= (\alpha^2 - \beta^2)^2(\alpha + \beta)baaaa, \\
b'a'a'b' &= (\alpha^2 - \beta^2)((\alpha^2 + \beta^2)\lambda + \alpha\beta\mu)baaaa, \\
b'a'b'b' + b'a'a'a' &= (\alpha^2 - \beta^2)((\alpha^2 + \beta^2)\mu + 4\alpha\beta\lambda)baaaa, \\
pa' &= (\alpha\nu + \beta\xi)baaaa, \\
pb' &= (\beta\nu + \alpha\xi)baaaa.
\end{aligned}$$

Considering a', b' as above with $\beta = 0$ we see that we can take $\lambda = 0$ or 1, and if $\lambda = 0$ then we can take $\mu = 0$ or 1. If $\lambda = 1$ we can still choose α, β so that $b'a'a'b' = 0$ provided $\mu \neq \pm 2$ and provided $\mu^2 - 4$ is a square. As we showed in the calculation of the descendants of 6.28 in the computation of the nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$, all values of μ such that $\mu^2 - 4$ is not a square give isomorphic algebras, and so we can assume that (λ, μ) equals $(0, 0)$, $(0, 1)$, $(1, 2)$, $(1, -2)$, $(1, \mu)$ where μ is any øxed value such that $\mu^2 - 4$ is not a square.

If $\lambda = \mu = 0$ then we can assume that $pa = 0$ or that $pa = \pm pb$. If $pa = 0$ but $pb \neq 0$ then we need $\beta = 0$, so we can assume that $\xi = 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. If $pa = pb = \nu baaaa$ then

$$pa' = pb' = (\alpha^2 - \beta^2)^{-2}\nu b'a'a'a'a'.$$

Now $(\alpha^2 - \beta^2)$ takes all possible values, so we can take $\nu = 0, 1, \omega$. If $pa = -pb = \nu baaaa$ then

$$pa' = -pb' = (\alpha - \beta)^{-1}(\alpha + \beta)^{-3}\nu b'a'a'a'a'.$$

But $\alpha - \beta$ can take all possible non-zero values with $\alpha + \beta = 1$, and so we can take $\nu = 0$ or 1.

If $\lambda = 0$, $\mu = 1$ then we need $\alpha = 0$ or $\beta = 0$. If $\beta = 0$ we also need $\alpha = 1$ and so

$$\begin{aligned}
pa' &= \nu b'a'a'a'a', \\
pb' &= \xi b'a'a'a'a'.
\end{aligned}$$

And if $\alpha = 0$ we need $\beta = -1$ giving

$$\begin{aligned}
pa' &= \xi b'a'a'a'a', \\
pb' &= \nu b'a'a'a'a'.
\end{aligned}$$

So we can assume that $0 \leq \nu \leq \xi < p$.

If $\lambda = 1$ and $\mu = 2$ then we need $\alpha - \beta = 1$ so that

$$\begin{aligned} b'a'a'a'a' &= (\alpha+\beta)^3 baaaa, \\ pa' &= (\alpha\nu + \beta\xi) baaaa, \\ pb' &= (\beta\nu + \alpha\xi) baaaa, \end{aligned}$$

where $\alpha + \beta$ can take any non-zero value. If $\nu \neq \pm\xi$ we can assume that $pa = 0$, though we then need $\alpha = 1$, $\beta = 0$ so that we have to take $0 \leq \xi < p$. If $pa = pb = \nu baaaa \neq 0$ then we can assume that $v = 1$ or ω. And if $pa = -pb = \nu baaaa \neq 0$ then we can assume that $\nu = 1$ or (if $p = 1 \bmod 3$) ω or ω^2.

If $\lambda = 1$ and $\mu = -2$ then we need $\alpha - \beta = (\alpha + \beta)^2$ giving

$$\begin{aligned} b'a'a'a'a' &= (\alpha+\beta)^7 baaaa, \\ pa' &= (\alpha\nu + \beta\xi) baaaa, \\ pb' &= (\beta\nu + \alpha\xi) baaaa \end{aligned}$$

where $\alpha + \beta$ can take any non-zero value. Again we can assume that $pa = 0$ unless $pa = \pm pb$. If $pa = 0$, $pb \neq 0$ then we need $\alpha = 1$, $\beta = 0$ so that we have to take $0 \leq \xi < p$. If $pa = pb = \nu baaaa \neq 0$ then we can assume that $v = 1$ or ω or (if $p = 1 \bmod 3$) $\omega^2, \omega^3, \omega^4$ or ω^5. And if $pa = -pb = \nu baaaa \neq 0$ then we can assume that $\nu = 1$ or (if $p = 1 \bmod 5$) $\omega, \omega^2, \omega^3$ or ω^4.

Finally, if $\lambda = 1$ and $\mu^2 - 4$ is not a square then we need $\alpha = 1$, $\beta = 0$, or $\alpha = 0$, $\beta = -1$ so that (as above) we can take $0 \leq \nu \leq \xi < p$.

24.3 Descendants of 6.404

Algebra 6.404 has $(p^3 + p^2 + 2p + 2 + (p+1)\gcd(p-1, 4) + \gcd(p-1, 8))/2$ descendants of order p^7 and p-class 5.

Algebra 6.404 has presentation

$$\langle a, b \mid baab, babb + \omega baaa,\ pa,\ pb, \text{ class } 4\rangle,$$

and so if L is a descendant of 6.404 of order p^7 then L/L_4 is free of characteristic p, L_4 is generated by $baaa$ modulo L_5, and L_5 has order p and is generated by $baaaa, baaab$. We have $baab$, $babb + \omega baaa$, pa, $pd \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \pm(\omega\beta a + \alpha b)$ modulo L_2 for some α, β which are not both zero. We then have

$$\begin{aligned} b'a'a'a'a' &= \pm(\alpha^2 - \omega\beta^2)^2(\alpha baaaa + \beta baaab), \\ b'a'a'a'b' &= (\alpha^2 - \omega\beta^2)^2(\omega\beta baaaa + \alpha baaab) \end{aligned}$$

so we can assume that $baaab = 0$ and that L_5 is generated by $baaaa$, though we then need $\beta = 0$.. If $baab = \lambda baaaa$, $babb + \omega baaa = \mu baaaa$, $pa = \nu baaaa$, $pb = \xi baaaa$ then

$$\begin{aligned} b'a'a'b' &= \alpha^4 \lambda baaaa = \pm\alpha^{-1}\lambda b'a'a'a'a', \\ b'a'b'b' + \omega b'a'a'a' &= \pm\alpha^4 \mu baaaa = \alpha^{-1}\mu b'a'a'a'a', \\ pa' &= \alpha\nu baaaa = \pm\alpha^{-4}\nu b'a'a'a'a', \\ pb' &= \pm\alpha\xi baaaa = \alpha^{-4}\xi b'a'a'a'a'. \end{aligned}$$

So this case is similar to the case when $baaab = 0$ in the descendants of 6.399. We can take $\mu = 0$ or 1. If $\mu = 0$ we can take $\lambda = 0$ or 1, and if $\mu = 1$ we can take $0 \leq \lambda \leq (p-1)/2$.

If $\lambda = \mu = 0$ then We can take $\xi = 0, 1, \omega$ or (if $p = 1 \bmod 4$) ω^2 or ω^3. If $\xi \neq 0$ we can take $0 \leq \nu \leq (p-1)/2$. If $\xi = 0$ and $p = 3 \bmod 4$ then we can take $\nu = 0, 1$; if $\xi = 0$ and $p = 5 \bmod 8$ then we can take $\nu = 0, 1, \omega$; and if $\xi = 0$ and $p = 1 \bmod 8$ then we can take $\nu = 0, 1, \omega, \omega^2$ or ω^3.

If $\lambda = 1$ and $\mu = 0$ then we need $\alpha = \pm 1$ so we can take $0 \leq \nu \leq (p-1)/2$ and $0 \leq \xi < p$.

If $\lambda = 0$ and $\mu = 1$ then we need $\alpha = 1$, and again we can take $0 \leq \nu \leq (p-1)/2$ and $0 \leq \xi < p$.

If $\lambda \neq 0$ and $\mu = 1$ then we need $\alpha = 1$ and we are restricted to $a' = a$ modulo L_2, $b' = b$ modulo L_2 so that we need to take $0 \leq \nu, \xi < p$.

25 Grandchildren of algebra 7 (5.41)

All øve descendants of 5.41 of order p^6 have descendants of order p^7 and p-class 5 (6.420 ˘ 6.424).

25.1 Descendants of 6.420

Algebra 6.420 has $5p + 1 + \gcd(p-1, 4)$ descendants of order p^7 and p-class 5.

Algebra 6.420 has presentation

$$\langle a, b \mid pa - baa,\ pb,\ \text{class } 4 \rangle,$$

and so if L is a descendant ot 6.420 of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $babb$ modulo L_5, and L_5 has order p and is generated by $babba$ and $babbb$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a$ modulo L_2 and $b' = \alpha^{-1} b$ modulo L_2 and we have

$$\begin{aligned} b'a'b'b'a' &= \alpha^{-1} babba, \\ b'a'b'b'b' &= \alpha^{-3} babbb. \end{aligned}$$

So we can assume that one of the following four relations holds: $babba = 0$, $babbb = 0$, $babbb = babba$, $babbb = \omega babba$.

25.1.1 $babba = 0$

If $babba = 0$ then L_5 is generated by $babbb$, and $pa - baa$, pb are scalar multiple of $babbb$. We have

$$\begin{aligned} b'a'b'b'b' &= \alpha^{-3} babbb, \\ pa' - b'a'a' &= \alpha(pa - baa), \\ pb' &= \alpha^{-1} pb, \end{aligned}$$

and so if $pb \neq 0$ we can assume that $pb = babbb$ or $\omega babbb$ and that $pa - baa = \lambda babbb$ with $0 \leq \lambda < p$. If $pb = 0$ then we can assume that $pa - baa = 0$, $babbb$, $\omega babbb$ or (if $p = 1 \bmod 4$) $\omega^2 babbb$ or $\omega^3 babbb$ ($2p + 1 + \gcd(p-1, 4)$ algebras).

25.1.2 $babbb = 0$, $babba$ or $\omega babba$

In these cases L_5 is generated by $babba$, and adding a suitable scalar multiple of bab to a we can assume that $pa = baa$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^{-1} babba, \\ pb' &= \alpha^{-1} pb, \end{aligned}$$

and so we can assume that $pb = \lambda babba$ with $0 \leq \lambda < p$ ($3p$ algebras).

25.2 Descendants of 6.421 ˇ 6.423

These algebras have $2p^2 - p - 1 + (p+1)\gcd(p-1,3)$ descendants of order p^7 and p-class 5.

Algebras 6.421 ˇ 6.423 have presentations

$$\langle a, b \mid pa - baa - kbabb,\, pb,\, \text{class } 4\rangle,$$

where $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. So if L is a descendant of one of these algebras of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $babb$ modulo L_5, and L_5 has order p and is generated by $babba$ and $babbb$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a$ modulo L_2 and $b' = \alpha^{-1} b$ modulo L_2 for some α with $\alpha^3 = 1$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^{-1} babba, \\ b'a'b'b'b' &= babbb. \end{aligned}$$

So we can assume that one of the following relations holds: $babba = 0$, $babbb = 0$, $babbb = \lambda babba$ where λ lies in a transversal for the cube roots of unity.

25.2.1 $babba = 0$

If $babba = 0$ then L_5 is generated by $babbb$, and $pa - baa - kbabb$, pb are scalar multiple of $babbb$. We have

$$\begin{aligned} b'a'b'b'b' &= babbb, \\ pa' - b'a'a' &= \alpha(pa - baa), \\ pb' &= \alpha^{-1} pb, \end{aligned}$$

so if $pa - baa - kbabb \neq 0$ we can assume that $pa - baa - kbabb = \mu babbb$ where μ lies in a transversal for the cube roots of unity, and we can assume that $pb = \nu babbb$ with $0 \leq \nu < p$. If $pa - baa - kbabb = 0$ we can assume that $pb = 0$ or that $pb = \mu babbb$ where μ lies in a transversal for the cube roots of unity ($p^2 - 1 + \gcd(p-1,3)$ algebras).

25.2.2 $babbb = 0$

If $babbb = 0$ then L_5 is generated by $babba$, and $pa - baa - kbabb$, pb are scalar multiple of $babba$. Adding a suitable scalar multiple of bab to a we can assume that $pa - baa - kbabb = 0$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^{-1} babba, \\ pb' &= \alpha^{-1} pb, \end{aligned}$$

and so we can assume that $pb = \lambda babba$ with $0 \leq \lambda < p$ ($p \gcd(p-1,3)$ algebras).

25.2.3 $babbb = \lambda babba$

If $babbb = \lambda babba$ (with λ in a transversal for the cube roots of unity) then L_5 is generated by $babba$, and $pa - baa - kbabb$, pb are scalar multiple of $babba$, though we now need $\alpha = 1$.

Adding a suitable scalar multiple of bab to a we can assume that $pa - baa - kbabb = 0$. We then have

$$\begin{aligned} b'a'b'b'a' &= babba, \\ pb' &= pb, \end{aligned}$$

and so we can assume that $pb = \mu babba$ with $0 \leq \mu < p$ ($p^2 - p$ algebras).

25.3 Descendants of 6.424

Algebra 6.424 has presentation

$$\langle a, b \mid pa - baa - \lambda babb,\ pb - babb,\ \text{class } 4\rangle\ (0 \leq \lambda < p),$$

so if L is a descendant of 6.424 of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $babb$ modulo L_5, and L_5 has order p and is generated by $babba$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = a$ modulo L_2 and $b' = b$ modulo L_2, and so we have

$$\begin{aligned} b'a'b'b'a' &= babba, \\ pb' - b'a'b'b' &= pb - babb. \end{aligned}$$

But adding a suitable scalar multiple of bab to a we can assume that $pa - baa - \lambda babb = 0$. So we have p^2 algebras

$$\langle a, b \mid pa - baa - \lambda babb,\ pb - babb - \mu babba,\ \text{class } 5\rangle\ (0 \leq \lambda, \mu < p).$$

26 Grandchildren of algebra 8 (5.39)

Algebra 5.39 has 6 descendants of order p^6 (6.408 ˘ 6.413), and they all have descendants of order p^7 and p-class 5.

26.1 Descendants of 6.408 ˘ 6.410

These algebras have presentation

$$\langle a, b \mid pa - bab,\ pb - kbaaa,\ \text{class } 4\rangle,$$

where $k = 1, \omega, \omega^2$, with $k = \omega, \omega^2$ only arising if $p = 1 \bmod 3$. If L is a descendant of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baaa$ modulo L_5, and L_5 is generated by $baaaa$. We have $pa - bab$, $pb - kbaaa \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a$ modulo L_2 where $\alpha^3 = 1$ and $b' = \pm b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a'a' &= \pm\alpha baaaa, \\ pa' - b'a'b' &= \alpha(pa - bab), \\ pb' - b'a'a'a' &= \pm(pb - baaa). \end{aligned}$$

So we can assume that $pa - bab = \lambda baaaa$ with $0 \leq \lambda \leq (p-1)/2$ and that $pb - baaa = 0$ or $\mu baaa$ where μ lies in a transversal for the cube roots of unity. if $p = 1 \bmod 3$ we $(p-1)/3 + 1$ values of μ for each of 3 values of k, and if $p = 2 \bmod 3$ then we have p values of μ for one value of k. So we have $\frac{1}{2}(p+1)(p-1+\gcd(p-1,3))$ algebras here:

$$\langle a, b \mid pa - bab - \lambda baaaa,\ pb - kbaaa - \mu baaaa,\ \text{class } 5\rangle,$$

where $0 \leq \lambda \leq (p-1)/2$, $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2, and $\mu = 0$ or μ lies in a transversal for the cube roots of unity.

Algebra 6.411 has $2+2\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7 and p-class 5.

Algebra 6.411 has presentation

$$\langle a,b \mid pa-bab,\, pb,\, \text{class } 4\rangle,$$

so if L is a descendant of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baaa$ modulo L_5, and L_5 is generated by $baaaa, baaab$. We have $pa-bab$, $pb \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \pm b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a'a' &= \pm\alpha^4 baaaa \pm \alpha^3\beta baaab, \\ b'a'a'a'b' &= \alpha^3 baaab \end{aligned}$$

and so we can assume that $baaaa = 0$ or that $baaab = 0$.

26.2.1 $baaaa = 0$

If $baaaa = 0$ then L_5 is generated by $baaab$, though we then need $\beta = 0$. Adding a suitable scalar multiple of baa to b we may assume that $pa = bab$. We then have

$$\begin{aligned} b'a'a'a'b' &= \alpha^3 baaab, \\ pb' &= \pm pb, \end{aligned}$$

and so we can assume that $pb = 0$, $baaab$ or (if $p = 1 \bmod 3$) $\omega baaab$ or $\omega^2 baaab$, giving

$$\langle a,b \mid baaaa,\, pa-bab,\, pb,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaaa,\, pa-bab,\, pb-baaab,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaaa,\, pa-bab,\, pb-\omega baaab,\, \text{class } 5\rangle\ (p=1 \bmod 3),$$

$$\langle a,b \mid baaaa,\, pa-bab,\, pb-\omega^2 baaab,\, \text{class } 5\rangle\ (p=1 \bmod 3).$$

26.2.2 $baaab = 0$

If $baaab = 0$ then L_5 is generated by $baaaa$, and we have

$$\begin{aligned} b'a'a'a'a' &= \pm\alpha^4 baaaa, \\ pa' - b'a'b' &= \alpha(pa-bab) + \beta pb, \\ pb' &= \pm pb \end{aligned}$$

so we can assume that $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. If $pb \neq 0$ we can assume that $pa = bab$, and if $pb = 0$ we can assume that $pa - bab = 0$, $baaaa$ or (if $p = 1 \bmod 3$) $\omega baaaa$ or $\omega^2 baaaa$ giving

$$\langle a,b \mid baaab,\, pa-bab,\, pb,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaab,\, pa-bab-baaaa,\, pb,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaab,\, pa-bab-\omega baaaa,\, pb,\, \text{class } 5\rangle\ (p=1 \bmod 3),$$

$$\langle a,b \mid baaab,\, pa-bab-\omega^2 baaaa,\, pb,\, \text{class } 5\rangle\ (p=1 \bmod 3),$$

$$\langle a,b \mid baaab,\, pa-bab,\, pb-baaaa,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaab,\, pa-bab,\, pb-\omega baaaa,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baaab,\, pa-bab,\, pb-\omega^2 baaaa,\, \text{class } 5\rangle\ (p=1 \bmod 4),$$

$$\langle a,b \mid baaab,\, pa-bab,\, pb-\omega^3 baaaa,\, \text{class } 5\rangle\ (p=1 \bmod 4).$$

These two algebras have $p-1+\frac{1}{2}(p+1)\gcd(p-1,4)$ descendants of order p^7 and p-class 5.

Algebras 6.412 and 6.413 have presentations

$$\langle a,b \mid pa-bab-kbaaa,\, pb,\, \text{class } 4\rangle$$

where $k=1,\omega$ with the case $k=\omega$ only arising if $p=1 \bmod 4$. If L is a descendant of one of these algebras of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 is generated by $baaaa, baaab$. If a', b' generate L and satisfy the same relations as a, b then $a'=\alpha a+\beta b$ modulo L_2 and $b'=\gamma b$ modulo L_2, where $\gamma=\pm 1$, and $\alpha^2=\gamma$. (So if $p=3 \bmod 4$ then $\gamma=1$ and $\alpha=\pm 1$.) We then have

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\gamma baaaa+\alpha^3\beta\gamma baaab, \\ b'a'a'a'b' &= \alpha^3 baaab \end{aligned}$$

so we can assume that $baaaa=0$ or that $baaab=0$.

26.3.1 $baaaa=0$

If $baaaa=0$ then L_5 is generated by $baaab$ though we need $\beta=0$. By adding a suitable scalar multiple of baa to b we may assume that $pa-bab-kbaaa=0$. We have

$$\begin{aligned} b'a'a'a'b' &= \alpha^3 baaab, \\ pb' &= \gamma pb \end{aligned}$$

and so if $p=1 \bmod 4$ we can take $pb=\lambda baaab$ where $\lambda=0$ or λ lies in a transversal for the fourth roots of unity, and if $p=3 \bmod 4$ we can take $pb=\lambda baaab$ where $0\leq\lambda\leq(p-1)/2$ so we have $(p-1+\gcd(p-1,4))/2$ algebras

$\langle a,b \mid baaaa,\, pa-bab-baaa,\, pb-\lambda baaab,\, \text{class } 5\rangle$ (λ,μ give same algebras if $\lambda^4=\mu^4$),

$\langle a,b \mid baaaa,\, pa-bab-\omega baaa,\, pb-\lambda baaab,\, \text{class } 5\rangle$ ($p=1 \bmod 4$, λ,μ give same algebras if $\lambda^4=\mu^4$).

26.3.2 $baaab=0$

If $baaab=0$ then L_5 is generated by $baaaa$ and we have

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\gamma baaaa, \\ pa'-b'a'b'-kb'a'a'a' &= 2k^2\alpha^2\beta\gamma baaaa+\alpha(pa-bab-kbaaa)+\beta pb, \\ pb' &= \gamma pb. \end{aligned}$$

Now $\alpha^4=1$ and $\alpha^2\gamma=1$, and so we can assume that $pb=\lambda baaaa$ where $0\leq\lambda<p$, and if $\lambda\neq-2k^2$ we can assume that $pa-bab-kbaaa=0$. But if $\lambda=-2k^2$ then we can assume that

$$pa-bab-kbaaa=\mu baaaa$$

where $\mu=0$ or μ lies in a transversal for the fourth roots of unity. So we have $(p-1+p\gcd(p-1,4))/2$ algebras

$$\langle a,b \mid baaab,\, pa-bab-baaa,\, pb-\lambda baaaa,\, \text{class } 5\rangle\ (0\leq\lambda<p),$$

$$\langle a,b \mid baaab,\, pa-bab-\omega baaa,\, pb-\lambda baaaa,\, \text{class } 5\rangle\ (p=1 \bmod 4,\ 0\leq\lambda<p),$$

$\langle a,b \mid baaab,\, pa-bab-baaa-\lambda baaaa,\, pb+2baaaa,\, \text{class } 5\rangle$ ($\lambda\neq 0$, λ,μ give same algebras if $\lambda^4=\mu^4$),

$$\langle a,b \mid baaab,\, pa-bab-\omega baaa-\lambda baaab,\, pb+2\omega^2 baaaa,\, \text{class } 5\rangle\ (p=1 \bmod 4)$$

where $\lambda\neq 0$, λ,μ give same algebras if $\lambda^4=\mu^4$.

27 Grandchildren of algebra 9 (5.40)

Algebra 5.40 has 6 descendants of order p^6 (6.414 ˘ 6.419), all of which have descendants of order p^7 and p-class 5.

27.1 Descendants of 6.414 ˘ 6.416

These algebras have presentations

$$\langle a, b \mid pa - \omega bab,\, pb - kbaaa,\, \text{class } 4\rangle,$$

with $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. This case is almost identical to the descendants of 6.408 ˘ 6.410 and we have $\frac{1}{2}(p+1)(p-1+\gcd(p-1,3))$ algebras here:

$$\langle a, b \mid pa - \omega bab - \lambda baaaa,\, pb - kbaaa - \mu baaaa,\, \text{class } 5\rangle,$$

where $0 \leq \lambda \leq (p-1)/2$, $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2, and $\mu = 0$ or μ lies in a transversal for the cube roots of unity.

27.2 Descendants of 6.417

Algebra 6.417 has presentation

$$\langle a, b \mid pa - \omega bab,\, pb,\, \text{class } 4\rangle,$$

and so this case is almost identical to the descendants of 6.411. So we have $2 + 2\gcd(p-1,3) + \gcd(p-1,4)$ descendants of order p^7 and p-class 5:

$$\langle a, b \mid baaaa,\, pa - \omega bab,\, pb,\, \text{class } 5\rangle,$$

$$\langle a, b \mid baaaa,\, pa - \omega bab,\, pb - baaab,\, \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\, pa - \omega bab,\, pb - \omega baaab,\, \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaaa,\, pa - \omega bab,\, pb - \omega^2 baaab,\, \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\, pa - \omega bab,\, pb,\, \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\, pa - \omega bab - baaaa,\, pb,\, \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\, pa - \omega bab - \omega baaaa,\, pb,\, \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\, pa - \omega bab - \omega^2 baaaa,\, pb,\, \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\, pa - \omega bab,\, pb - baaaa,\, \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\, pa - \omega bab,\, pb - \omega baaaa,\, \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\, pa - \omega bab,\, pb - \omega^2 baaaa,\, \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid baaab,\, pa - \omega bab,\, pb - \omega^3 baaaa,\, \text{class } 5\rangle\ (p = 1 \bmod 4).$$

27.3 Descendants of 6.418 and 6.419

Algebras 6.418 and 6.419 have presentations

$$\langle a, b \mid pa - \omega bab - kbaaa,\ pb,\ \text{class } 4\rangle,$$

where $k = 1$ or (if $p = 1 \bmod 4$) ω, so this case is almost identical to the descendants of 6.412 and 6.413. So we have $p - 1 + \frac{1}{2}(p+1)\gcd(p-1, 4)$ descendants of order p^7 and p-class 5:

$\langle a, b \mid baaaa,\ pa - \omega bab - baaa,\ pb - \lambda baaab,\ \text{class } 5\rangle$ (λ, μ give same algebras if $\lambda^4 = \mu^4$),

$\langle a, b \mid baaaa,\ pa - \omega bab - \omega baaa,\ pb - \lambda baaab,\ \text{class } 5\rangle$ ($p = 1 \bmod 4$, λ, μ give same algebras if $\lambda^4 = \mu^4$),

$$\langle a, b \mid baaab,\ pa - \omega bab - baaa,\ pb - \lambda baaaa,\ \text{class } 5\rangle\ (0 \le \lambda < p),$$

$$\langle a, b \mid baaab,\ pa - \omega bab - \omega baaa,\ pb - \lambda baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4,\ 0 \le \lambda < p),$$

$\langle a, b \mid baaab,\ pa - \omega bab - baaa - \lambda baaaa,\ pb + 2\omega^{-1} baaaa,\ \text{class } 5\rangle$ ($\lambda \neq 0$, λ, μ give same algebras if $\lambda^4 = \mu^4$),

$$\langle a, b \mid baaab,\ pa - \omega bab - \omega baaa - \lambda baaab,\ pb + 2\omega baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4)$$

where $\lambda \neq 0$, λ, μ give same algebras if $\lambda^4 = \mu^4$.

28 Grandchildren of algebra 10 (5.42)

Algebra 5.42 has two descendants of order p^6 (6.425 and 6.426), both of which have descendants of order p^7.

28.1 Descendants of 6.425

Algebra 6.425 has presentation

$$\langle a, b \mid pa - baa,\ pb + bab,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.425 of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baab$ modulo L_5 and L_5 is generated by $baaba$, $baabb$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then either $a' = \alpha a$ modulo L_2 and $b' = \alpha^{-1} b$ modulo L_2 or $a' = \alpha b$ modulo L_2 and $b' = \alpha^{-1} a$ modulo L_2 and

$$\begin{aligned} b'a'a'b'a' &= \alpha baaba, \\ b'a'a'b'b' &= \alpha^{-1} baabb \end{aligned}$$

or

$$\begin{aligned} b'a'a'b'a' &= -\alpha baabb, \\ b'a'a'b'b' &= -\alpha^{-1} baaba. \end{aligned}$$

Considering a', b' of the ørst kind we can assume that $baaba = 0$ or that $baabb = 0$, $baaba$ or $\omega baaba$. However, by considering a', b' of the second kind we see that the case $baaba = 0$ is equivalent to the case $baabb = 0$. If $baabb = baaba$ then taking a', b' of the second kind we have

$$\begin{aligned} b'a'a'b'a' &= -\alpha baaba, \\ b'a'a'b'b' &= -\alpha^{-1} baaba, \end{aligned}$$

so the cases $baabb = baaba$ and $baabb = \omega baaba$ are distinct. So we may assume that $babba = 0$, $baaba$ or $\omega baaba$, and that L_5 is generated by $baaba$.

28.1.1 $baabb = 0$

If $baabb = 0$ then we are restricted to a', b' of the form $\alpha a, \alpha^{-1} b$ modulo L_2. Adding a suitable scalar multiple of baa to a we may assume that $pa = baa$, and adding a suitable scalar multiple of baa to b we may assume that $pb = -bab$. So we have

$$\langle a, b \mid baabb, pa - baa,\ pb + bab, \text{ class } 5\rangle.$$

28.1.2 $baabb = baaba$ or $\omega baaba$

We are now restricted to a', b' as above (both types), with $\alpha = \pm 1$. As above we can assume that $pa = baa$, $pb = -bab$, giving

$$\langle a, b \mid baabb - baaba, pa - baa,\ pb + bab, \text{ class } 5\rangle,$$

$$\langle a, b \mid baabb - \omega baaba, pa - baa,\ pb + bab, \text{ class } 5\rangle.$$

28.2 Descendants of 6.426

Algebra 6.426 has presentation

$$\langle a, b \mid pa - baa - baab,\ pa + bab - \nu baab, \text{ class } 4\rangle\ (0 \le \nu < p),$$

and so if L is a descendant of 6.425 of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baab$ modulo L_5 and L_5 is generated by $baaba$. If we replace a by $a + \lambda baa$ for suitable λ we can assume that $pa - baa - baab = 0$, and if we replace b by $b + \mu baa$ for suitable μ we can assume that $pa + bab - \nu baab = 0$. So we have

$$\langle a, b \mid pa - baa - baab,\ pa + bab - \nu baab, \text{ class } 5\rangle\ (0 \le \nu < p).$$

29 Grandchildren of algebra 11 (5.45)

Algebra 5.45 has one two parameter family of descendants of order p^6 (6.427), and these all have descendants of order p^7.

29.1 Descendants of 6.427

As the parameters of 6.427 vary there are p diœerent algebras of order p^6, and all but one of them has exactly one descendant of order p^7. When $\lambda = \mu = 0$ (see below) the algebra has two descendants. So there are $p + 1$ descendants of order p^7.

Algebra 6.427 has presentation

$$\langle a, b \mid pa + bab - \lambda baaa,\ pb + \omega baa - \mu baaa, \text{ class } 4\rangle\ (0 \le \lambda, \mu < p),$$

with the isomorphism class depending only on the value of $\mu^2 - \omega\lambda^2$. So if L is a descendant of 6.427 of order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa, bab modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 is generated by $baaaa$ and $baaab$. If $\mu \ne 0$ then (as a consequence of the identities $[pa, a] = [pb, b] = 0$) $baaab$ is a scalar multiple of $baaaa$. And if $\lambda \ne 0$ and $\mu = 0$ then the same identities imply that $baaaa = 0$.

If $\mu \ne 0$ then adding suitable scalar multipes of baa and bab to a we can assume that

$$pa + bab - \lambda baaa = pb + \omega baa - \mu baaa = 0.$$

And if $\mu = 0$ and $\lambda \neq 0$ then adding suitable scalar multipes of baa and bab to b we can similarly assume that

$$pa + bab - \lambda baaa = pb + \omega baa - \mu baaa = 0.$$

So consider the case when $\lambda = \mu = 0$. In the covering algebra for 6.427 the elements $baaaa$ and $baaab$ are linearly independent. But if a', b' generate L and satisfy the same relations as a, b modulo L_4, then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \pm(\omega\beta a + \alpha b) \text{ modulo } L_2 \end{aligned}$$

for some α, β with $\alpha^2 - \omega\beta^2 = 1$. This gives

$$\begin{aligned} b'a'a'a'a' &= \pm\alpha baaaa \pm \beta baaab, \\ b'a'a'a'b' &= \omega\beta baaaa + \alpha baaab. \end{aligned}$$

The elements $baaaa$ and $baaab$ are linearly dependant and so we have

$$\omega\mu baaaa + \lambda baaab = 0$$

for some λ, μ which are not both zero. If $\lambda^2 - \omega\mu^2$ is a square then we can choose λ, μ so that $\lambda^2 - \omega\mu^2 = 1$, and so we can assume that $baaab = 0$. However, if $\lambda^2 - \omega\mu^2$ is not a square, then we cannot assume that $baaab = 0$. We show that all the cases when $\lambda^2 - \omega\mu^2$ is not a square are equivalent up to isomorphism. We see that

$$\begin{pmatrix} baaaa \\ baaab \end{pmatrix} = \begin{pmatrix} \pm\alpha & \pm\beta \\ \omega\beta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} b'a'a'a'a' \\ b'a'a'a'b' \end{pmatrix}$$

so that

$$(\omega\mu, \lambda) \longmapsto (\omega\mu, \lambda) \begin{pmatrix} \pm\alpha & \pm\beta \\ \omega\beta & \alpha \end{pmatrix}^{-1}$$

and we show that there are three orbits of 2-vectors (including the vector $(0, 0)$) under this group of transformations. (For this calculation we remove the restriction $\alpha^2 - \omega\beta^2 = 1$, and replace it by the restriction that $\alpha^2 - \omega\beta^2$ is a square. Note that this gives a group of order $(p^2 - 1)/2$.)

The group element $\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ øxes the 2-vector (x, y) if

$$\begin{aligned} \alpha x + \omega\beta y &= x, \\ \beta x + \alpha y &= y. \end{aligned}$$

So the number of 2-vectors øxed by this group element is p^n, where n is the nullity of

$$\begin{pmatrix} \alpha - 1 & \omega\beta \\ \beta & \alpha - 1 \end{pmatrix}.$$

However this matrix has nullity 0 unless $\alpha = 1$ and $\beta = 0$, when it has nullity 2. So as we run over the $p^2 - 1$ non-singular matrices $\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ we get one matrix øxing p^2 2-vectors, and $(p^2 - 3)/2$ matrices øxing 1 2-vector. So

$$\sum_{g \in G} fix(g) = 3(p^2 - 1)/2.$$

So there are three orbits under the group of matrices $\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$ with $\alpha^2 - \omega\beta^2$ a square. A similar calculation shows that there are still three orbits under the bigger group of matrices $\begin{pmatrix} \pm\alpha & \pm\beta \\ \omega\beta & \alpha \end{pmatrix}$ with $\alpha^2 - \omega\beta^2$ a square.

So we can assume that $baaab = 0$, or that $\omega\mu baaaa + \lambda baaab = 0$ for one choice of λ, μ with $\lambda^2 - \omega\mu^2 = \omega$. As above we can assume that

$$pa + bab - \lambda baaa = pb + \omega baa - \mu baaa = 0.$$

So if λ, μ are not both zero we have $p-1$ algebras

$$\langle a, b \mid pa + bab - \lambda baaa,\ pb + \omega baa - \mu baaa,\ \text{class } 5\rangle\ (0 \leq \lambda, \mu < p),$$

and if $\lambda = \mu = 0$ we have

$$\langle a, b \mid baaab,\ pa + bab,\ pb + \omega baa,\ \text{class } 5\rangle,$$

and one algebra

$$\langle a, b \mid \omega\mu baaaa + \lambda baaab,\ pa + bab,\ pb + \omega baa,\ \text{class } 5\rangle\ (\lambda^2 - \omega\mu^2 = \omega).$$

30 Grandchildren of algebra 12 (5.47)

Algebras 5.47 has two descendants of order p^6 (6.428 and 6.429), both of which have descendants of order p^7.

30.1 Descendants of 6.428

Algebra 6.428 has presentation

$$\langle a, b \mid ba,\ p^2b,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.428 of order p^7 then L_2 is generated by pa, pb modulo L_2, L_3 is generated by p^2a modulo L_4, L_4 is generated by p^3a modulo L_5 and L_5 is generated by p^4a. Adding a suitable scalar multiple of p^2a to b we may assume that $p^2b = 0$, but we have $ba = \lambda p^4a$ for some λ. Clearly we can take $\lambda = 0$ or 1, giving

$$\langle a, b \mid ba,\ p^2b,\ \text{class } 5\rangle,$$

$$\langle a, b \mid ba - p^4a,\ p^2b,\ \text{class } 5\rangle.$$

30.2 Descendants of 6.429

Algebra 6.429 has presentation

$$\langle a, b \mid ba - p^3a,\ p^2b,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.428 of order p^7 then L_2 is generated by pa, pb modulo L_2, L_3 is generated by p^2a modulo L_4, L_4 is generated by p^3a modulo L_5 and L_5 is generated by p^4a. Adding a suitable scalar multiple of p^2a to b we may assume that $p^2b = 0$, and adding a suitable scalar multiple of pb to b we may assume that $ba = p^3a$. So we have

$$\langle a, b \mid ba - p^3a,\ p^2b,\ \text{class } 5\rangle.$$

31 Grandchildren of algebra 13 (5.48)

Algebras 5.48 has one descendants of order p^6, which is terminal.

32 Grandchildren of algebra 14 (5.49)

Algebras 5.49 has four descendants of order p^6, but all but one (6.431) are terminal.

32.1 Descendants of 6.431

Algebra 6.431 has 4 descendants of order p^7 and p-class 5.

Algebra 6.431 has presentation

$$\langle a, b \mid baa,\ bab,\ pb,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.431 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by p^2a modulo L_4, L_4 is generated by p^3a modulo L_5 and L_5 is generated by p^4a. The elements baa, bab, pb all lie in L_5, but adding a suitable scalar multiple of p^3a to b we may assume that $pb = 0$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2, $b' = \gamma b$ modulo L_2, and

$$\begin{aligned} p^4a' &= \alpha p^4 a, \\ b'a'a' &= \alpha^2\gamma baa + \alpha\beta\gamma bab, \\ b'a'b' &= \alpha\gamma^2 bab. \end{aligned}$$

So we can assume that $baa = 0$ and that $bab = 0$, p^4a or ωp^4a, or that $bab = 0$ and $baa = p^4a$. so we have

$$\langle a, b \mid baa,\ bab,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa,\ bab - p^4a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa,\ bab - \omega p^4a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa - p^4a,\ bab,\ pb,\ \text{class } 5\rangle.$$

33 Grandchildren of algebra 15 (5.50)

Algebra 5.50 has 13 descendants of order p^6, but only eight of these (6.435, 6.436, 6.442 ˘ 6.447) have descendants of order p^7.

33.1 Descendants of 6.435

Algebra 6.435 has $7 + 2\gcd(p-1, 3) + 2\gcd(p-1, 4)$ descendants of order p^7 and p-class 5.

Algebra 6.435 has presentation

$$\langle a, b \mid bab,\ p^2a,\ pb,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.435 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 has order p and is generated by $baaaa, baaab$. We have bab, p^2a and $pb \in L_5$. If a', b' generate L

and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\gamma baaaa + \alpha^3\beta\gamma baaab, \\ b'a'a'a'b' &= \alpha^3\gamma^2 baaab \end{aligned}$$

and so we can assume that $baaaa = 0$ or that $baaab = 0$.

33.1.1 $baaaa = 0$

If $baaaa = 0$ then L_5 is generated by $baaab$ though we need $\beta = 0$. Adding a suitable scalar multiple of baa to b we may assume that $bab = 0$, and if $p^2a \neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb = 0$. Now

$$\begin{aligned} b'a'a'a'b' &= \alpha^3\gamma^2 baaab, \\ p^2a' &= \alpha pa, \end{aligned}$$

and so we can assume that $p^2a = 0$, $baaab$ or $\omega baaab$. If $p^2a = 0$ then $pb' = \gamma pb$ so we can assume that $pb = 0$ or $baaab$. We have

$$\langle a, b \mid baaaa,\ bab,\ p^2a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaaa,\ bab,\ p^2a - baaab,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab,\ p^2a - \omega baaab,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab,\ p^2a,\ pb - baaab,\ \text{class } 5\rangle.$$

33.1.2 $baaab = 0$

If $baaab = 0$ then L_5 is generated by $baaaa$. We then have

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\gamma baaaa, \\ b'a'b' &= \alpha\gamma^2 bab, \\ p^2a' &= \alpha pa. \end{aligned}$$

If $p^2a \neq 0$ we can assume that $p^2a = baaaa$ (though we then need $\gamma = \alpha^{-3}$), as above we can assume that $pb = 0$, and we can assume that $bab = 0$, $baaaa$, $\omega baaaa$, or (if $p = 1 \bmod 3$) $\omega^2 baaaa$, $\omega^3 baaaa$, $\omega^4 baaaa$ or $\omega^5 baaaa$. If $p^2a = 0$ we have $pb' = \gamma pb$. So if $p^2a = 0$ we can assume that $bab = 0$ or $baaaa$. If $p^2a = bab = 0$ then we can assume that $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. If $p^2a = 0$, $bab = baaaa$ then we need $\gamma = \alpha^3$, and again we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. We have

$$\langle a, b \mid baaab,\ bab,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab - \omega baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab - \omega^2 baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\ bab - \omega^3 baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\ bab - \omega^4 baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b \mid baaab,\ bab - \omega^5 baaaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b \mid baaab,\ bab,\ p^2a,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab,\ p^2a,\ pb - baaaa,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a,\ pb - baaaa,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab,\ p^2a,\ pb - \omega baaaa,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a,\ pb - \omega baaaa,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaab,\ bab,\ p^2a,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid baaab,\ bab,\ p^2a,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid baaab,\ bab - baaaa,\ p^2a,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4).$$

33.2 Descendants of 6.436

Algebra 6.436 has $2 + 2\gcd(p-1, 3) + \gcd(p-1, 4) + 2\gcd(p-1, 5)$ descendants of order p^7 and p-class 5.

Algebra 6.436 has presentation

$$\langle a, b \mid bab - baaa,\ p^2a,\ pb,\ \text{class } 4\rangle,$$

so if L is a descendant of 6.436 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 has order p and is generated by $baaaa, baaab$. We have $bab - baaa$, p^2a and $pb \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \alpha^2 b$ modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^6 baaaa + \alpha^5\beta baaab, \\ b'a'a'a'b' &= \alpha^7 baaab \end{aligned}$$

and so we can assume that $baaaa = 0$ or that $baaab = 0$.

33.2.1 $baaaa = 0$

If $baaaa = 0$ then L_5 is generated by $baaab$, though we then need $\beta = 0$. Adding a suitable scalar multiple of baa to b we may assume that $bab = baaa$. We then have

$$\begin{aligned} b'a'a'a'b' &= \alpha^7 baaab, \\ p^2a' &= \alpha pa, \end{aligned}$$

and so we can assume that $p^2a = 0$, $baaab$, $\omega baaab$ or (if $p = 1 \bmod 3$) $\omega^2 baaab$, $\omega^3 baaab$, $\omega^4 baaab$ or $\omega^5 baaab$. If $p^2a \neq 0$ we can add a suitable scalar multiple of pa to b so that $pb = 0$. And if $p^2a = 0$ then we have $pb' = \alpha^2 pb$ so that we can assume that $pb = 0$, $baaab$ or (if $p = 1 \bmod 5$) $\omega baaab$, $\omega^2 baaab$, $\omega^3 baaab$ or $\omega^4 baaab$. We have

$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a - baaab,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a - \omega baaab,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a - xbaaab,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 3, x = \omega^2, \omega^3, \omega^4, \omega^5),$$
$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a,\ pb - baaab,\ \text{class } 5\rangle,$$
$$\langle a, b \mid baaaa,\ bab - baaa,\ p^2a,\ pb - xbaaab,\ \text{class } 5\rangle\ (p = 1 \bmod 5, x = \omega, \omega^2, \omega^3, \omega^4).$$

33.2.2 $baaab = 0$

If $baaab = 0$ then L_5 is generated by $baaaa$. We then have

$$\begin{aligned} b'a'a'a'a' &= \alpha^6 baaaa, \\ b'a'b' - b'a'a'a' &= -2\alpha^4\beta baaaa + \alpha^5(bab - baaa), \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

So we can assume that $bab = baaa$, and we can assume that $p^2a = 0$, $baaaa$ or (if $p = 1 \bmod 5$) $\omega baaaa$, $\omega^2 baaaa$, $\omega^3 baaaa$ or $\omega^4 baaaa$. If $p^2a \neq 0$ we can add a suitable scalar multiple of pa to b so that $pb = 0$, and if $p^2a = 0$ we have $pb' = \alpha^2 pb$ so that we can assume that $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$.

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a - baaaa,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a - xbaaaa,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 5, x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a,\ pb - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a,\ pb - \omega baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a,\ pb - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid baaab,\ bab - baaa,\ p^2a,\ pb - \omega^3 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4).$$

33.3 Descendants of 6.442 ˇ 6.444

Algebras 6.442 ˇ 6.444 have $2p - 2 + (p + 2)\gcd(p - 1, 3)$ descendants of order p^7 and p-class 5.

Algebras 6.442 ˇ 6.444 have presentations

$$\langle a, b \mid bab,\ p^2a,\ pb - kbaaa,\ \text{class } 4\rangle,$$

where $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. So if L is a descendant of 6.442 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 has order p and is generated by $baaaa$. We have bab, p^2a and $pb - kbaaa \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2 where $\alpha^3 = 1$, and

$$\begin{aligned} b'a'a'a'a' &= \alpha\gamma baaaa, \\ b'a'b' &= \alpha\gamma^2 bab, \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

So we can assume that $bab = 0$ or $baaaa$. If $bab = 0$ then we can assume that $p^2a = 0$ or $baaaa$, but if $bab = baaaa$ then we need $\gamma = 1$ and so we have $p^2a = xbaaaa$ where $0 \leq x < p$. If p^2a is non-zero then adding a suitable scalar multiple of pa to b we may assume that $pb - kbaaa = 0$. If $p^2a = 0$ then we have

$$pb' - kb'a'a'a' = \gamma(pb - baaa)$$

and so we may assume that $pb - kbaaa = 0$ or $xbaaaa$, where x lies in a transversal for the cube roots of unity. We have

$$\langle a, b \mid bab,\ p^2a,\ pb - kbaaa,\ \text{class } 5\rangle,$$

$$\langle a, b \mid bab,\, p^2a - baaaa,\, pb - kbaaa, \text{ class } 5\rangle,$$

$$\langle a, b \mid bab - baaaa,\, p^2a - xbaaaa,\, pb - kbaaa, \text{ class } 5\rangle\ (0 \le x < p),$$

$$\langle a, b \mid bab,\, p^2a,\, pb - kbaaa - xbaaaa, \text{ class } 5\rangle\ (x \neq 0,\ x, y \text{ give same algebra if } x^3 = y^3),$$

$$\langle a, b \mid bab-baaaa,\, p^2a,\, pb-kbaaa-xbaaaa, \text{ class } 5\rangle\ (x \neq 0,\ x, y \text{ give same algebra if } x^3 = y^3).$$

(In these algebras $k = 1$ if $p = 2 \bmod 3$ and $k = 1, \omega, \omega^2$ if $p = 1 \bmod 3$.)

33.4 Descendants of 6.445 ˇ 6.447

Algebras 6.445 ˇ 6.447 have $2p - 2 + \gcd(p - 1, 3)$ descendants of order p^7 and p-class 5.

Algebras 6.445 ˇ 6.447 have presentations

$$\langle a, b \mid bab - baaa,\, p^2a,\, pb - kbaaa, \text{ class } 4\rangle,$$

where $k = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. So if L is a descendant of 6.445 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 has order p and is generated by $baaaa$. We have $bab - baaa$, p^2a and $pb - kbaaa \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \alpha^2 b$ modulo L_2 where $\alpha^3 = 1$, and

$$\begin{aligned} b'a'a'a'a' &= baaaa, \\ b'a'b' &= \alpha^2 bab - 2\alpha\beta baaaa, \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

So we can assume that $bab = baaa$ though we then need $\beta = 0$. If $p^2a \neq 0$ then adding a suitable scalar multiple of pa to b we can assume that $pb = kbaaa$. If $p^2a = 0$ then

$$pb' - kb'a'a'a' = \alpha^2(pb - kbaaa).$$

So if $p \neq 1 \bmod 3$ we have $2p - 1$ algebras

$$\langle a, b \mid bab - baaa,\, p^2a - xbaaaa,\, pb - baaa, \text{ class } 5\rangle\ (0 \le x < p),$$

$$\langle a, b \mid bab - baaa,\, p^2a,\, pb - baaa - xbaaaa, \text{ class } 5\rangle\ (0 < x < p),$$

and if $p = 1 \bmod 3$ we again have $2p + 1$ algebras

$$\langle a, b \mid bab - baaa,\, p^2a - xbaaaa,\, pb - kbaaa, \text{ class } 5\rangle,$$

$$\langle a, b \mid bab - baaa,\, p^2a,\, pb - kbaaa - xbaaaa, \text{ class } 5\rangle\ (x \neq 0)$$

where $k = 1, \omega, \omega^2$ and where $x = 0$ or x is in a tranversal for the cube roots of unity.

34 Grandchildren of algebras 16 and 17 (5.51 and 5.52)

Algebras 5.51 and 5.52 have three descendants each of order p^6 (6.448 ˇ 6.453) but only 6.448 and 6.451 have descendants or order p^7.

34.1 Descendants of 6.448 and 6.451

Each distinct algebra from this two parameter family of $p+1$ algebras has a unique descendant of order p^7 and p-class 5, so there are $p+1$ descendants in all.

Algebra 6.448 has presentation

$$\langle a, b \mid bab,\, p^2a,\, pb - kbaa - \lambda baaa, \text{ class } 4\rangle \ (0 \le \lambda \le (p-1)/2),$$

with $k = 1, \omega$ and with λ and $-\lambda$ giving isomorphic algebras. So if L is a descendant of 6.448 or 6.451 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5 and L_5 has order p and is generated by $baaaa$. We have bab, p^2a and $pb - baa - \lambda baaa \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \pm a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2, with $a' = -a + \beta b$ only arising when $\lambda = 0$.

Adding a suitable scalar multiple of pa to b we can assume that $bab = 0$, with suitable choice of β we can assume that $p^2a = 0$ and adding a suitable scalar multiple of pa to a we can assume that $pb - baa - \lambda baaa$. So we have

$$\langle a, b \mid bab,\, p^2a,\, pb - kbaa - \lambda baaa, \text{ class } 5\rangle \ (0 \le \lambda \le (p-1)/2)$$

with $k = 1, \omega$ and with λ and $-\lambda$ giving isomorphic algebras.

35 Grandchildren of algebra 18 (5.54)

Algebra 5.54 has 7 descendants of order p^6 (6.454, 6.455, 6.460 ˘ 6.463) with descendants of order p^7.

35.1 Descendants of 6.454

Algebra 6.454 has $8 + 2\gcd(p-1, 3) + 4\gcd(p-1, 4)$ descendants of order p^7 and p-class 5.

Algebra 6.454 has presentation

$$\langle a, b \mid baa,\, p^2a,\, pb, \text{ class } 4\rangle,$$

and so if L is a descendant of 6.454 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab modulo L_4, L_4 is generated by $babb$ modulo L_5 and L_5 has order p and is generated by $babba$ and $babbb$. We have baa, p^2a and $pb \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha a$ modulo L_2 and $b' = \gamma b$ modulo L_2 and

$$\begin{aligned} b'a'b'b'a' &= \alpha^2\gamma^3 babba, \\ b'a'b'b'b' &= \alpha\gamma^4 babbbb. \end{aligned}$$

So we can assume that $babba = 0$ or that $babbb = 0$ or that $babbb = babba$.

35.1.1 $babba = 0$

If $babba = 0$ then L_5 is generated by $babbb$ and we have

$$\begin{aligned} b'a'b'b'b' &= \alpha\gamma^4 babbbb, \\ b'a'a' &= \alpha^2\gamma baa, \\ p^2a' &= \alpha p^2a. \end{aligned}$$

So we can assume that $baa = 0$ or $babbb$ and we can assume that $p^2a = 0$, $babbb$, $\omega babbb$ or (if $p = 1 \bmod 4$) $\omega^2 babbb$ or $\omega^3 babbb$. If $p^2a \neq 0$ then we can add a suitable scalar multiple of pa to b so that $pb = 0$. If $baa = p^2a = 0$ then we can assume that $pb = 0$ or $babbb$, but if $baa = babbb$ and $p^2a = 0$ then we need $\alpha = \gamma^3$ so that we have

$$\begin{aligned} b'a'b'b'b' &= \gamma^7 babbbb, \\ pb' &= \gamma pb \end{aligned}$$

and we can assume that $pb = 0$, $babbb$, $\omega babbb$ or (if $p = 1 \bmod 3$) $\omega^2 babbb$, $\omega^3 babbb$, $\omega^4 babbb$ or $\omega^5 babbb$. So we have

$$\langle a, b \mid babba,\ baa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babba,\ baa,\ p^2a,\ pb - babbb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a,\ pb - babbb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a,\ pb - \omega babbb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a,\ pb - kbabbb,\ \text{class } 5\rangle\ (p = 1 \bmod 3,\ k = \omega^2, \omega^3, \omega^4, \omega^5),$$
$$\langle a, b \mid babba,\ baa,\ p^2a - babbb,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa,\ p^2a - \omega babbb,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa,\ p^2a - \omega^2 babbb,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid babba,\ baa,\ p^2a - \omega^3 babbb,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a - babbb,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a - \omega babbb,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a - \omega^2 babbb,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b \mid babba,\ baa - babbb,\ p^2a - \omega^3 babbb,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4).$$

35.1.2 $babbb = 0$

If $babbb = 0$ then L_5 is generated by $babba$. Adding a suitable scalar multiple of bab to a we can assume that $baa = 0$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^2\gamma^3 babba, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

and so we can assume that $p^2a = 0$ or $babba$. If $p^2a \neq 0$ then (as above) we may assume that $pb = 0$, and if $p^2a = 0$ then we have

$$pb' = \gamma pb$$

and so we can assume that $pb = 0$, $babba$ or $\omega babba$. So we have

$$\langle a, b \mid babbb,\ baa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb,\ baa,\ p^2a - babba,\ pb,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babbb,\ baa,\ p^2a,\ pb - babba,\ \text{class } 5\rangle,$$
$$\langle a, b \mid babbb,\ baa,\ p^2a,\ pb - \omega babba,\ \text{class } 5\rangle.$$

35.1.3 $babbb = babba$

If $babbb = babba$ then L_5 is generated by $babba$, though we need $\alpha = \gamma$. As above, we can assume that $baa = 0$, and if $p^2a \neq 0$ we can assume that $pb = 0$. We have

$$\begin{aligned} b'a'b'b'a' &= \alpha^5 babba, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

and so we can assume that $p^2a = 0$, $babba$, $\omega babba$ or (if $p = 1 \bmod 4$) $\omega^2 babba$ or $\omega^3 babba$, and if $p^2a = 0$ we can assume that $pb = 0$, $babba$, $\omega babba$ or (if $p = 1 \bmod 4$) $\omega^2 babba$ or $\omega^3 babba$. So we have

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a - babba,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a - \omega babba,\ pb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a - \omega^2 babba,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a - \omega^3 babba,\ pb,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a,\ pb - babba,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a,\ pb - \omega babba,\ \text{class } 5\rangle,$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a,\ pb - \omega^2 babba,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid babbb - babba,\ baa,\ p^2a,\ pb - \omega^3 babba,\ \text{class } 5\rangle\ (p = 1 \bmod 4).$$

35.2 Descendants of 6.455

Algebra 6.455 has 4 descendants of order p^7 and p-class 5.

Algebra 6.455 has presentation

$$\langle a, b \mid baa,\ p^2a,\ pb - babb,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.455 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab modulo L_4, L_4 is generated by $babb$ modulo L_5 and L_5 has order p and is generated by $babba$. We have baa, p^2a and $pb - babb \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a' = \alpha^{-2}a$ modulo L_2 and $b' = \alpha b$ modulo L_2. Adding a suitable scalar multiple of bab to a we can assume that $baa = 0$, and if $p^2a \neq 0$ then adding a suitable scalar multiple of pa to b we can assume that $pb = 0$. If a', b' are as described above then

$$\begin{aligned} b'a'b'b'a' &= \alpha^{-1} babba, \\ p^2a' &= \alpha^{-2} p^2 a \end{aligned}$$

so we can assume that $p^2a = 0$ or $babba$. if $p^2a = 0$ then $pb' = \alpha pb$ so we can assume that $pb = 0$, $babba$ or $\omega babba$. This gives

$$\langle a, b \mid baa,\ p^2a,\ pb - babb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa,\ p^2a - babba,\ pb - babb,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa,\ p^2a,\ pb - babb - babba,\ \text{class } 5\rangle,$$

$$\langle a, b \mid baa,\ p^2a,\ pb - babb - \omega babba,\ \text{class } 5\rangle.$$

35.3 Descendants of 6.459

Algebra 6.459 has $4p+2\gcd(p-1,3)+\gcd(p-1,4)+2\gcd(p-1,5)$ descendants of order p^7 and p-class 5.

Algebra 6.459 has presentation

$$\langle a,b \mid baa-babb,\ p^2a,\ pb,\ \text{class } 4\rangle,$$

and so if L is a descendant of 6.459 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab modulo L_4, L_4 is generated by $babb$ modulo L_5 and L_5 has order p and is generated by $babba$ and $babbb$. We have $baa-babb$, p^2a and $pb \in L_5$. If a', b' generate L and satisfy the same relations as a, b modulo L_5 then $a'=\alpha^2a$ modulo L_2 and $b'=\alpha b$ modulo L_2, and

$$\begin{aligned} b'a'b'b'a' &= \alpha^7 babba, \\ b'a'b'b'b' &= \alpha^6 babbb. \end{aligned}$$

So we may assume that $babba=0$ or $babbb=0$ or $babbb=babba$.

35.3.1 $babba=0$

If $babba=0$ then L_5 is generated by $babbb$, and we have

$$\begin{aligned} b'a'b'b'b' &= \alpha^6 babbb, \\ b'a'a'-b'a'b'b' &= \alpha^5(baa-babb), \\ p^2a' &= \alpha^2p^2a. \end{aligned}$$

So we can assume tht $baa-babb=0$ or $babbb$. If $baa-babb=0$ then we can assume that $p^2a=0$, $babbb$, $\omega babbb$ or (if $p=1 \bmod 4$) $\omega^2 babbb$ or $\omega^3 babbb$. If $baa-babb=p^2a=0$ then we have $pb'=\alpha pb$ and so we can assume that $pb=0$, $babbb$ or (if $p=1 \bmod 5$) $\omega babbb$, $\omega^2 babbb$, $\omega^3 babbb$ or $\omega^4 babbb$. If $p^2a \neq 0$ then we can assume that $pb=0$. If $baa-babb=babbb$ then we need $\alpha=1$, and so we have $p^2a=xbabbb$ with $0 \leq x < p$, $pb=0$ if $p^2a\neq 0$, and $pb=xbabbb$ with $0\leq x<p$ if $p^2a=0$.

35.3.2 $babbb=0$

If $babbb=0$ then L_5 is generated by $babba$. Adding a suitable scalar multiple of bab to a we may assume that $baa-babb=0$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^7 babba, \\ p^2a' &= \alpha^2p^2a \end{aligned}$$

and so we can assume that $p^2a=0$, $babba$ or (if $p=1 \bmod 5$) $\omega babba$, $\omega^2 babba$, $\omega^3 babba$ or $\omega^4 babba$. If $p^2a\neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb=0$. And if $p^2a=0$ then we have $pb'=\alpha pb$ and so we can assume that $pb=0$, $babba$, $\omega babba$ or (if $p=1 \bmod 3$) $\omega^2 babba$, $\omega^3 babba$, $\omega^4 babba$ or $\omega^5 babba$.

35.3.3 $babbb=babba$

If $babbb=babba$ then L_5 is generated by $babba$, though we need $\alpha=1$. Adding a suitable scalar multiple of bab to a we may assume that $baa-babb=0$. We then have

$$\begin{aligned} b'a'b'b'a' &= babba, \\ p^2a' &= p^2a \end{aligned}$$

and so we can assume that $p^2a=xbabba$ with $0\leq x<p$. As above, if $p^2a\neq 0$ we can assume that $pb=0$, and if $p^2a=0$ then we can assume that $pb=xbabba$ with $0\leq x<p$.

35.4 Descendants of 6.460 ˘ 6.463

Algebras 6.460 ˘ 6.463 have $3p-3+\gcd(p-1,4)$ descendants of order p^7 and p-class 5.

Algebras 6.460 ˘ 6.463 have presentations

$$\langle a,b \mid baa-babb,\, p^2a,\, pb-kbabb,\, \text{class } 4\rangle,$$

with $k=1,\omega$ or (if $p=1 \bmod 4$) ω^2 or ω^3. So if L is a descendant of 6.460 ˘ 6.463 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by bab modulo L_4, L_4 is generated by $babb$ modulo L_5 and L_5 has order p and is generated by $babba$. We have $baa-babb$, p^2a and $pb-kbabb \in L_5$. If a',b' generate L and satisfy the same relations as a,b modulo L_5 then $a'=\alpha^2a$ modulo L_2 and $b'=\alpha b$ modulo L_2 with $\alpha^4=1$. Adding a suitable scalar multiple of bab to a we can assume that $baa=babb$. We then have

$$\begin{aligned} b'a'b'b'a' &= \alpha^3 babba, \\ p^2a' &= \alpha^2p^2a \end{aligned}$$

and so we can assume that $p^2a=0$, or that $p^2a=xbabba$ where x lies in a transversal for the fourth roots of unity. If $p^2a\neq 0$ then adding a suitable scalar multiple of pa to b we may assume that $pb=0$. And if $p^2a=0$ then we have $pb'=\alpha pb$, so that we can assume that $pb=xbabba$ where $0\le x\le (p-1)/2$ if $p=1 \bmod 4$, and $0\le x<p$ if $p=3 \bmod 4$. So if $p=3 \bmod 4$ we have

$$\langle a,b \mid baa-babb,\, p^2a-xbabba,\, pb-kbabb,\, \text{class } 4\rangle\ (k=1,\omega,\ 0\le x\le (p-1)/2),$$

$$\langle a,b \mid baa-babb,\, p^2a,\, pb-kbabb-xbabba,\, \text{class } 4\rangle\ (k=1,\omega,\ 0<x<p),$$

and if $p=1 \bmod 4$ then we have

$$\langle a,b \mid baa-babb,\, p^2a-xbabba,\, pb-kbabb,\, \text{class } 4\rangle\ (k=1,\omega,\omega^2,\omega^3)$$

with $0\le x<p$ where x and y give isomorphic algebras if $x^4=y^4, and$

$$\langle a,b \mid baa-babb,\, p^2a,\, pb-kbabb-xbabba,\, \text{class } 4\rangle\ (k=1,\omega,\omega^2,\omega^3,\ 0<x\le (p-1)/2).$$

36 Grandchildren of algebra 19 (5.58)

Algebra 5.58 has two descendants of order p^6, but only 6.467 has descendants of order p^7.

36.1 Descendants of 6.467

Algebra 6.467 has two descendants of order p^7 and p-class 5.

Algebra 6.467 has presentation

$$\langle a,b \mid baa,\, pb-ba,\, \text{class } 4\rangle,$$

and so if L is a descendant of 6.467 of order p^7 then L_2 is generated by ba, pa modulo L_3, L_3 is generated by p^2a modulo L_4, L_4 is generated by p^3a modulo L_5 and L_5 has order p and is generated by p^4a. We have baa, $p^2b-ba\in L_5$. Adding a suitable scalar multiple of p^3a to b we can assume taht $pb=ba$. If a',b' generate L and satisfy the same relations as a,b modulo L_5 then $a'=a+\beta b$ modulo L_2 and $b'=\gamma b$ modulo L_2. We then have

$$\begin{aligned} p^4a' &= p^4a, \\ b'a'a' &= \gamma baa \end{aligned}$$

and so we can assume that $baa=0$ or p^4a, giving

$$\langle a,b \mid baa,\, pb-ba,\, \text{class } 5\rangle,$$

$$\langle a,b \mid baa-p^4a,\, pb-ba,\, \text{class } 5\rangle.$$

37 Grandchildren of algebra 20 (5.73)

Algebra 5.73 has two descendants of order p^6 (6.518 and 6.519), but 6.519 is terminal.

37.1 Descendants of 6.518

Algebra 6.518 has presentation

$$\langle a, b \mid ba,\, pb,\, \text{class } 5\rangle,$$

and so if L is a descendant of 6.518 of order p^7, then L_2 is generated by pa, and L_6 is generated by p^5a. We have ba and $pb \in L_6$. Adding a suitable scalar multiple of p^4a to b we may assume that $pb = 0$. If a', b' generate L, and satisfy the same relations as a, b modulo L_6 then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2, and

$$\begin{aligned} p^5a' &= \alpha p^5a, \\ b'a' &= \alpha\gamma ba. \end{aligned}$$

So we may assume that $ba = 0$ or p^5a, giving

$$\langle a, b \mid ba,\, pb,\, \text{class } 6\rangle,$$

$$\langle a, b \mid ba - p^5a,\, pb,\, \text{class } 6\rangle.$$

38 Grandchildren of algebra 21 (5.60)

Algebra 5.60 has 25 descendants of order p^6, but all but 6.469 and 6.475 are terminal.

38.1 Descendants of 6.469

Algebra 6.469 has $4 + 2\gcd(p-1,3) + 3\gcd(p-1,5) + \gcd(p-1,8)$ descendants of order p^7 and p-class 6.

Algebra 6.469 has presentation

$$\langle a, b \mid bab,\, baaab,\, pa,\, pb,\, \text{class } 5\rangle,$$

so if L is a descendant of 6.469 or order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5, L_5 is generated by $baaaa$ modulo L_6 and L_6 has order p and is generated by $baaaaa$. We have bab, $baaab$, pa, $pb \in L_6$. If a', b' generate L and satisfy the same relations as a, b modulo L_6, then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \gamma b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a'a'a' &= \alpha^5\gamma baaaaa, \\ b'a'a'a'b' &= \alpha^3\gamma^2 baaab, \\ pa' &= \alpha pa + \beta pb, \\ pb' &= \gamma pb \end{aligned}$$

so we can assume that $baaab = 0$ or $baaaaa$ and that $pb = 0$, $baaaaa$ or (if $p = 1 \bmod 5$) $\omega baaaaa$, $\omega^2 baaaaa$, $\omega^3 baaaaa$ or $\omega^4 baaaaa$.

If $baaab \neq 0$ then we can subtract a suitable scalar multiple of baa from b so that $bab = 0$. But if $baaab = 0$ then we have

$$b'a'b' = \alpha\gamma^2 bab$$

so we can assume that $bab = 0$ or $baaaaa$.

If $pb \neq 0$ then we can choose β so that $pa' = 0$. If $baaab = pb = 0$ then we can assume that $pa = 0$ or $baaaaa$. If $baaab = baaaaa$ and $pb = 0$ then we need $\gamma = \alpha^2$ so we can assume that $pa = 0$, $baaaaa$, $\omega baaaaa$ or (if $p = 1 \bmod 3$) $\omega^2 baaaaa$, $\omega^3 baaaaa$, $\omega^4 baaaaa$ or $\omega^5 baaaaa$. If $baaab = 0$, $bab = baaaaa$, $pb = 0$ then we have $\gamma = \alpha^4$ and so we can assume that $pa = 0$, $baaaaa$, $\omega baaaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaaa$ or $\omega^3 baaaaa$ or (if $p = 1 \bmod 8$) $\omega^4 baaaaa$, $\omega^5 baaaaa$, $\omega^6 baaaaa$ or $\omega^7 baaaaa$. We have

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - baaaaa,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab,\ pa,\ pb - xbaaaaa,\ \text{class } 6\rangle\ (p = 1 \bmod 5,\ x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b \mid bab,\ baaab,\ pa - baaaaa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa,\ pb - baaaaa,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa,\ pb - xbaaaaa,\ \text{class } 6\rangle\ (p = 1 \bmod 5,\ x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa - baaaaa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa - \omega baaaaa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab,\ baaab - baaaaa,\ pa - xbaaaaa,\ pb,\ \text{class } 6\rangle\ (p = 1 \bmod 3,\ x = \omega^2, \omega^3, \omega^4, \omega^5),$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa,\ pb - baaaaa,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa,\ pb - xbaaaaa,\ \text{class } 6\rangle\ (p = 1 \bmod 5,\ x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa - baaaaa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa - \omega baaaaa,\ pb,\ \text{class } 6\rangle,$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa - \omega^2 baaaaa,\ pb,\ \text{class } 6\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa - \omega^3 baaaaa,\ pb,\ \text{class } 6\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b \mid bab - baaaaa,\ baaab,\ pa - xbaaaaa,\ pb,\ \text{class } 6\rangle\ (p = 1 \bmod 8,\ x = \omega^4, \omega^5, \omega^6, \omega^7).$$

38.2 Descendants of 6.475

Algebra 6.475 has $4p - 1 + \gcd(p-1, 5) + \gcd(p-1, 7)$ descendants of order p^7 and p-class 5.

Algebra 6.475 has presentation

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb,\ \text{class } 5\rangle,$$

so if L is a descendant of 6.475 or order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5, L_5 is generated by $baaaa$ modulo L_6 and L_6 has order p and is generated by $baaaaa$. We have $bab - baaaa$, $baaab$, pa, $pb \in L_6$, but if $baaab \neq 0$ then by adding a suitable scalar multiple of baa to

b we may assume that $bab = baaaa$. If a', b' generate L and satisfy the same relations as a, b modulo L_6, then $a' = \alpha a + \beta b$ modulo L_2 and $b' = \alpha^3 b$ modulo L_2. We then have

$$\begin{aligned} b'a'a'a'a'a' &= \alpha^8 baaaaa, \\ b'a'a'a'b' &= \alpha^9 baaab, \\ pa' &= \alpha pa + \beta pb, \\ pb' &= \alpha^3 pb \end{aligned}$$

so we can assume that $baaab = 0$ or $baaaaa$. If $baaab = 0$ then

$$b'a'b' - b'a'a'a'a' = \alpha^7 (bab - baaaa)$$

and so we can assume that $bab - baaaa = 0$ or $baaaaa$.

If $bab - baaaa = baaab = 0$ then we can assume that $pb = 0$, $baaaaa$ or (if $p = 1 \bmod 5$) $\omega baaaaa$, $\omega^2 baaaaa$, $\omega^3 baaaaa$ or $\omega^4 baaaaa$. If $pb \neq 0$ then we can choose β so that $pa = 0$. If $baaab = pb = 0$ then we can assume that $pa = 0$ or $baaaaa$ or (if $p = 1 \bmod 7$) $xbaaaaa$ with $x = \omega, \omega^2, \ldots, \omega^6$.

If $baaab = baaaaa$ or $bab - baaaa = baaaaa$ then we need $\alpha = 1$ and so we can assume that $pa = 0$, $pb = xbaaaaa$ with $0 \leq x < p$ or that $pa = xbaaaaa$ with $0 < x < p$ and $pb = 0$.

So we have

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb,\ \text{class } 6 \rangle,$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - baaaaa,\ \text{class } 6 \rangle,$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa,\ pb - xbaaaaa,\ \text{class } 6 \rangle\ (p = 1 \bmod 5,\ x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - baaaaa,\ pb,\ \text{class } 6 \rangle,$$

$$\langle a, b \mid bab - baaaa,\ baaab,\ pa - xbaaaaa,\ pb,\ \text{class } 6 \rangle\ (p = 1 \bmod 7,\ x = \omega, \omega^2, \ldots, \omega^6),$$

$$\langle a, b \mid bab - baaaa,\ baaab - baaaaa,\ pa,\ pb - xbaaaaa,\ \text{class } 6 \rangle\ (0 \leq x < p),$$

$$\langle a, b \mid bab - baaaa,\ baaab - baaaaa,\ pa - xbaaaaa,\ pb,\ \text{class } 6 \rangle\ (0 < x < p),$$

$$\langle a, b \mid bab - baaaa - baaaaa,\ baaab,\ pa,\ pb - xbaaaaa,\ \text{class } 6 \rangle\ (0 \leq x < p),$$

$$\langle a, b \mid bab - baaaa - baaaaa,\ baaab,\ pa - xbaaaaa,\ pb,\ \text{class } 6 \rangle\ (0 < x < p).$$

39 Grandchildren of algebra 22 (5.65)

Algebra 5.65 has 24 families of descendants of order p^6, but all of them are terminal except for 6.507.

39.1 Descendants of 6.507

Algebra 6.507 has $2p^2 + p + 2p \gcd(p-1, 3) + p \gcd(p-1, 5)$ descendants of order p^7 and p-class 6.

Algebra 6.507 has presentation

$$\langle a, b \mid bab - baaa,\ baaab,\ pa,\ pb,\ \text{class } 5 \rangle,$$

and so if L is a descendant of 6.507 or order p^7 then L_2 is generated by ba modulo L_3, L_3 is generated by baa modulo L_4, L_4 is generated by $baaa$ modulo L_5, L_5 is generated by $baaaa$ modulo L_6 and L_6 has order p and is generated by $baaaaa$. We have $bab - baaa$, $baaab$, pa, $pb \in L_6$, but adding a suitable scalar multiple of baa to b we may assume that

$bab = baaa$. If a', b' generate L and satisfy the same relations as a, b modulo L_6, then $a' = \alpha a$ modulo L_2 and $b' = \alpha^2 b$ modulo L_2. We then have

$$\begin{aligned}
b'a'a'a'a'a' &= \alpha^7 baaaaa, \\
b'a'a'a'b' &= \alpha^7 baaab, \\
pa' &= \alpha pa, \\
pb' &= \alpha^2 pb.
\end{aligned}$$

So we can assume that $baaab = xbaaaaa$ with $0 \leq x < p$. If $x \neq 1$ we can add a suitable scalar multiple of baa to b so that $bab = baaa$. But if $x = 1$ then

$$b'a'b' - b'a'a'a' = \alpha^5(bab - baaa)$$

so we can assume that $bab - baaa = 0$, $baaaaa$ or $\omega baaaaa$.

First consider the case when $bab - baaa = 0$. Then we can assume that $pb = 0$, $baaaaa$ or (if $p = 1 \bmod 5$) $\omega baaaaa$, $\omega^2 baaaaa$, $\omega^3 baaaaa$ or $\omega^4 baaaaa$. If $pb = 0$ then we can choose $pa = 0$, $baaaaa$ or $\omega baaaaa$ or (if $p = 1 \bmod 6$) $\omega^2 baaaaa$, $\omega^3 baaaaa$, $\omega^4 baaaaa$ or $\omega^5 baaaaa$. But if $pb \neq 0$ then we need $\alpha^5 = 1$ and so we can take $pa = 0$ or $pa = xbaaaaa$ where x lies in a transversal for the øfth roots of unity.

If $bab - baaa = baaaaa$ or $\omega baaaaa$ and $baaab = baaaaa$ then we need $\alpha^2 = 1$. So we can take $pa = xbaaaaa$ with $0 \leq x < p$ and $pb = ybaaaaa$ with $0 \leq y \leq (p-1)/2$.

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa,\ pb,\ \text{class } 6\rangle\ (0 \leq x < p),$$

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa - zbaaaaa,\ pb - baaaaa,\ \text{class } 6\rangle\ (0 \leq x < p,\ z^5 \sim z'^5),$$

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa - zbaaaaa,\ pb - ybaaaaa,\ \text{class } 6\rangle\ (p = 1 \bmod 5,\ 0 \leq x < p,\ y = \omega, \ldots, \omega^4,\ z^5 \sim z$$

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa - baaaaa,\ pb,\ \text{class } 6\rangle\ (0 \leq x < p),$$

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa - \omega baaaaa,\ pb,\ \text{class } 6\rangle\ (0 \leq x < p),$$

$$\langle a, b \mid bab - baaa,\ baaab - xbaaaaa,\ pa - ybaaaaa,\ pb,\ \text{class } 6\rangle\ (p = 1 \bmod 3,\ 0 \leq x < p,\ y = \omega^3, \ldots, \omega^5),$$

$$\langle a, b \mid bab - baaa - baaaaa,\ baaab - baaaaa,\ pa - xbaaaaa,\ pb - ybaaaaa,\ \text{class } 6\rangle\ (0 \leq x < p,\ 0 \leq y \leq (p-1)/2),$$

$$\langle a, b \mid bab - baaa - \omega baaaaa,\ baaab - baaaaa,\ pa - xbaaaaa,\ pb - ybaaaaa,\ \text{class } 6\rangle\ (0 \leq x < p,\ 0 \leq y \leq (p-1)/2).$$

40 Grandchildren of algebra 23 (3.1)

Algebra 3.1 has $3p + 27$ immediate descendants of order p^6, with presentations 6.85 ˇ 6.117. All these algebras are capable, and between them they have

$$2p^2 + 63p + 362 + (p + 19)\gcd(p-1, 3) + 5\gcd(p-1, 4) + \gcd(p-1, 5)$$

descendants of order p^7 and p-class 3.

40.1 Descendants of 6.85

$$\langle a, b, c \mid ba, ca, cb, \text{class } 2\rangle$$

Algebra 6.85 has 3 descendants of order p^7 and p-class 3.

If L is a descendant of 6.85 of order p^7 then L_3 is generated by p^2a, p^2b, p^2c. The automorphism group of 6.85 induces $GL(3, p)$ on L/L_2, and so we may assume that L_3 is generated by p^2a and that $p^2b = p^2c = 0$. We then have ba, ca, cb as scalar multiples of p^2a. If a', b', c' generate L and if $p^2b' = p^2c' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2a' &= \alpha p^2 a, \\ b'a' &= \alpha\delta ba + \alpha\varepsilon ca + (\beta\varepsilon - \gamma\delta)cb, \\ c'a' &= \alpha\nu ba + \alpha\xi ca + (\beta\xi - \gamma\nu)cb, \\ c'b' &= (\delta\xi - \varepsilon\nu)cb. \end{aligned}$$

If $cb \neq 0$ we can assume that $cb = p^2a$ and that $ba = ca = 0$. If $cb = 0$ then we can take $ba = 0$ and $ca = 0$ or p^2a. So we have 3 algebras.

$$\langle a, b, c \mid ba, ca, cb, p^2b, p^2c, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid ba, ca - p^2a, cb, p^2b, p^2c, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid ba, ca, cb - p^2a, p^2b, p^2c, \text{ class } 3\rangle.$$

40.2 Descendants of 6.86

$$\langle a, b, c \mid ca, cb, pc, \text{ class } 2\rangle$$

Algebra 6.86 has $p + 17$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.86 of order p^7 then L_3 has order p and is generated by baa, bab, pba, p^2a, p^2b, and $ca, cb, pc \in L_3$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha(\alpha\varepsilon - \beta\delta)baa + \beta(\alpha\varepsilon - \beta\delta)bab, \\ b'a'b' &= \delta(\alpha\varepsilon - \beta\delta)baa + \varepsilon(\alpha\varepsilon - \beta\delta)bab, \\ pb'a' &= (\alpha\varepsilon - \beta\delta)pba, \\ p^2a' &= \alpha p^2a + \beta p^2b, \\ p^2b' &= \delta p^2a + \varepsilon p^2b. \end{aligned}$$

One possibility is that $baa = bab = p^2a = p^2b = 0$ and then L_3 is generated by pba.

If $baa = bab = 0$ but p^2a, p^2b are not both zero then we can assume that $p^2a = 0$ and that $p^2b \neq 0$ (though we then need $\beta = 0$). We can then take $pba = 0$ or p^2a.

If baa and bab are not both zero then we can assume that $baa = 0$ and that L_3 is generated by bab (though we then need $\beta = 0$). We then have

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ pb'a' &= \alpha\varepsilon pba, \\ p^2a' &= \alpha p^2a, \\ p^2b' &= \delta p^2a + \varepsilon p^2b. \end{aligned}$$

So we can take $pba = 0$ or bab. If $pba = 0$ we can take $p^2a = 0$, bab or ωbab. If $pba = p^2a = 0$ then we can take $p^2b = 0$ or bab, and if $pba = 0$, $p^2a \neq 0$ we can take $p^2b = 0$. If $pba = bab$ then we need $\varepsilon = 1$ and so we can take $p^2a = xbab$ with $0 \leq x < p$. Again, if $p^2a \neq 0$ we can take $p^2b = 0$, and if $p^2a = 0$ we can take $p^2b = 0$ or bab. So we have the following possibilities:

$$\begin{aligned}
baa &= bab = p^2a = p^2b = 0, \\
baa &= bab = p^2a = pba = 0, \\
baa &= bab = p^2a = 0,\ p^2b = pba, \\
baa &= pba = p^2a = p^2b = 0, \\
baa &= pba = p^2a = 0,\ p^2b - bab, \\
baa &= pba = p^2b = 0,\ p^2a = bab, \\
baa &= pba = p^2b = 0,\ p^2a = \omega bab, \\
baa &= p^2b = 0,\ p^2a = xbab,\ pba = bab,\ (0 \leq x < p), \\
baa &= p^2a = 0,\ pba = bab,\ p^2b = bab.
\end{aligned}$$

For each of these possibilities we give the most general form of generators a', b', c' for L which satisfy the speciøed relations, and then compute the possibilities for ca, cb, pc.

40.2.1 Case 1

Let $baa = bab = p^2a = p^2b = 0$. Then L_3 is generated by pba and adding suitable scalar multiples of ba, pa, pb to c we can take $ca = cb = pc = 0$. So we have

$$\langle a, b, c \mid baa,\ bab,\ p^2a,\ p^2b,\ ca,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.2.2 Case 2

Let $baa = bab = p^2a = pba = 0$. Then L_3 is generated by p^2b, and adding a suitable scalar multiple of pb to c we can take $pc = 0$. We then have

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c, \\
b' &= \delta a + \varepsilon b + \eta c, \\
c' &= \xi c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
p^2b' &= \varepsilon p^2b, \\
c'a' &= \alpha\xi ca, \\
c'b' &= \delta\xi ca + \varepsilon\xi cb.
\end{aligned}$$

So we can take $ca = 0$ or p^2b. If $ca = 0$ we can take $cb = 0$ or p^2b, and if $ca = p^2b$ we can take $cb = 0$. So we have

$$\langle a, b, c \mid baa,\ bab,\ pba,\ p^2a,\ ca,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ bab,\ pba,\ p^2a,\ ca,\ cb - p^2b,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ bab,\ pba,\ p^2a,\ ca - p^2b,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.2.3 Case 3

Let $baa = bab = p^2a = 0$, $p^2b = pba$. Then L_3 is generated by pba and adding suitable scalar multiples of ba, pa, pb to c we can take $ca = cb = pc = 0$. So we have

$$\langle a, b, c \mid baa,\ bab,\ p^2a,\ p^2b - pba,\ ca,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.2.4 Case 4

Let $baa = pba = p^2a = p^2b = 0$. Then L_3 is generated by bab, and adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ c'a' &= \alpha\xi ca, \\ pc' &= \xi pc. \end{aligned}$$

So we can independantly take ca and pc equal to 0 or bab.

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b,\ ca,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b,\ ca,\ cb,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b,\ ca - bab,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b,\ ca - bab,\ cb,\ pc - bab,\ \text{class } 3\rangle.$$

40.2.5 Case 5

Let $baa = pba = p^2a = 0$, $p^2b - bab$. Then L_3 is generated by bab, and adding suitable scalar multiple of ba and pb to c we can take $cb = pc = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta a + \alpha^{-1} b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^{-1} bab, \\ c'a' &= \alpha\xi ca. \end{aligned}$$

So we can take $ca = 0$ or bab.

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b - bab,\ ca,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a,\ p^2b - bab,\ ca - bab,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.2.6 Cases 6 & 7

Let $baa = pba = p^2b = 0$, $p^2a = kbab$ with $k = 1, \omega$. Then L_3 is generated by bab, and adding suitable scalar multiples of ba and pa to c we can take $cb = pc = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \pm b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ c'a' &= \alpha\xi ca. \end{aligned}$$

So we can take $ca = 0$ or bab.

$$\langle a, b, c \mid baa,\ pba,\ p^2a - bab,\ p^2b,\ ca,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a - bab,\ p^2b,\ ca - bab,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a - \omega bab,\ p^2b,\ ca,\ cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa,\ pba,\ p^2a - \omega bab,\ p^2b,\ ca - bab,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.2.7 Case 8

Let $baa = p^2b = 0$, $p^2a = xbab$, $pba = bab$ with $0 \le x < p$. Then L_3 is generated by bab and adding a suitable scalar multiple of pb to c we can take $ca = 0$, and adding a suitable scalar multiple of pa to c we can take $cb = 0$. If $x \ne -1$ then we can add a suitable scalar multiple of ba to c so that $pc = 0$. But if $x = -1$ we can take $pc = 0$ or bab. So we have $p + 1$ algebras

$$\langle a, b, c \mid baa,\ pba - bab,\ p^2a - xbab,\ p^2b,\ ca,\ cb,\ pc,\ \text{class } 3\rangle\ (0 \le x < p),$$

$$\langle a, b, c \mid baa,\ pba - bab,\ p^2a + bab,\ p^2b,\ ca,\ cb,\ pc - bab,\ \text{class } 3\rangle.$$

40.2.8 Case 9

Let $baa = p^2a = 0$, $pba = bab$, $p^2b = bab$. Then L_3 is generated by bab and adding suitable scalar multiples of ba, pa, pb to c we can take $ca = cb = pc = 0$. So we have

$$\langle a, b, c \mid baa,\ pba - bab,\ p^2a,\ p^2b - bab,\ ca,\ cb,\ pc,\ \text{class } 3\rangle.$$

40.3 Descendants of 6.87

$$\langle a, b, c \mid ca,\ cb,\ pc - ba,\ \text{class } 2\rangle$$

Algebra 6.87 has 5 descendants of order p^7 and p-class 3.

If L is a descendant of 6.87 of order p^7 then L_3 is generated by pba, p^2a, p^2b, and $ca, cb, pc - ba \in L_3$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= (\alpha\varepsilon - \beta\delta)c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} pb'a' &= (\alpha\varepsilon - \beta\delta)pba, \\ p^2a' &= \alpha p^2a + \beta p^2b + \gamma pba, \\ p^2b' &= \delta p^2a + \varepsilon p^2b + \eta pba. \end{aligned}$$

So if $pba \neq 0$ we can assume that $p^2a = p^2b = 0$, and if $pba = 0$ we can assume that $p^2a = 0$ and that L_3 is generated by p^2b.

40.3.1 $pba \neq 0$, $p^2a = p^2b = 0$

If L_3 is generated by pba and $p^2a = p^2b = 0$ then adding suitable scalar multiples of ba, pa, pb to c we can take $ca = cb = pc - ba = 0$ and so we have one algebra

$$\langle a, b, c \,|\, p^2a, p^2b, ca, cb,\ pc - ba,\ \text{class } 3\rangle.$$

40.3.2 $pba = p^2a = 0$

If $pba = p^2a = 0$ and L_3 is generated by p^2b then adding a suitable scalar multiple of pb to c we can take $pc - ba = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3, and if $pb'a' = p^2a' = pc' - b'a' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \alpha\varepsilon c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2b' &= \varepsilon p^2b, \\ c'a' &= \alpha^2\varepsilon ca, \\ c'b' &= \alpha\delta\varepsilon ca + \alpha\varepsilon^2 cb. \end{aligned}$$

So we can take $ca = 0$, p^2b or ωp^2b, and if $ca \neq 0$ we can take $cb = 0$. If $ca = 0$ we can take $cb = 0$ or p^2b.

$$\langle a, b, c \,|\, p^2a, pba, ca, cb,\ pc - ba,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, p^2a, pba, ca, cb - p^2b,\ pc - ba,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, p^2a, pba, ca - p^2b, cb,\ pc - ba,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, p^2a, pba, ca - \omega p^2b, cb,\ pc - ba,\ \text{class } 3\rangle.$$

40.4 Descendants of 6.88

$$\langle a, b, c \,|\, ca, cb,\ pa,\ \text{class } 2\rangle$$

Algebra 6.88 has 26 descendants of order p^7 and p-class 3.

If L is a descendant of 6.88 or order p^7 then L_3 has order p and is generated by baa, bab, p^2b, p^2c. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\varepsilon baa, \\ b'a'b' &= \alpha\delta\varepsilon baa + \alpha\varepsilon^2 bab, \\ p^2b &= \varepsilon p^2b + \eta p^2c, \\ p^2c &= \xi p^2c. \end{aligned}$$

So we can assume that $baa = bab = 0$, or that $baa = 0$, $bab \neq 0$, or that $baa \neq 0$, $bab = 0$, and we can also assume that at most one of p^2b, p^2c is non-zero. However at least one of baa, bab, p^2b, p^2c must be non-zero. So we have the following possibilities.

$$\begin{aligned} baa &= bab = p^2b = 0, \\ baa &= bab = p^2c = 0, \\ baa &= p^2b = p^2c = 0, \\ baa &= p^2b = 0,\ p^2c = bab, \\ baa &= p^2c = 0,\ p^2b = bab, \\ bab &= p^2b = p^2c = 0, \\ bab &= p^2b = 0,\ p^2c = baa, \\ bab &= p^2c = 0,\ p^2b = baa, \\ bab &= p^2c = 0,\ p^2b = \omega baa. \end{aligned}$$

For each of these possibilities we consider the most general possible generators a', b', c' which satisfy the same relations as a, b, c modulo L_3 and also satisfy the corresponding weight 3 relations speciøed.

40.4.1 Case 1

Let $baa = bab = p^2b = 0$, so that L_3 is generated by p^2c. Adding a suitable scalar multiple of pc to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \varepsilon b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2c' &= \xi p^2c, \\ c'a' &= \alpha\xi ca, \\ c'b' &= \delta\xi ca + \varepsilon\xi cb. \end{aligned}$$

We can take $ca = 0$ or p^2c and if $ca = p^2c$ we can take $cb = 0$. If $ca = 0$ we can take $cb = 0$ or p^2c. So we have 3 algebras.

$$\langle a, b, c \,|\, baa, bab, p^2b, ca, cb,\ pa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, p^2b, ca, cb - p^2c,\ pa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, p^2b, ca - p^2c, cb,\ pa,\ \text{class } 3\rangle.$$

40.4.2 Case 2

Let $baa = bab = p^2c = 0$, so that L_3 is generated by p^2b. Adding a suitable scalar multiple of pb to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2b' &= \varepsilon p^2 b, \\ c'a' &= \alpha\xi ca, \\ c'b' &= \delta\xi ca + \varepsilon\xi cb. \end{aligned}$$

We can take $ca = 0$ or p^2b and if $ca = p^2b$ we can take $cb = 0$. If $ca = 0$ we can take $cb = 0$ or p^2b. So we have 3 algebras.

$$\langle a, b, c \mid baa, bab, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, p^2c, ca, cb - p^2b, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, p^2c, ca - p^2b, cb, pa, \text{ class } 3\rangle.$$

40.4.3 Case 3

Let $baa = p^2b = p^2c = 0$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ c'a' &= \alpha\xi ca, \\ pa' &= \alpha pa. \end{aligned}$$

We can take $ca = 0$ or bab and (independantly) take $pa = 0$, bab or ωbab. So we have 6 algebras

$$\langle a, b, c \mid baa, p^2b, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c, ca - bab, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c, ca, cb, pa - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c, ca - bab, cb, pa - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c, ca, cb, pa - \omega bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c, ca - bab, cb, pa - \omega bab, \text{ class } 3\rangle.$$

40.4.4 Case 4

Let $baa = p^2b = 0$, $p^2c = bab$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$, and adding a suitable scalar multiple of pc to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \varepsilon b, \\ c' &= \alpha\varepsilon^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ c'a' &= \alpha^2\varepsilon^2 ca \end{aligned}$$

so we can assume that $ca = 0$ or bab.

$$\langle a, b, c \mid baa, p^2b, p^2c - bab, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b, p^2c - bab, ca - bab, cb, pa, \text{ class } 3\rangle.$$

40.4.5 Case 5

Let $baa = p^2c = 0$, $p^2b = bab$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$, and adding a suitable scalar multiple of pb to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + \alpha^{-1} b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^{-1} bab, \\ c'a' &= \alpha\xi ca \end{aligned}$$

so we can assume that $ca = 0$ or bab.

$$\langle a, b, c \mid baa, p^2b - bab, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2b - bab, p^2c, ca - bab, cb, pa, \text{ class } 3\rangle.$$

40.4.6 Case 6

Let $bab = p^2b = p^2c = 0$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ba to c we can take $ca = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\varepsilon baa, \\ c'b' &= \varepsilon\xi cb, \\ pa' &= \alpha pa. \end{aligned}$$

We can take cb and pa (independantly) equal to 0 or baa so we have 4 algebras

$$\langle a, b, c \,|\, bab, p^2b, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b, p^2c, ca, cb, pa - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b, p^2c, ca, cb - baa, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b, p^2c, ca, cb - baa, pa - baa, \text{ class } 3\rangle.$$

40.4.7 Case 7

Let $bab = p^2b = 0$, $p^2c = baa$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ba to c we can take $ca = 0$, and adding a suitable scalar multiple of pc to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \varepsilon b, \\ c' &= \alpha^2 \varepsilon c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2 \varepsilon baa, \\ c'b' &= \alpha^2 \varepsilon^2 cb. \end{aligned}$$

So we can take $cb = 0$ or baa and we have

$$\langle a, b, c \,|\, bab, p^2b, p^2c - baa, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b, p^2c - baa, ca, cb - baa, pa, \text{ class } 3\rangle$$

40.4.8 Cases 8 & 9

Let $bab = p^2c = 0$, $p^2b = kbaa$ with $k = 1, \omega$, so that L_3 is generated by baa. Adding a suitable scalar multiple of ba to c we can take $ca = 0$, and adding a suitable scalar multiple of pb to a we can take $pa = 0$. We then have

$$\begin{aligned} a' &= \pm a, \\ b' &= \varepsilon b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \varepsilon baa, \\ c'b' &= \varepsilon \xi cb. \end{aligned}$$

So we can take $cb = 0$ or baa, and we have 4 algebras.

$$\langle a, b, c \,|\, bab, p^2b - baa, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b - baa, p^2c, ca, cb - baa, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b - \omega baa, p^2c, ca, cb, pa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, p^2b - \omega baa, p^2c, ca, cb - baa, pa, \text{ class } 3\rangle.$$

$$\langle a, b, c \mid ca, cb, pa - ba, \text{ class } 2\rangle$$

Algebra 6.89 has $p+6$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.89 of order p^7 then L_3 has order p and is generated by bab, p^2b, p^2c. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_3 then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ p^2b' &= -\delta bab + p^2b + \eta p^2c, \\ p^2c' &= \xi p^2c. \end{aligned}$$

If either of bab or p^2c are non-zero then we can assume that $p^2b = 0$, but if $bab = p^2c = 0$ then L_3 is generated by p^2b. So we can assume that one of the following sets of relations holds:

$$\begin{aligned} p^2b &= p^2c = 0, \\ bab &= p^2c = 0, \\ bab &= p^2b = 0, \\ p^2c &= bab,\ p^2b = 0. \end{aligned}$$

In each case we consider the most general set of generators a', b', c' for L of the form above which satisfy the relevant additional set of two relations.

40.5.1 $p^2b = p^2c = 0$

If $p^2b = p^2c = 0$ then L_3 is generated by bab and adding suitable scalar multiples of pb and ba to c we can take $ca = cb = 0$, and adding a suitable scalar multiple of pb to b we can take $pa - ba = 0$. So we have

$$\langle a, b, c \mid p^2b, p^2c, ca, cb, pa - ba, \text{ class } 3\rangle.$$

40.5.2 $bab = p^2c = 0$

If $bab = p^2c = 0$ then L_3 is generated by p^2b, and adding a suitable scalar multiple of pb to a we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + b + \eta c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2b' &= p^2b, \\ c'a' &= \alpha\xi ca, \\ c'b' &= \delta\xi ca + \xi cb. \end{aligned}$$

So we can take $ca = 0$ or p^2b and if $ca = p^2b$ we can tkae $cb = 0$. If $ca = 0$ we can take $cb = 0$ or p^2b. So we have 3 algebras

$$\langle a, b, c \mid bab, p^2c, ca, cb,\ pa - ba,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, p^2c, ca, cb - p^2b,\ pa - ba,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, p^2c, ca - p^2b, cb,\ pa - ba,\ \text{class } 3\rangle.$$

40.5.3 $\quad bab = p^2b = 0$

If $bab = p^2b = 0$ then L_3 is generated by p^2c. Adding a suitable scalar multiple of pc to a we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta a + b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2c' &= \xi p^2c, \\ c'a' &= \alpha\xi ca, \\ c'b' &= \delta\xi ca + \xi cb. \end{aligned}$$

So we can take $ca = 0$ or p^2c, and if $ca = p^2c$ we can take $cb = 0$. But if $ca = 0$ then we have to take $cb = xp^2c$ with $0 \leq x < p$. We have $p + 1$ algebras

$$\langle a, b, c \mid bab, p^2b, ca, cb - xp^2c,\ pa - ba,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid bab, p^2b, ca - p^2c, cb,\ pa - ba,\ \text{class } 3\rangle.$$

40.5.4 $\quad p^2c = bab,\ p^2b = 0$

If $p^2c = bab$, $p^2b = 0$ then L_3 is generated by bab. Adding suitable scalar multiples of pb and ba to c we can take $ca = cb = 0$, and adding a suitable scalar multiple of pc to a we can take $pa - ba = 0$. So we have 1 algebra

$$\langle a, b, c \mid p^2b, p^2c - bab, ca, cb,\ pa - ba,\ \text{class } 3\rangle.$$

40.6 Descendants of 6.90

$$\langle a, b, c \mid cb,\ pb,\ pc,\ \text{class } 2\rangle$$

Algebra 6.90 has 30 descendants of order p^7 and p-class 3.

If L is a descendant of 6.90 of order p^7 then L_3 is generated by baa, bab, bac, caa, cac, p^2a. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_2 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{pmatrix} b'a'a' \\ b'a'b' \\ b'a'c' \\ c'a'a' \\ c'a'c' \end{pmatrix} = \begin{pmatrix} \alpha^2\delta & \alpha\beta\delta & \alpha\beta\varepsilon+\alpha\gamma\delta & \alpha^2\varepsilon & \alpha\gamma\varepsilon \\ 0 & \alpha\delta^2 & 2\alpha\delta\varepsilon & 0 & \alpha\varepsilon^2 \\ 0 & \alpha\delta\nu & \alpha\delta\xi+\alpha\varepsilon\nu & 0 & \alpha\varepsilon\xi \\ \alpha^2\nu & \alpha\beta\nu & \alpha\beta\xi+\alpha\gamma\nu & \alpha^2\xi & \alpha\gamma\xi \\ 0 & \alpha\nu^2 & 2\alpha\nu\xi & 0 & \alpha\xi^2 \end{pmatrix} \begin{pmatrix} baa \\ bab \\ bac \\ caa \\ cac \end{pmatrix}$$

and $p^2a' = \alpha p^2a$. One possibility is that $baa = bab = bac = caa = cac = 0$ and that L_3 is generated by p^2a. If $bab = bac = cac = 0$ then we can take $baa = 0$ and assume that L_3 is generated by caa. So consider the case when bab, bac, cac are not all zero. If one of bab, cac is non-zero the interchanging b and c if necessary we can take $cac \neq 0$, and if $bab = cac = 0$ then $bac \neq 0$, and taking $c' = b + c$ we have $c'ac' = 2bac \neq 0$. So we can assume that $cac \neq 0$, and then taking $b' = \delta b + \varepsilon c$, $c' = \xi c$ we have

$$\begin{aligned} b'ac' &= \alpha\delta\xi bac + \alpha\varepsilon\xi cac, \\ c'ac' &= \alpha\xi^2 cac \end{aligned}$$

and we can choose ε so that $b'ac' = 0$. So we can take $bac = 0$, $cac \neq 0$, and considering $a' = a$, $b' = b$, $c' = \xi c$ we can take $bab = 0$, or we can take $cac = -bab$ or $-\omega bab$. We show that these three possibilities give non-isomorphic algebras.

First consider the case when $bab = bac = 0$, $cac \neq 0$. Then taking a', b', c' as above, to ensure that $c'a'c' \neq 0$ we need $\alpha\xi^2 \neq 0$, and to ensure that $b'a'c' = 0$ we need $\alpha\varepsilon\xi = 0$. So we need $\varepsilon = 0$ and this gives $b'a'b' = 0$.

Next consider the case when $bac = 0$ and $cac = -bab$. Then to ensure that $b'a'c' = 0$ we need $\delta\nu - \varepsilon\xi = 0$. One possibility is that $\varepsilon = \nu = 0$, but then

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ c'a'c' &= \alpha\xi^2 cac = -\alpha\xi^2 bab \end{aligned}$$

which cannot give $b'a'b' = 0$ or $c'a'c' = -\omega b'a'b'$. If $\varepsilon \neq 0$ then we have $\xi = \delta\varepsilon^{-1}\nu$ and

$$\begin{aligned} b'a'b' &= \alpha(\delta^2 - \varepsilon^2) bab, \\ c'a'c' &= -\alpha\nu^2\varepsilon^{-2}(\delta^2 - \varepsilon^2) bab \end{aligned}$$

which again cannot give $b'a'b' = 0$ or $c'a'c' = -\omega b'a'b'$. (Note that we cannot have $\delta^2 - \varepsilon^2 = \delta\nu - \varepsilon\xi = 0$.)

So if bab, bac, cac are not all zero then we can take $cac \neq 0$, $bac = 0$, and take $bab = 0$ or $cac = -bab$ or $cac = -\omega bab$.

Let $bab = bac = 0$, $cac \neq 0$. Then to ensure that $b'a'b' = b'a'c' = 0$ we need $\varepsilon = 0$, and we then have

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ b'a'a' &= \alpha^2\delta baa, \\ c'a'a' &= \alpha^2\nu baa + \alpha^2\xi caa + \alpha\gamma\xi cac \end{aligned}$$

so we can take $caa = 0$ and $baa = 0$ or cac.

Now let $cac = -kbab$ where $k = 1$ or ω, and let $bac = 0$. Then taking

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= b, \\ c' &= c \end{aligned}$$

we have $c'a'c' = -kb'a'b'$ and $b'a'c' = 0$ and we have

$$\begin{aligned}
b'a'a' &= baa + \beta bab, \\
c'a'a' &= caa - k\gamma bab
\end{aligned}$$

so we can take $baa = caa = 0$.

So we can assume that one of the following six sets of commutator relations holds.

$$\begin{aligned}
baa &= bab = bac = caa = cac = 0, \\
baa &= bab = bac = cac = 0,\ caa \neq 0, \\
baa &= bab = bac = caa = 0,\ cac \neq 0, \\
bab &= bac = caa = 0,\ baa = cac \neq 0, \\
baa &= bac = caa = 0,\ cac = -bab \neq 0, \\
baa &= bac = caa = 0,\ cac = -\omega bab \neq 0,
\end{aligned}$$

In each of these cases we give the most general form for generators a', b', c' which generate L and satisfy the same commutator relations as a, b, c and also satisfy $pb', pc' \in L_3$.

40.6.1 Case 1

Let $baa = bab = bac = caa = cac = 0$. Then L_3 is generated by p^2a, and adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. We have

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c, \\
b' &= \delta b + \varepsilon c, \\
c' &= \nu b + \xi c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
p^2a' &= \alpha p^2 a, \\
c'b' &= (\delta\xi - \varepsilon\nu)cb
\end{aligned}$$

so we can take $cb = 0$ or p^2a and we have 2 algebras.

$$\langle a, b, c \mid baa, bab, bac, caa, cac, cb, pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, caa, cac, cb - p^2a,\ pb,\ pc, \text{ class } 3\rangle.$$

40.6.2 Case 2

Let $baa = bab = bac = cac = 0$, $caa \neq 0$ so that L_3 is generated by caa. We have

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c, \\
b' &= \delta b, \\
c' &= \nu b + \xi c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
c'a'a' &= \alpha^2 \xi caa, \\
p^2a' &= \alpha p^2 a, \\
c'b' &= \delta\xi cb
\end{aligned}$$

so we can take p^2a and cb independantly equal to 0 or caa. If $p^2a = caa$ then adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. And if $p^2a = 0$ then we have

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ c'b' &= \delta\xi cb, \\ pb' &= \delta pb \\ pc' &= \nu pb + \xi pc \end{aligned}$$

If $cb = 0$ we can take $pb = 0$ or caa. If $cb = pb = 0$ we can take $pc = 0$, caa or ωcaa and if $cb = 0$, $pb = caa$ we can take $pc = 0$. If $cb = caa$ we need $\delta = \alpha^2$, but again we can take $pb = 0$ or caa, and if $pb \neq 0$ we can take $pc = 0$. If $cb = caa$, $pb = 0$ we can take $pc = 0$, caa or ωcaa. So we have 10 algebras

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb,\ pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb - caa,\ pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb,\ pb,\ pc - caa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb - caa,\ pb,\ pc - caa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb,\ pb,\ pc - \omega caa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb - caa,\ pb,\ pc - \omega caa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb,\ pb - caa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a, cb - caa,\ pb - caa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a - caa, cb,\ pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, p^2a - caa, cb - caa,\ pb,\ pc, \text{ class } 3\rangle.$$

40.6.3 Case 3

Let $baa = bab = bac = caa = 0$, $cac \neq 0$ so that L_3 is generated by cac. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \delta b, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

so we can take $p^2a = 0$, cac or ωcac. If $p^2a \neq 0$ then adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$, and if $p^2a = 0$ we have

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ pb' &= \delta pb \\ pc' &= \nu pb + \xi pc \end{aligned}$$

so we can take $pb = 0$, $pc = 0$ or cac, or we can take $pb = cac$, $pc = 0$. So we have 5 algebras.

$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2a,\ pb,\ pc - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2a,\ pb - cac,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2a - cac,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2a - \omega cac,\ pb,\ pc,\ \text{class } 3\rangle.$$

40.6.4 Case 4

Let $bab = bac = caa = 0$, $baa = cac \neq 0$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^{-1}\xi^2 b, \\ c' &= -\alpha^{-1}\gamma\xi b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha\xi^2 cac, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

so we can take $p^2a = 0$, baa or ωbaa. If $p^2a \neq 0$ then by adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. If $p^2a = 0$ we have

$$\begin{aligned} b'a'a' &= \alpha\xi^2 baa, \\ pb' &= \alpha^{-1}\xi^2 pb \\ pc' &= -\alpha^{-1}\gamma\xi pb + \xi pc \end{aligned}$$

so we can take $pb = 0$, $pc = 0$ or baa, or we can take $pb = baa$ or ωbaa, $pc = 0$. So we have 6 algebras.

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a,\ pb,\ pc - baa,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a - baa,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb,\ p^2a - \omega baa,\ pb,\ pc,\ \text{class } 3\rangle.$$

40.6.5 Case 5

Let $baa = bac = caa = 0$, $cac = -bab \neq 0$ with, so that L_3 is generated by bab. Adding a suitable scalar multiple of ba or ca to c we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha(\delta^2 - \omega\varepsilon^2)bab, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

so we can take $p^2a = 0$ or bab. If $p^2a \neq 0$ then by adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. If $p^2a = 0$ we have

$$\begin{aligned} b'a'b' &= \alpha(\delta^2 - \varepsilon^2)bab, \\ pb' &= \delta pb + \varepsilon pc \\ pc' &= \pm\varepsilon pb \pm \delta pc. \end{aligned}$$

We can take $pb = 0$, $pc = 0$ or bab, or we can take $pb = pc = bab$. So we have four algebras

$$\langle a, b, c \,|\, baa, bac, caa, cac + bab, cb,\ p^2a,\ pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + bab, cb,\ p^2a,\ pb,\ pc - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + bab, cb,\ p^2a,\ pb - bab,\ pc - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + bab, cb,\ p^2a - bab,\ pb,\ pc, \text{ class } 3\rangle.$$

40.6.6 Case 6

Let $baa = bac = caa = 0$, $cac = -\omega bab \neq 0$ with, so that L_3 is generated by bab. Adding a suitable scalar multiple of ba or ca to c we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\varepsilon\omega b \pm \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha(\delta^2 - \omega\varepsilon^2)bab, \\ p^2a' &= \alpha p^2 a \end{aligned}$$

so we can take $p^2a = 0$ or bab. If $p^2a \neq 0$ then by adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. If $p^2a = 0$ we have

$$\begin{aligned} b'a'b' &= \alpha(\delta^2 - \omega\varepsilon^2)bab, \\ pb' &= \delta pb + \varepsilon pc \\ pc' &= \pm\omega\varepsilon pb \pm \delta pc \end{aligned}$$

so we can take $pb = 0$, $pc = 0$ or bab. So we have 3 algebras.

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, cb,\ p^2a,\ pb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, cb,\ p^2a,\ pb,\ pc - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, cb,\ p^2a - bab,\ pb,\ pc, \text{ class } 3\rangle.$$

$$\langle a, b, c \mid cb,\ pa,\ pc,\ \text{class } 2\rangle$$

Algebra 6.91 has $2p + 88 + \gcd(p-1, 4)$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.91 of order p^7. Then L_3 has order p and is generated by baa, bab, bac, caa, cac, p^2b. If a', b', c' generate L and satisfy $c'b' = pa' = pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa + \alpha\gamma\delta bac + \alpha^2\varepsilon caa + \alpha\gamma\varepsilon cac, \\ b'a'b' &= \alpha\delta^2 bab + 2\alpha\delta\varepsilon bac + \alpha\varepsilon^2 cac, \\ b'a'c' &= \alpha\delta\xi bac + \alpha\varepsilon\xi cac, \\ c'a'a' &= \alpha^2\xi caa + \alpha\gamma\xi cac, \\ c'a'c' &= \alpha\xi^2 cac. \end{aligned}$$

One possibility is that $baa = bab = bac = caa = cac = 0$ and that L_3 is generated by p^2b. So consider the possibilities when at least one of baa, bab, bac, caa, cac is non-zero.

If $cac \neq 0$ we can take $bac = caa = 0$, though we then need $\gamma = \varepsilon = 0$. One possibility is that $baa = bab = 0$. If $baa = 0$ but $bab \neq 0$ we can take $cac = -bab$ or $cac = -\omega bab$. And if $baa \neq 0$ we can take $bab = 0$, $cac = baa$, or $bab = baa$, $cac = -baa$, or $bab = baa$, $cac = -\omega baa$. (6 possibilities)

If $cac = 0$ and $bac \neq 0$ then we can take $baa = bab = 0$, though we then need $\gamma = \varepsilon = 0$. We can then take $caa = 0$ or bac. (2 possibilities)

If $cac = bac = 0$, but $caa \neq 0$ then we can take $baa = 0$ and we can take $bab = 0$ or $caa = bab$. (2 possibilities)

If $bac = caa = cac = 0$ then we can have $baa = 0$, $bab \neq 0$ or $baa \neq 0$, $bab = 0$, or $bab = baa \neq 0$. (3 possibilities)

So we can assume that one of the following 14 sets of commutator relations holds

$$\begin{aligned} baa &= bab = bac = caa = cac = 0, \\ baa &= bab = bac = caa = 0,\ cac \neq 0, \\ baa &= bac = caa = 0,\ cac = -bab \neq 0, \\ baa &= bac = caa = 0,\ cac = -\omega bab \neq 0, \\ bab &= bac = caa = 0,\ cac = baa \neq 0, \\ bac &= caa = 0,\ bab = baa \neq 0,\ cac = -baa, \\ bac &= caa = 0,\ bab = baa \neq 0,\ cac = -\omega baa, \\ baa &= bab = caa = cac = 0,\ bac \neq 0, \\ baa &= bab = cac = 0,\ caa = bac \neq 0, \\ baa &= bab = bac = cac = 0,\ caa \neq 0, \\ baa &= bac = cac = 0,\ caa = bab \neq 0, \\ baa &= bac = caa = cac,\ bab \neq 0, \\ bab &= bac = caa = cac,\ baa \neq 0, \\ bac &= caa = cac,\ bab = baa \neq 0. \end{aligned}$$

In each of these cases we consider the most general generators a', b', c' for L satisfying the commutator relations under consideration, and also satisfying $c'b' = pa' = pc' \in L_3$.

40.7.1 Case 1

Let $baa = bab = bac = caa = cac = 0$. Then L_3 is generated by p^2b and adding suitable scalar multiples of pb to a and c we can take $pa = pc = 0$. We then have $p^2b' = \delta p^2 b$, $c'b' = \delta\xi cb$, so we can take $cb = 0$ or p^2b giving 2 algebras

$$\langle a, b, c \,|\, baa, bab, bac, caa, cac, cb, pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, caa, cac, cb - p^2b,\ pa,\ pc,\ \text{class } 3\rangle.$$

40.7.2 Case 2

Let $baa = bab = bac = caa = 0$, $cac \neq 0$ so that L_3 is generated by cac. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

So we can take $p^2b = 0$ or cac. If $p^2b = cac$ we can take $pa = pc = 0$. If $p^2b = 0$ then we have $pa' = \alpha pa$, $pc' = \xi pc$ so we can take $pa = 0$, cac or ωcac and independantly take $pc = 0$ or cac. So we have 7 algebras

$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa - cac,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa - \omega cac,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa,\ pc - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa - cac,\ pc - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb, p^2b,\ pa - \omega cac,\ pc - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, bab, bac, caa, cb,\ p^2b - cac, pa,\ pc,\ \text{class } 3\rangle.$$

40.7.3 Case 3

Let $baa = bac = caa = 0$, $cac = -bab \neq 0$, so that L_3 is generated by bab. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \pm\delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\delta^2 cac, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

So we can take $p^2b = 0$ or bab. If $p^2b = bab$ we can take $pa = pc = 0$. If $p^2b = 0$ then we have $pa' = \alpha pa$, $pc' = \pm\delta pc$ so we can take $pa = 0$, bab or ωbab and independantly take $pc = 0$ or bab. So we have 7 algebras

$$\langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\begin{gathered} \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa - bab,\ pc, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa - \omega bab,\ pc, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa - bab,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b,\ pa - \omega bab,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + bab, cb, p^2b - bab,\ pa,\ pc, \text{ class } 3\rangle. \end{gathered}$$

40.7.4 Case 4

Let $baa = bac = caa = 0$, $cac = -\omega bab \neq 0$, so that L_3 is generated by bab. This case is almost identical to Case 3, and we have 7 algebras

$$\langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\begin{gathered} \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa - bab,\ pc, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa - \omega bab,\ pc, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa - bab,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b,\ pa - \omega bab,\ pc - bab, \text{ class } 3\rangle, \\ \langle a, b, c \mid baa, bac, caa, cac + \omega bab, cb, p^2b - bab,\ pa,\ pc, \text{ class } 3\rangle. \end{gathered}$$

40.7.5 Case 5

Let $bab = bac = caa = 0$, $cac = baa \neq 0$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^{-1}\xi^2 b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha\xi^2 baa, \\ p^2b' &= \alpha^{-1}\xi^2 p^2 b. \end{aligned}$$

So we can take $p^2b = 0$, baa or ωbaa. If $p^2b \neq 0$ we can take $pa = pc = 0$, and if $p^2b = 0$ we have $pa' = \alpha pa$, $pc' = \xi pc$ so we can take $pa = 0$, baa or ωbaa and independantly take $pc = 0$ or baa. So we have 8 algebras

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa - baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa - \omega baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa,\ pc - baa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa - baa,\ pc - baa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b,\ pa - \omega baa,\ pc - baa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b - baa,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, bac, caa, cac - baa, cb, p^2b - \omega baa,\ pa,\ pc,\ \text{class } 3\rangle.$$

40.7.6 Case 6

Let $bac = caa = 0$, $bab = baa \neq 0$, $cac = -baa$, so that L_3 is generated by baa. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha b, \\ c' &= \pm\alpha c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ p^2b' &= \alpha p^2 b. \end{aligned}$$

So we can take $p^2b = 0$, baa or ωbaa. If $p^2b \neq 0$ we can take $pa = pc = 0$, and if $p^2b = 0$ we have $pa' = \alpha pa$, $pc' = \pm\alpha pc$ so we can take $pa = 0$, baa or ωbaa. If $pa = 0$ we can take $pc = 0$, baa or (if $p = 1 \bmod 4$) ωbaa, and if $pa \neq 0$ we can take $pc = xbaa$ with $0 \leq x \leq (p-1)/2$. So we have $p + 4 + \gcd(p-1, 4)/2$ algebras

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b,\ pa,\ pc - baa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b,\ pa,\ pc - \omega baa,\ \text{class } 3\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b,\ pa - baa,\ pc - xbaa,\ \text{class } 3\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b,\ pa - \omega baa,\ pc - xbaa,\ \text{class } 3\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b - baa,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + baa, cb, p^2b - \omega baa,\ pa,\ pc,\ \text{class } 3\rangle.$$

40.7.7 Case 7

Let $bac = caa = 0$, $bab = baa \neq 0$, $cac = -\omega baa$. This case is almost identical to Case 6, and we have $p + 4 + \gcd(p-1, 4)/2$ algebras

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + \omega baa, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + \omega baa, cb, p^2b,\ pa,\ pc - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + \omega baa, cb, p^2b,\ pa,\ pc - \omega baa, \text{ class } 3\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, bab{-}baa, bac, caa, cac{+}\omega baa, cb, p^2b,\ pa{-}baa,\ pc{-}xbaa, \text{ class } 3\rangle\ (0 \leq x \leq (p{-}1)/2),$$

$$\langle a, b, c \,|\, bab{-}baa, bac, caa, cac{+}\omega baa, cb, p^2b,\ pa{-}\omega baa,\ pc{-}xbaa, \text{ class } 3\rangle\ (0 \leq x \leq (p{-}1)/2),$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + \omega baa, cb, p^2b - baa,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac + \omega baa, cb, p^2b - \omega baa,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.8 Case 8

Let $baa = bab = caa = cac = 0$, $bac \neq 0$, so that L_3 is generated by bac. Adding a suitable scalar multiple of ca to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'c' &= \alpha\delta\xi bac, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

So we can take $p^2b = 0$ or bac. If $p^2b \neq 0$ we can take $pa = pc = 0$, and if $p^2b = 0$ we have $pa' = \alpha pa$, $pc' = \xi pc$ so we can take $pa = 0$, bac and independantly take $pc = 0$ or bac. So we have 5 algebras

$$\langle a, b, c \,|\, baa, bab, caa, cac, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa, cac, cb, p^2b,\ pa,\ pc - bac, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa, cac, cb, p^2b,\ pa - bac,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa, cac, cb, p^2b,\ pa - bac,\ pc - bac, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa, cac, cb, p^2b - bac,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.9 Case 9

Let $baa = bab = cac = 0$, $caa = bac \neq 0$ so that L_3 is generated by bac. Adding a suitable scalar multiple of ca to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'c' &= \alpha^2\xi bac, \\ p^2b' &= \alpha p^2 b. \end{aligned}$$

So we can take $p^2b = 0$ or bac. If $p^2b \neq 0$ we can take $pa = pc = 0$, and if $p^2b = 0$ we have $pa' = \alpha pa$, $pc' = \xi pc$ so we can take $pa = 0$, bac and independantly take $pc = 0$, bac or ωbac. So we have 7 algebras

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa,\ pc - bac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa,\ pc - \omega bac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa - bac,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa - bac,\ pc - bac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b,\ pa - bac,\ pc - \omega bac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, caa - bac, cac, cb, p^2b - bac,\ pa,\ pc,\ \text{class } 3\rangle.$$

40.7.10 Case 10

Let $baa = bab = bac = cac = 0$, $caa \neq 0$ so that L_3 is generated by caa. We have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ p^2b' &= \delta p^2 b, \\ c'b' &= \delta\xi cb. \end{aligned}$$

So we can take $p^2b = 0$ or caa and independantly take $cb = 0$ or caa. If $p^2b = caa$ then we can take $pa = pc = 0$. If $p^2b = 0$ then we have

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ c'b' &= \delta\xi cb, \\ pa' &= \alpha pa + \gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

So we can take cb and pc (independantly) equal to 0, caa or ωcaa. If $pc = caa$ we can take $pa = 0$, and if $pc = 0$ we can take $pa = 0$ or caa. So we have 10 algebras

$$\langle a, b, c \,|\, baa, bab, bac, cac, cb, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, cb, p^2b,\ pa,\ pc - caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, cb, p^2b,\ pa,\ pc - \omega caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, cb - caa, p^2b,\ pa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, bac, cac, cb - caa, p^2b,\ pa,\ pc - caa,\ \text{class } 3\rangle,$$

$$\langle a,b,c \mid baa, bab, bac, cac, cb-caa, p^2b,\ pa,\ pc-\omega caa, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bab, bac, cac, cb, p^2b,\ pa-caa,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bab, bac, cac, cb-caa, p^2b,\ pa-caa,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bab, bac, cac, cb, p^2b-caa,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bab, bac, cac, cb-caa, p^2b-caa,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.11 Case 11

Let $baa = bac = cac = 0$, $caa = bab \neq 0$, so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b, \\ c' &= \alpha^{-1}\delta^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

We can take $p^2b = 0$ or bab, and if $p^2b = bab$ we can take $pa = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ pa' &= \alpha pa + \gamma pc, \\ pc' &= \alpha^{-1}\delta^2 pc. \end{aligned}$$

We can take $pc = 0$, bab or ωbab and if $pc \neq 0$ we can take $pa = 0$. If $pc = 0$ we can take $pa = 0$, bab or ωbab. This gives 6 algebras

$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b,\ pa,\ pc-bab, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b,\ pa,\ pc-\omega bab, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b,\ pa-bab,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b,\ pa-\omega bab,\ pc, \text{ class } 3\rangle,$$
$$\langle a,b,c \mid baa, bac, caa-bab, cac, cb, p^2b-bab,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.12 Case 12

Let $baa = bac = caa = cac$, $bab \neq 0$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

We can take $p^2b = 0$ or bab, and if $p^2b = bab$ we can take $pa = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ pa' &= \alpha pa + \gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $pc = 0$, or bab and if $pc \neq 0$ we can take $pa = 0$. If $pc = 0$ we can take $pa = 0$, bab or ωbab. This gives 5 algebras

$$\langle a, b, c \mid baa, bac, caa, cac, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, bac, caa, cac, cb, p^2b,\ pa,\ pc - bab, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, bac, caa, cac, cb, p^2b,\ pa - bab,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, bac, caa, cac, cb, p^2b,\ pa - \omega bab,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, bac, caa, cac, cb, p^2b - bab,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.13 Case 13

Let $bab = bac = caa = cac$, $baa \neq 0$ so that L_3 is generated by baa. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ p^2b' &= \delta p^2 b, \\ c'b' &= \delta\xi cb. \end{aligned}$$

We can take p^2b equal to 0, baa or ωbaa and independantly take $cb = 0$ or baa. If $p^2b \neq 0$ we can take $pa = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ c'b' &= \delta\xi cb, \\ pa' &= \alpha pa + \gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take cb and pc independantly equal to 0 or baa, and if $pc \neq 0$ we can take $pa = 0$. If $pc = 0$ we can take cb and pa independantly equal to 0 or baa. So we have 10 algebras

$$\langle a, b, c \mid bab, bac, caa, cac, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb, p^2b,\ pa,\ pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb, p^2b,\ pa - baa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb, p^2b - baa,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb, p^2b - \omega baa,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb - baa, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb - baa, p^2b,\ pa,\ pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb - baa, p^2b,\ pa - baa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb - baa, p^2b - baa,\ pa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, caa, cac, cb - baa, p^2b - \omega baa,\ pa,\ pc, \text{ class } 3\rangle.$$

40.7.14 Case 14

Finally, let $bac = caa = cac$, $bab = baa \neq 0$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ p^2 b' &= \alpha p^2 b, \\ c'b' &= \delta\xi cb. \end{aligned}$$

We can take p^2b equal to 0, baa or ωbaa and if $p^2b \neq 0$ we can take $pa = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ pa' &= \alpha pa + \gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

We can pc equal to 0 or baa, and if $pc \neq 0$ we can take $pa = 0$. If $pc = 0$ we can pa equal to 0, baa or ωbaa. So we have 6 algebras

$$\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b,\ pa,\ pc, \text{ class } 3\rangle,$$

$$\begin{aligned} &\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b,\ pa - baa,\ pc, \text{ class } 3\rangle, \\ &\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b,\ pa - \omega baa,\ pc, \text{ class } 3\rangle, \\ &\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b,\ pa,\ pc - baa, \text{ class } 3\rangle, \\ &\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b - baa,\ pa,\ pc, \text{ class } 3\rangle, \\ &\langle a, b, c \,|\, bab - baa, bac, caa, cac, cb, p^2b - \omega baa,\ pa,\ pc, \text{ class } 3\rangle. \end{aligned}$$

40.8 Descendants of 6.92

$$\langle a, b, c \,|\, cb,\ pb - ba,\ pc, \text{ class } 2\rangle$$

Algebra 6.92 has $2p + 13$ descendants of order p^7 and p-class 3.

If L is a descendant of algebra 6.92 of order p^7 then L_3 has order p and is generated by baa, caa, cac, p^2a. If a', b', c' generate L and if $c'b'$, $pb' - b'a'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ c'a'a' &= \xi caa + \gamma\xi cac, \\ c'a'c' &= \xi^2 cac. \end{aligned}$$

If $baa \neq 0$ then L_3 is generated by baa and we can take $caa = cac = 0$ or $caa = 0$, $cac = baa$ or $cac = 0$, $caa = baa$.

If $baa = 0$ then we can always take one of caa and $cac = 0$.

So we have 6 possible commutator structures

$$\begin{aligned} baa &= caa = cac = 0, \\ baa &= caa = 0,\ cac \neq 0, \\ baa &= cac = 0,\ caa \neq 0, \\ caa &= cac = 0,\ baa \neq 0, \\ caa &= 0,\ cac = baa \neq 0, \\ cac &= 0,\ caa = baa \neq 0. \end{aligned}$$

40.8.1 Case 1

Let $baa = caa = cac = 0$, so that L_3 is generated by p^2a. Then we can take $pb-ba = pc = 0$ and we can take $cb = 0$ or p^2a, giving 2 algebras

$$\langle a, b, c \mid baa, caa, cac, cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, cac, cb - p^2a,\ pb - ba,\ pc,\ \text{class } 3\rangle.$$

40.8.2 Case 2

Let $baa = caa = 0$, and let L_3 be generated by cac. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We then have

$$\begin{aligned} a' &= a + \beta b, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \xi^2 cac, \\ p^2a' &= p^2a \end{aligned}$$

so we can take $p^2a = 0$, cac or ωcac. If $p^2a \neq 0$ we can take $pb - ba = pc = 0$. If $p^2a = 0$ then we have

$$\begin{aligned} c'a'c' &= \xi^2 cac, \\ pb' - b'a' &= \delta(pb - ba), \\ pc' &= \xi pc, \end{aligned}$$

so we can take $pb - ba$ and pc to be 0 or cac. So we have 6 algebras

$$\langle a, b, c \mid baa, caa, p^2a, cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, p^2a, cb,\ pb - ba,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, p^2a, cb,\ pb - ba - cac,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, p^2a, cb,\ pb - ba - cac,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, p^2a - cac, cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, p^2a - \omega cac, cb,\ pb - ba,\ pc,\ \text{class } 3\rangle.$$

40.8.3 Case 3

Let $baa = cac = 0$ and let L_3 be generated by caa. We have

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a' &= \xi caa, \\ p^2a' &= p^2a, \\ c'b' &= \delta\xi cb. \end{aligned}$$

We can take p^2a and cb equal to 0 or caa, and if $p^2a = caa$ we can take $pb - ba = pc = 0$. If $p^2a = 0$ then adding a suitable scalar multiple of ca to b we can take $pb - ba = 0$ and we have

$$\begin{aligned} c'a'a' &= \xi caa, \\ c'b' &= \delta\xi cb, \\ pc' &= \xi pc. \end{aligned}$$

So if $p^2a = 0$ we can take $cb = 0$ or caa, $pb - ba = 0$ and $pc = xcaa$ with $0 \leq x < p$. This gives $2p + 2$ algebras

$$\langle a, b, c \mid baa, cac, p^2a, cb, pb - ba,\ pc - xcaa, \text{ class } 3\rangle \ (0 \leq x < p),$$

$$\langle a, b, c \mid baa, cac, p^2a, cb - caa,\ pb - ba,\ pc - xcaa, \text{ class } 3\rangle \ (0 \leq x < p),$$

$$\langle a, b, c \mid baa, cac, p^2a - caa, cb,\ pb - ba,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, cac, p^2a - caa, cb - caa,\ pb - ba,\ pc, \text{ class } 3\rangle.$$

40.8.4 Case 4

Let $caa = cac = 0$, $baa \neq 0$, with L_3 generated by baa. Adding a suitable scalar multiple of pa to c we can take $cb = 0$, adding a suitable scalar multiple of b to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to a we can take $pb - ba = 0$, and adding a suitable scalar multiple of ba to c we can take $pc = 0$. So we have 1 algebra

$$\langle a, b, c \mid caa, cac, p^2a, cb,\ pb - ba,\ pc, \text{ class } 3\rangle.$$

40.8.5 Case 5

Let $caa = 0$, $cac = baa \neq 0$, with L_3 generated by baa. Just as in Case 4 we have 1 algebra

$$\langle a, b, c \mid caa, cac - baa, p^2a, cb,\ pb - ba,\ pc, \text{ class } 3\rangle.$$

40.8.6 Case 6

Let $cac = 0$, $caa = baa \neq 0$, with L_3 generated by baa. Just as in Case 4 we have 1 algebra

$$\langle a, b, c \mid caa - baa, cac, p^2a, cb,\ pb - ba,\ pc, \text{ class } 3\rangle.$$

$$\langle a, b, c \mid cb,\ pb - ca,\ pc, \text{ class } 2\rangle$$

Algebra 6.93 has $p + 15 + 2\gcd(p-1,3) + \gcd(p-1,4)$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.93 of order p^7 then L_3 has order p and is generated by baa, bab, caa, p^2a. If a', b', c' generate L and if $c'b'$, $pb' - c'a'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha\xi b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3\xi baa + \alpha^2\beta\xi bab + \alpha^2\varepsilon caa, \\ b'a'b' &= \alpha^3\xi^2 bab, \\ c'a'a' &= \alpha^2\xi caa. \end{aligned}$$

So we can assume that L satisøes one of the following 5 sets of commutator relations:

$$\begin{aligned} baa &= bab = caa = 0, \\ baa &= bab = 0,\ caa \neq 0, \\ baa &= caa = 0,\ bab \neq 0, \\ baa &= 0,\ caa = bab \neq 0, \\ bab &= caa = 0,\ baa \neq 0. \end{aligned}$$

40.9.1 Case 1

Let $baa = bab = caa = 0$, so that L_3 is generated by p^2a. Adding suitable scalar multiples of pa to b and c we can take $pb - ca = pc = 0$. We have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha\xi b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2a' &= \alpha p^2 a, \\ c'b' &= \alpha\xi^2 cb. \end{aligned}$$

So we can take $cb = 0$, p^2a or ωp^2a, giving 3 algebras

$$\langle a, b, c \mid baa, bab, caa, cb,\ pb - ca,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, caa, cb - p^2a,\ pb - ca,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, caa, cb - \omega p^2a,\ pb - ca,\ pc, \text{ class } 3\rangle.$$

40.9.2 Case 2

Let $baa = bab = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of pa to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha\xi b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ p^2a' &= \alpha p^2 a, \end{aligned}$$

so we can take $p^2a = 0$ or caa. If $p^2a = caa$ then we can take $pb - ca = pc = 0$. If $p^2a = 0$ then we can add a suitable scalar multiple of ca to c so that $pb - ca = 0$, and we can add a suitable scalar multiple of ba to c so that $pc = 0$. So we have 2 algebras

$$\langle a, b, c \,|\, baa, bab, p^2a, cb,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bab, p^2a - caa, cb,\ pb - ca,\ pc,\ \text{class } 3\rangle.$$

40.9.3 Case 3

Let $baa = caa = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha\xi b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^3\xi^2 caa, \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

So we can take $p^2a = 0$, bab or ωbab. If $p^2a \neq 0$ we can take $pb - ca = pc = 0$. If $p^2a = 0$ then we have

$$\begin{aligned} b'a'b' &= \alpha^3\xi^2 bab, \\ pb' - c'a' &= \alpha\xi(pb - ca) + \varepsilon pc, \\ pc' &= \xi pc \end{aligned}$$

so we can take $pc = 0$ or bab, and if $pc = bab$ we can take $pb - ca = 0$. If $pc = 0$ we can take $pb - ca = 0$ or bab. So we have 5 algebras

$$\langle a, b, c \,|\, baa, caa, p^2a, cb,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, p^2a, cb,\ pb - ca,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, p^2a, cb,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, p^2a - bab, cb,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, p^2a - \omega bab, cb,\ pb - ca,\ pc,\ \text{class } 3\rangle.$$

40.9.4 Case 4

Let $baa = 0$ and let $caa = bab \neq 0$. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= b - \alpha^{-1}\beta c, \\ c' &= \alpha^{-1} c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ p^2 a' &= \alpha p^2 a. \end{aligned}$$

So we can take $pa = xbab$ with $0 \leq x < p$, and if $x \neq -1$ we can take $pb - ca = pc = 0$. If $x = -1$ we can take $pb - ca = 0$ and $pc = 0$, bab or ωbab. So we have $p + 2$ algebras

$$\langle a, b, c \,|\, baa, caa - bab, p^2 a - xbab, cb, \; pb - ca, \; pc, \text{ class } 3\rangle \; (0 \leq x < p),$$

$$\langle a, b, c \,|\, baa, caa - bab, p^2 a + bab, cb, \; pb - ca, \; pc - bab, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa - bab, p^2 a + bab, cb, \; pb - ca, \; pc - \omega bab, \text{ class } 3\rangle.$$

40.9.5 Case 5

Let $bab = caa = 0$, and let L_3 be generated by baa. The we have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha\xi b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 \xi baa, \\ p^2 a' &= \alpha p^2 a, \\ c'b' &= \alpha\xi^2 cb. \end{aligned}$$

So we can take $p^2 a = 0$ or baa. If $p^2 a = 0$ we can take $cb = 0$ or baa, and if $p^2 a = baa$ then we need $\xi = \alpha^{-2}$ and so we can take $cb = 0$, baa or ωbaa or (if $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. If $p^2 a \neq 0$ we can take $pb - ca = pc = 0$. And if $p^2 a = 0$ then adding a suitable scalar multiple of ba to c we can take $pb - ca = 0$. We then have

$$\begin{aligned} b'a'a' &= \alpha^3 \xi baa, \\ c'b' &= \alpha\xi^2 cb, \\ pc' &= \xi pc \end{aligned}$$

so we can take $cb = 0$ or baa and take $pc = 0$, baa or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. We have $3 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, bab, caa, p^2 a, cb, \; pb - ca, \; pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2 a, cb, \; pb - ca, \; pc - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2 a, cb, \; pb - ca, \; pc - \omega baa, \text{ class } 3\rangle \; (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab, caa, p^2a, cb, pb-ca, pc-\omega^2 baa, \text{class } 3\rangle \ (p=1 \bmod 3),$$

$$\langle a,b,c \mid bab, caa, p^2a, cb-baa, pb-ca, pc, \text{class } 3\rangle,$$

$$\langle a,b,c \mid bab, caa, p^2a, cb-baa, pb-ca, pc-baa, \text{class } 3\rangle,$$

$$\langle a,b,c \mid bab, caa, p^2a, cb-baa, pb-ca, pc-\omega baa, \text{class } 3\rangle \ (p=1 \bmod 3),$$

$$\langle a,b,c \mid bab, caa, p^2a, cb-baa, pb-ca, pc-\omega^2 baa, \text{class } 3\rangle \ (p=1 \bmod 3),$$

$$\langle a,b,c \mid bab, caa, p^2a-baa, cb, pb-ca, pc, \text{class } 3\rangle,$$

$$\langle a,b,c \mid bab, caa, p^2a-baa, cb-baa, pb-ca, pc, \text{class } 3\rangle,$$

$$\langle a,b,c \mid bab, caa, p^2a-baa, cb-\omega baa, pb-ca, pc, \text{class } 3\rangle,$$

$$\langle a,b,c \mid bab, caa, p^2a-baa, cb-\omega^2 baa, pb-ca, pc, \text{class } 3\rangle \ (p=1 \bmod 4),$$

$$\langle a,b,c \mid bab, caa, p^2a-baa, cb-\omega^3 baa, pb-ca, pc, \text{class } 3\rangle \ (p=1 \bmod 4).$$

40.10 Descendants of 6.94

$$\langle a,b,c \mid cb, pa, pc-ba, \text{class } 2\rangle$$

Algebra 6.94 has $2p+15+2\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7 and p-class 3.

Let L be a descendant of 6.94 or order p^7. Then L_3 is generated by caa, cac, p^2b. If a', b', c' generate L and if $c'a'a'$, $c'a'c'$, $p^2b' \in L_3$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^3\delta caa, \\ c'a'c' &= \alpha^3\delta^2 cac \end{aligned}$$

so we can assume that one of the following four sets of commutator relations holds

$$\begin{aligned} caa &= cac = 0, \\ caa &= 0,\ cac \neq 0, \\ cac &= 0,\ caa \neq 0, \\ caa &= cac \neq 0. \end{aligned}$$

40.10.1 Case 1

Let $caa = cac = 0$, and let L_3 be generated by p^2b. Then we can take $pa = pc - ba = 0$ and we can take $cb = 0$ or p^2b, giving 2 algebras

$$\langle a,b,c \mid caa, cac, cb, pa, pc-ba, \text{class } 3\rangle,$$

$$\langle a,b,c \mid caa, cac, cb-p^2b, pa, pc-ba, \text{class } 3\rangle.$$

40.10.2 Case 2

Let $caa = 0$, and let L_3 be generated by cac. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha^3\delta^2 cac, \\ p^2b' &= \delta p^2 b. \end{aligned}$$

So we can take $p^2b = 0$ or cac, and if $p^2b \neq 0$ we can take $pa = pc - ba = 0$. If $p^2b = 0$ we have

$$\begin{aligned} c'a'c' &= \alpha^3\delta^2 cac, \\ pa' &= \alpha pa, \\ pc' - b'a' &= \alpha\delta(pc - ba) \end{aligned}$$

and so we can take $pa = 0$, cac or ωcac and $pc - ba = 0$ or cac. We have 7 algebras

$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa,\ pc - ba,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa - cac,\ pc - ba,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa - \omega cac,\ pc - ba,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa,\ pc - ba - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa - cac,\ pc - ba - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, caa, p^2b, cb,\ pa - \omega cac,\ pc - ba - cac,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, caa, p^2b - cac, cb,\ pa,\ pc - ba,\ \text{class } 3\rangle.$$

40.10.3 Case 3

Let $cac = 0$, and let L_3 be generated by caa. Adding a suitable scalar multiple of ca to b we can take $pc - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^3\delta caa, \\ p^2b' &= \delta p^2 b, \\ c'b' &= \alpha\delta^2 cb \end{aligned}$$

so we can take $p^2b = 0$, caa or (if $p = 1 \bmod 3$) ωcaa or $\omega^2 caa$ and we can take $cb = 0$ or caa. If $p^2b \neq 0$ we can take $pa = 0$. If $p^2b = 0$ we have

$$\begin{aligned} c'a'a' &= \alpha^3\delta caa, \\ c'b' &= \alpha\delta^2 cb, \\ pa' &= \alpha pa \end{aligned}$$

so we can take $cb = 0$ or caa. If $cb = 0$ we can take $pa = 0$ or caa, and if $cb = caa$ then we can take $pa = 0$, caa, ωcaa or (if $p = 1 \bmod 4$) $\omega^2 caa$ or $\omega^3 caa$. We have $3 + 2\gcd(p-1,3) + \gcd(p-1,4)$ algebras

$$\langle a, b, c \,|\, cac, p^2b, cb, pa, pc - ba, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, cac, p^2b, cb, pa - caa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b, cb - caa, pa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b, cb - caa, pa - caa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b, cb - caa, pa - \omega caa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b, cb - caa, pa - \omega^2 caa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, cac, p^2b, cb - caa, pa - \omega^3 caa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, cac, p^2b - caa, cb, pa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b - \omega caa, cb, pa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cac, p^2b - \omega^2 caa, cb, pa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cac, p^2b - caa, cb - caa, pa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac, p^2b - \omega caa, cb - caa, pa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cac, p^2b - \omega^2 caa, cb - caa, pa, pc - ba, \text{ class } 3\rangle \ (p = 1 \bmod 3).$$

40.10.4 Case 4

Let $caa = cac \neq 0$, so that L_3 is generated by caa. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \alpha c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^3 caa, \\ p^2b' &= p^2b, \\ c'b' &= \alpha cb \end{aligned}$$

so we can take $p^2b = 0$, caa or (if $p = 1 \bmod 3$) ωcaa or $\omega^2 caa$ and if $p^2b \neq 0$ we can take $pa = pc - ba = 0$. If $p^2b = 0$ we have

$$\begin{aligned} c'a'a' &= \alpha^3 caa, \\ pa' &= \alpha pa, \\ pc' - b'a' &= \alpha(pc - ba). \end{aligned}$$

So we can take $pa = 0$, caa or ωcaa. If $pa = 0$ we can take $pc - ba = 0$, caa or ωcaa, and if $pa \neq 0$ we can take $pc - ba = xcaa$ with $0 \leq x < p$. This gives $2p + 3 + \gcd(p-1,3)$ algebras

$$\langle a, b, c \,|\, cac - caa, p^2b, cb, pa, pc - ba, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, cac - caa, p^2b, cb, pa, pc - ba - caa, \text{ class } 3\rangle,$$

$$\langle a,b,c\,|\,cac-caa, p^2b, cb,\ pa,\ pc-ba-\omega caa,\ \text{class } 3\rangle,$$

$$\langle a,b,c\,|\,cac-caa, p^2b, cb,\ pa-caa,\ pc-ba-xcaa,\ \text{class } 3\rangle\ (0\le x<p),$$

$$\langle a,b,c\,|\,cac-caa, p^2b, cb,\ pa-\omega caa,\ pc-ba-xcaa,\ \text{class } 3\rangle\ (0\le x<p),$$

$$\langle a,b,c\,|\,cac-caa, p^2b-caa, cb,\ pa,\ pc-ba,\ \text{class } 3\rangle,$$

$$\langle a,b,c\,|\,cac-caa, p^2b-\omega caa, cb,\ pa,\ pc-ba,\ \text{class } 3\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c\,|\,cac-caa, p^2b-\omega^2 caa, cb,\ pa,\ pc-ba,\ \text{class } 3\rangle\ (p=1 \bmod 3).$$

40.11 Descendants of 6.95

$$\langle a,b,c\,|\,cb,\ pa,\ pc-ca,\ \text{class } 2\rangle$$

Algebra 6.95 has $5p+10$ descendants of order p^7 and p-class 3.

If L is a descendant of algebra 6.95 of order p^7 then L_3 has order p and is generated by baa, bab and p^2b. If a', b', c' generate L and if $c'b'$, pa', $pc'-c'a' \in L_3$ then

$$\begin{aligned} a' &= a, \\ b' &= \delta b+\varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ b'a'b' &= \delta^2 bab. \end{aligned}$$

We can assume that one of the following four sets of commutator relations holds:

$$\begin{aligned} baa &= bab=0, \\ baa &= 0,\ bab\neq 0, \\ bab &= 0,\ baa\neq 0, \\ baa &= bab\neq 0. \end{aligned}$$

40.11.1 Case 1

Let $baa=bab=0$, and let L_3 be generated by p^2b. Then we can take $pa=pc-ca=0$ and we can take $cb=0$ or p^2b. So we have 2 algebras

$$\langle a,b,c\,|\,baa, bab, cb,\ pa,\ pc-ca,\ \text{class } 3\rangle,$$

$$\langle a,b,c\,|\,baa, bab, cb-p^2b,\ pa,\ pc-ca,\ \text{class } 3\rangle.$$

40.11.2 Case 2

Let $baa=0$ and let L_3 be generated by bab. Adding a suitable scalar multiple ba to c we can take $cb=0$. We then have

$$\begin{aligned} a' &= a, \\ b' &= \delta b+\varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \delta^2 bab, \\ p^2b' &= \delta p^2 b, \end{aligned}$$

and so we can take $p^2b = 0$ or bab. If $p^2b = bab$ then we can take $pa = pc - ca = 0$. If $p^2b = 0$ we have

$$\begin{aligned} pa' &= pa, \\ pc' - c'a' &= \xi(pc - ca) \end{aligned}$$

so we can take $pa = 0$, bab or ωbab and we can take $pc - ca = 0$ or bab. So we have 7 algebras

$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa,\ pc - ca,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa - bab,\ pc - ca,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa - \omega bab,\ pc - ca,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa,\ pc - ca - bab,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa - bab,\ pc - ca - bab,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b, cb,\ pa - \omega bab,\ pc - ca - bab,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, baa, p^2b - bab, cb,\ pa,\ pc - ca,\ \text{class } 3\rangle.$$

40.11.3 Case 3

Let $bab = 0$ and let L_3 be generated by baa. We have

$$\begin{aligned} a' &= a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ p^2b' &= \delta p^2 b, \\ c'b' &= \delta\xi cb \end{aligned}$$

and so we can take $p^2b = xbaa$ with $0 \leq x < p$ and we can take $cb = 0$ or baa. If $p^2b \neq 0$ we can take $pa = pc - ca = 0$. If $p^2b = 0$, then adding a suitable scalar multiple of ba to c we can take $pc - ca = 0$. We then have $pa' = pa$ and so we can take $pa = 0$ or baa. We have $2p + 2$ algebras

$$\langle a, b, c \,|\, bab, p^2b - xbaa, cb,\ pa,\ pc - ca,\ \text{class } 3\rangle\ (0 \leq x < p),$$
$$\langle a, b, c \,|\, bab, p^2b - xbaa, cb - baa,\ pa,\ pc - ca,\ \text{class } 3\rangle\ (0 \leq x < p),$$
$$\langle a, b, c \,|\, bab, p^2b, cb,\ pa - baa,\ pc - ca,\ \text{class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, p^2b, cb - baa,\ pa - baa,\ pc - ca,\ \text{class } 3\rangle.$$

40.11.4 Case 4

Let $baa = bab \neq 0$ so that L_3 is generated by baa. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We have

$$\begin{aligned} a' &= a, \\ b' &= b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= baa, \\ p^2 b' &= p^2 b, \end{aligned}$$

and so we can take $p^2 b = xbaa$ with $0 \leq x < p$. If $p^2 b \neq 0$ we can take $pa = pc - ca = 0$. If $p^2 b = 0$ we have

$$\begin{aligned} pa' &= pa, \\ pc' - c'a' &= \xi(pc - ca) \end{aligned}$$

and so we can take $pa = xbaa$ with $0 \leq x < p$ and we can take $pc - ca = 0$ or baa. We have $3p - 1$ algebras

$$\langle a, b, c \mid bab - baa, p^2 b - xbaa, cb,\ pa,\ pc - ca,\ \text{class } 3\rangle\ (0 < x < p),$$

$$\langle a, b, c \mid bab - baa, p^2 b, cb,\ pa - xbaa,\ pc - ca,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid bab - baa, p^2 b, cb,\ pa - xbaa,\ pc - ca - baa,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.12 Descendants of 6.96

$$\langle a, b, c \mid cb,\ pa - ba,\ pc,\ \text{class } 2\rangle$$

Algebra 6.96 has $2p + 26$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.96 of order p^7 then L_3 is generated by bab, caa, cac and $p^2 b$. If a', b', c' generate L and if $c'b'$, $pa' - b'c'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ c'a'a' &= \alpha^2 \xi caa + \alpha\gamma\xi cac, \\ c'a'c' &= \alpha\xi^2 cac. \end{aligned}$$

So we can assume that a, b, c satsify one of the following 7 sets of commutator relations:

$$\begin{aligned} bab &= caa = cac = 0, \\ bab &= caa = 0,\ cac \neq 0, \\ bab &= cac = 0,\ caa \neq 0, \\ caa &= cac = 0,\ bab \neq 0, \\ caa &= 0,\ cac = -bab \neq 0, \\ caa &= 0,\ cac = -\omega bab \neq 0, \\ cac &= 0,\ caa = bab \neq 0. \end{aligned}$$

40.12.1 Case 1

Let $bab = caa = cac = 0$, and let L_3 be generated by p^2b. Then we can take $pa - ba = pc = 0$ and we can take $cb = 0$ or p^2b, giving 2 algebras

$$\langle a, b, c \,|\, bab, caa, cac, cb, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, cac, cb - p^2b, pa - ba, pc, \text{class } 3\rangle.$$

40.12.2 Case 2

Let $bab = caa = 0$, $cac \neq 0$, so that L_3 is generated by cac. Adding a suitable scalar multiple of ca to b we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ p^2b' &= p^2b \end{aligned}$$

so we can take $p^2b = 0$ or cac. If $p^2b = cac$ we can take $pa - ba = pc = 0$. And if $p^2b = 0$ then we have

$$\begin{aligned} pa' - b'a' &= \alpha(pa - ba), \\ pc' &= \xi pc \end{aligned}$$

so we can take $pa - ba = 0$, cac or ωcac and we can take $pc = 0$ or cac. We have 7 algebras

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba - cac, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba - \omega cac, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba, pc - cac, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba - cac, pc - cac, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b, cb, pa - ba - \omega cac, pc - cac, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, p^2b - cac, cb, pa - ba, pc, \text{class } 3\rangle.$$

40.12.3 Case 3

Let $bab = cac = 0$, $caa \neq 0$ so that L_3 is generated by caa. Adding a suitable scalar multiple of ca to b we may assume that $pb - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ p^2b' &= p^2b, \\ c'b' &= \xi cb \end{aligned}$$

so we can take $p^2b = 0$ or caa and we can take $cb = 0$, caa or ωcaa. If $p^2b = caa$ we can take $pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} c'b' &= \xi cb, \\ pc' &= \xi pc \end{aligned}$$

and so if $cb = 0$ we can take $pc = 0$, caa or ωcaa, and if $cb \neq 0$ we can take $pc = xcaa$ with $0 \leq x < p$. We have $2p + 6$ algebras

$$\langle a, b, c \,|\, bab, cac, p^2b, cb, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, cac, p^2b, cb, pa - ba, pc - caa, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, cac, p^2b, cb, pa - ba, pc - \omega caa, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, cac, p^2b, cb - caa, pa - ba, pc - xcaa, \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, bab, cac, p^2b, cb - \omega caa, pa - ba, pc - xcaa, \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, bab, cac, p^2b - caa, cb, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, cac, p^2b - caa, cb - caa, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, cac, p^2b - caa, cb - \omega caa, pa - ba, pc, \text{class } 3\rangle.$$

40.12.4 Case 4

Let $caa = cac = 0$, $bab \neq 0$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ p^2b' &= p^2b \end{aligned}$$

and so we can take $p^2b = 0$ or bab. If $p^2b = bab$ we can take $pa - ba = pc = 0$. If $p^2b = 0$ then adding a suitable scalar multiple of pb to b we can take $pa - ba = 0$, and we then have

$$\begin{aligned} b'a'b' &= \alpha bab, \\ pc' &= \xi bab. \end{aligned}$$

So if $p^2b = 0$ we can take $pc = 0$ or bab. We have 3 algebras

$$\langle a, b, c \,|\, caa, cac, p^2b, cb, pa - ba, pc, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, caa, cac, p^2b, cb, pa - ba, pc - bab, \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, caa, cac, p^2b - bab, cb, pa - ba, pc, \text{class } 3\rangle.$$

40.12.5 Case 5

Let $caa = 0$, $cac = -bab \neq 0$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \pm c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ p^2b' &= p^2b \end{aligned}$$

and so we can take $p^2b = 0$ or bab. Adding a suitable scalar multiple of pb to b we can take $pa - ba = 0$, and adding suitable scalar multiple of ba to b and c we can take $pc = 0$. We have 2 algebras

$$\langle a, b, c \,|\, caa, cac + bab, p^2b, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, caa, cac + bab, p^2b - bab, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle.$$

40.12.6 Case 6

Let $caa = 0$, $cac = -\omega bab \neq 0$ so that L_3 is generated by bab. This case is almost identical to Case 5 and we have 2 algebras

$$\langle a, b, c \,|\, caa, cac + \omega bab, p^2b, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, caa, cac + \omega bab, p^2b - bab, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle.$$

40.12.7 Case 7

Let $cac = 0$, $caa = bab \neq 0$ so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$, and adding a suitable scalar multiple of pb to b we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= b, \\ c' &= \alpha^{-1}c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ p^2b' &= p^2b \end{aligned}$$

so we can take $p^2b = 0$ or bab. If $p^2b = bab$ we can take $pc = 0$, and if $p^2b = 0$ we have $pc' = \alpha^{-1}pc$ so we can take $pc = 0$, bab or ωbab. We have 4 algebras

$$\langle a, b, c \,|\, cac, caa - bab, p^2b, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cac, caa - bab, p^2b, cb,\ pa - ba,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cac, caa - bab, p^2b, cb,\ pa - ba,\ pc - \omega bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cac, caa - bab, p^2b - bab, cb,\ pa - ba,\ pc,\ \text{class } 3\rangle.$$

$$\langle a, b, c \mid cb,\ pa - ca,\ pc,\ \text{class } 2\rangle$$

Algebra 6.97 has $3p + 18 + \gcd(p - 1, 3)$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.97 of order p^7 then L_3 has order p and is generated by baa, bab, bac and p^2b. If a', b', c' generate L and if $c'b'$, $pa' - c'a'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa + \alpha\gamma\delta bac, \\ b'a'b' &= \alpha\delta^2 bab + 2\alpha\delta\varepsilon bac, \\ b'a'c' &= \alpha\delta bac. \end{aligned}$$

So we can assume that a, b, c satsify one of the following 5 sets of commutator relations:

$$\begin{aligned} baa &= bab = bac = 0, \\ baa &= bab = 0,\ bac \neq 0, \\ baa &= bac = 0,\ bab \neq 0, \\ bab &= bac = 0,\ baa \neq 0, \\ bac &= 0,\ baa = bab \neq 0. \end{aligned}$$

40.13.1 Case 1

Let $baa = bab = bac = 0$ and let L_3 be generated by p^2b. We have

$$\begin{aligned} p^2b' &= \delta p^2 b, \\ c'b' &= \delta cb \end{aligned}$$

and so we can take $cb = xp^2b$ with $0 \leq x < p$. We can take $pa - ca = pc = 0$, and so we have p algebras

$$\langle a, b, c \mid baa, bab, bac, cb - xp^2b,\ pa - ca,\ pc,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.13.2 Case 2

Let $baa = bab = 0$ and let L_3 be generated by bac. Adding a suitable scalar multiple of ca to c we can take $cb = 0$, adding a suitable scalar multiple of pb to c we can take $pa - ca = 0$, and adding a suitable scalar multiple of ba to c we can take $pc = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\delta bac, \\ p^2b' &= \delta p^2 b \end{aligned}$$

and so we can take $p^2b = 0$ or bac, giving 2 algebras

$$\langle a, b, c \mid baa, bab, p^2b, cb,\ pa - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, p^2b - bac, cb,\ pa - ca,\ pc,\ \text{class } 3\rangle.$$

40.13.3 Case 3

Let $baa = bac = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab, \\ p^2b' &= \delta p^2 b \end{aligned}$$

and so we can take $p^2b = 0$ or bab. If $p^2b = bab$ then we can take $pa - ca = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} pa' - c'a' &= \alpha(pa - ca) + \gamma pc, \\ pc' &= pc \end{aligned}$$

so we can take $pc = 0$ and $pa - ca = 0$, bab or ωbab or $pa - ca = 0$ and $pc = bab$. This gives 5 algebras

$$\langle a, b, c \mid baa, bac, p^2b, cb,\ pa - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, p^2b, cb,\ pa - ca - bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, p^2b, cb,\ pa - ca - \omega bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, p^2b, cb,\ pa - ca,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, p^2b - bab, cb,\ pa - ca,\ pc,\ \text{class } 3\rangle.$$

40.13.4 Case 4

Let $bab = bac = 0$ and let L_3 be generated by baa. Adding a suitable scalar multiple of ba to c we can take $pa - ca = 0$. We have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ p^2b' &= \delta p^2 b, \\ c'b' &= \delta cb \end{aligned}$$

and so we can take $p^2b = 0$, baa or ωbaa. If $p^2b \neq 0$ we can take $cb = xbaa$ with $0 \leq x < p$ and we can take $pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ c'b' &= \delta cb, \\ pc' &= pc \end{aligned}$$

and so we can take $cb = 0$, baa or ωbaa and we can take $pc = 0$ or baa. We have $2p + 6$ algebras

$$\langle a, b, c \mid bab, bac, p^2b, cb, pa - ca, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b, cb - baa, pa - ca, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b, cb - \omega baa, pa - ca, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b, cb, pa - ca, pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b, cb - baa, pa - ca, pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b, cb - \omega baa, pa - ca, pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, bac, p^2b - baa, cb - xbaa, pa - ca, pc, \text{ class } 3\rangle \ (0 \leq x < p),$$
$$\langle a, b, c \mid bab, bac, p^2b - \omega baa, cb - xbaa, pa - ca, pc, \text{ class } 3\rangle \ (0 \leq x < p).$$

40.13.5 Case 5

Let $bac = 0$ and let L_3 be generated by baa with $bab = baa$. Adding a suitable scalar multiple of ba to c we can take $cb = 0$. We have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ p^2b' &= \alpha p^2 b \end{aligned}$$

and so we can take $p^2b = 0$, baa or ωbaa. If $p^2b \neq 0$ then we can take $pa - ca = pc = 0$. If $p^2b = 0$ we have

$$\begin{aligned} pa' - c'a' &= \alpha(pa - ca) + \gamma pc, \\ pc' &= pc \end{aligned}$$

so we can take $pc = 0$ and $pa - ca = 0$, baa or ωbaa, or we can take $pa - ca = 0$ and $pc = baa$ or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. We have

$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca - baa, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca - \omega baa, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca, pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca, pc - \omega baa, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab - baa, bac, p^2b, cb, pa - ca, pc - \omega^2 baa, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab - baa, bac, p^2b - baa, cb, pa - ca, pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab - baa, bac, p^2b - \omega baa, cb, pa - ca, pc, \text{ class } 3\rangle.$$

$$\langle a, b, c \mid cb,\ pb - ba,\ pc - \mu ca,\ \text{class } 2\rangle\ (\mu \neq 0,\ \mu, \mu^{-1} \text{ give isomorphic algebras})$$

Algebra 6.98 has $\frac{5}{2}p+\frac{1}{2}$ descendants of order p^7 and p-class 3. There are 4 descendants when $\mu = 1$, 4 when $\mu = -1$, and 5 when $\mu \neq \pm 1$.

If L is a descendant of algebra 6.98 of order p^7 then L_3 is generated by baa, caa and p^2a. The automorphism group of algebra 6.98 depends on the value of μ, and we need to distinguish between the cases $\mu = 1$, $\mu = -1$ and $\mu \neq \pm 1$.

40.14.1 $\mu = 1$

If a', b', c' generate L and if $c'b'$, $pb' - b'a'$, $pc' - c'a' \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \sigma b + \tau c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa + \varepsilon caa, \\ c'a'a' &= \sigma baa + \tau caa \end{aligned}$$

and so we can assume that a, b, c satisfy one the following 2 sets of commutator relations

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\ caa \neq 0. \end{aligned}$$

Case 1 Let $baa = caa = 0$ and let L_3 be generated by p^2a. We can take $pb - ba = pc - ca = 0$. We have

$$\begin{aligned} p^2a' &= p^2a, \\ c'b' &= (\delta\tau - \varepsilon\sigma)cb \end{aligned}$$

and so we can take $cb = 0$ or p^2a giving 2 algebras

$$\langle a, b, c \mid baa, caa, cb,\ pb - ba,\ pc - ca,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, cb - p^2a,\ pb - ba,\ pc - ca,\ \text{class } 3\rangle.$$

Case 2 Let $baa = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of pa to b we can take $cb = 0$, and adding a suitable scalar multiple of c to a we can take $p^2a = 0$, and adding a suitable scalar multiple of pa to a we can take $pc - ca = 0$. We then have 2 algebras

$$\langle a, b, c \mid baa, p^2a, cb,\ pb - ba,\ pc - ca,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2a, cb,\ pb - ba - caa,\ pc - ca,\ \text{class } 3\rangle.$$

40.14.2 $\mu \neq 1, -1$

If a', b', c' generate L and if $c'b'$, $pb' - b'a'$, $pc' - \mu c'a' \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \tau c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ c'a'a' &= \tau caa, \end{aligned}$$

and so we can assume that a, b, c satisfy one of the following 4 sets of commutator relations

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\ caa \neq 0, \\ caa &= 0,\ baa \neq 0, \\ baa &= caa \neq 0. \end{aligned}$$

Case 3 Let $baa = caa = 0$, and let L_3 be generated by p^2a. Then we can take $pb - ba = pc - \mu ca = 0$ and $cb = 0$ or p^2a, giving 2 algebras

$\langle a, b, c \,|\, baa, caa, cb,\ pb{-}ba,\ pc{-}\mu ca,$ class 3$\rangle$ ($\mu \neq 0, \pm 1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \,|\, baa, caa, cb{-}p^2a,\ pb{-}ba,\ pc{-}\mu ca,$ class 3$\rangle$ ($\mu \neq 0, \pm 1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 4 Let $baa = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of c to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to b we can take $cb = 0$, adding a suitable scalar multiple of ca to p we can take $pb - ba = 0$, and adding a suitable scalar multiple of pa to a we can take $pc - \mu ca = 0$. This gives 1 algebra

$\langle a, b, c \,|\, baa, p^2a, cb,\ pb{-}ba,\ pc{-}\mu ca,$ class 3$\rangle$ ($\mu \neq 0, \pm 1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 5 Let $caa = 0$ and let L_3 be generated by baa. As in Case 4 we have 1 algebra

$\langle a, b, c \,|\, caa, p^2a, cb,\ pb{-}ba,\ pc{-}\mu ca,$ class 3$\rangle$ ($\mu \neq 0, \pm 1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 6 Let L_3 be generated by baa and let $caa = baa$. As in the previous two cases we have 1 algebra

$\langle a, b, c \,|\, caa{-}baa, p^2a, cb,\ pb{-}ba,\ pc{-}\mu ca,$ class 3$\rangle$ ($\mu \neq 0, \pm 1,\ \mu, \mu^{-1}$ give isomorphic algebras).

40.14.3 $\mu = -1$

If a', b', c' generate L and if $c'b'$, $pb' - b'a'$, $pc' + c'a' \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \tau c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ c'a'a' &= \tau caa, \end{aligned}$$

or

$$\begin{aligned} a' &= -a+\beta b+\gamma c, \\ b' &= \varepsilon c, \\ c' &= \sigma b \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \varepsilon caa, \\ c'a'a' &= \sigma baa, \end{aligned}$$

and so we can assume that a, b, c satisfy one of the following 3 sets of commmutator relations

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\, caa \neq 0, \\ baa &= caa \neq 0. \end{aligned}$$

As in the case when $\mu \neq 1, -1$ we have 4 algebras

$$\langle a, b, c \mid baa, caa, cb,\ pb - ba,\ pc + ca,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, cb - p^2a,\ pb - ba,\ pc + ca,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, p^2a, cb,\ pb - ba,\ pc + ca,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa - baa, p^2a, cb,\ pb - ba,\ pc + ca,\ \text{class } 3\rangle.$$

40.15 Descendants of 6.99

$$\langle a, b, c \mid cb,\ pb - ba - ca,\ pc - ca,\ \text{class } 2\rangle$$

Algebra 6.99 has $p+4$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.99 of order p^7 then L_3 has order p and is generated by baa, caa and p^2a. If a', b', c' generate L and if $c'b'$, $pb' - b'a' - c'a'$, $pc' - c'a' \in L_3$ then

$$\begin{aligned} a' &= a+\beta b+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa + \varepsilon caa, \\ c'a'a' &= \delta caa. \end{aligned}$$

So we can assume that a, b, c satisfy one of the following 3 sets of commutator relations:

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\, caa \neq 0, \\ caa &= 0,\, baa \neq 0. \end{aligned}$$

40.15.1 Case 1

Let $baa = caa = 0$ and let p^2a generate L_3. Then we can take $pb - ba - ca = pc - ca = 0$ and we can take $cb = 0$, p^2a or ωp^2a so we have 3 algebras

$$\langle a, b, c \mid baa, caa, cb,\ pb - ba - ca,\ pc - ca, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, caa, cb - p^2a,\ pb - ba - ca,\ pc - ca, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, caa, cb - \omega p^2a,\ pb - ba - ca,\ pc - ca, \text{ class } 3\rangle.$$

40.15.2 Case 2

Let $baa = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of c to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to c we can take $cb = 0$, and adding suitable scalar multiples of ca and ba to c we can take $pb - ba - ca = pc - ca = 0$. So we have 1 algebra

$$\langle a, b, c \mid baa, p^2a, cb,\ pb - ba - ca,\ pc - ca, \text{ class } 3\rangle.$$

40.15.3 Case 3

Let $caa = 0$ and let L_3 be generated by baa. Adding a suitable scalar multiple of b to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to c we can take $cb = 0$, and adding a suitable scalar multiple of ba to c we can take $pb - ba - ca = 0$. We then have

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa, \\ pc' - c'a' &= \delta(pc - ca) \end{aligned}$$

and so we can take $pc - ca = xbaa$ with $0 \leq x < p$. We have p algebras

$$\langle a, b, c \mid caa, p^2a, cb,\ pb - ba - ca,\ pc - ca - xbaa, \text{ class } 3\rangle\ (0 \leq x < p).$$

40.16 Descendants of 6.100

$$\langle a, b, c \mid cb,\ pb - \omega ca,\ pc - ba, \text{ class } 2\rangle$$

Algebra 6.100 has 3 descendants of order p^7 and p-class 3.

If L is a descendants of algebra 6.100 of order p^7 then L_3 has order p and is generated by baa, caa and p^2a. If a', b', c' generate L and if $c'b'$, $pb' - \omega c'a'$, $pc' - b'a' \in L_3$ then

$$\begin{aligned} a' &= \pm a + \beta b + \gamma c, \\ b' &= \delta b \pm \omega\nu c, \\ c' &= \nu b \pm \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta baa \pm \omega\nu caa, \\ c'a'a' &= \nu baa \pm \delta caa. \end{aligned}$$

So we can assume that a, b, c satisfy one of the following two sets of commutator relations

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\ caa \neq 0. \end{aligned}$$

40.16.1 Case 1

Let $baa = caa = 0$ and let L_3 be generated by p^2a. Then we can take $pb-\omega ca = pc-ba = 0$ and we can take $cb = 0$ or p^2a, giving 2 algebras

$$\langle a, b, c \mid baa, caa, cb,\ pb - \omega ca,\ pc - ba,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, caa, cb - p^2a,\ pb - \omega ca,\ pc - ba,\ \text{class } 3\rangle.$$

40.16.2 Case 2

Let $baa = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of c to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to c we can take $cb = 0$, and adding suitable scalar multiple of ca and ba to c we can take $pb - \omega ca = pc - ba = 0$. We have 1 algebra

$$\langle a, b, c \mid baa, p^2a, cb,\ pb - \omega ca,\ pc - ba,\ \text{class } 3\rangle.$$

40.17 Descendants of 6.101

$$\langle a, b, c \mid cb,\ pb - \mu ca,\ pc - ba - ca,\ \text{class } 2\rangle\ (x^2 - x - \mu \text{ irreducible})$$

Algebra 6.101 has 3 descendants of order p^7 and p-class 3 for each value of x, giving $\frac{3}{2}(p-1)$ algebras in all.

If L is a descendant of 6.101 of order p^7 then L_3 has order p and is generated by baa, caa and p^2a. If a', b', c' generate L and if $c'b'$, $pb - \mu ca$, $pc - ba - ca \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= (\xi - \nu)b + \mu\nu c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= (\xi - \nu)baa + \mu\nu caa, \\ c'a'a' &= \nu baa + \xi caa. \end{aligned}$$

So we can assume that a, b, c satisfy one of the following two sets of commutator relations:

$$\begin{aligned} baa &= caa = 0, \\ baa &= 0,\ caa \neq 0. \end{aligned}$$

40.17.1 Case 1

Let $baa = caa = 0$ and let L_3 be generated by p^2a. Then we can take $pb - \mu ca = pc - ba - ca = 0$ and we have $pc - ba - ca \in L_3$ then

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= (\xi - \nu)b + \mu\nu c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2a' &= p^2a, \\ c'b' &= (\xi^2 - \nu\xi - \mu\nu^2)cb. \end{aligned}$$

Now $\xi^2 - \nu\xi - \mu\nu^2 = (\xi - \frac{\nu}{2})^2 - (\frac{1}{4} + \mu)\nu^2$, and so (since $(\frac{1}{4} + \mu) \neq 0$) this takes on all possible non-zero values. So we can take $cb = 0$ or p^2a, giving 2 algebras for each of the $(p-1)/2$ values of μ.

$$\langle a, b, c \mid baa, caa, cb, pb - \mu ca, pc - ba - ca, \text{class } 3\rangle \ (x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \mid baa, caa, cb - p^2a, pb - \mu ca, pc - ba - ca, \text{class } 3\rangle \ (x^2 - x - \mu \text{ irreducible}).$$

40.17.2 Case 2

Let $baa = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of b to a we can take $p^2a = 0$, adding a suitable scalar multiple of pa to c we can take $cb = 0$, and adding suitable scalar multiple of ca and ba to b we can take $pb - \mu ca = pc - ba - ca = 0$. We have 1 algebra for each of the $(p-1)/2$ values of μ.

$$\langle a, b, c \mid baa, p^2a, cb, pb - \mu ca, pc - ba - ca, \text{class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}).$$

40.18 Descendants of 6.102

$$\langle a, b, c \mid cb, pa - ca, pc - ba, \text{class } 2\rangle$$

Algebra 6.102 has $p + 3$ descendants of order p^7 and p-class 3.

If L is a descendant of algebra 6.102 of order p^7 then L_3 has order p and is generated by baa and p^2b. If a', b', c' generate L and if $c'b'$, $pa' - c'a'$, $pc' - b'a' \in L_3$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^{-1}b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha baa, \\ p^2b' &= \alpha^{-1}p^2b \end{aligned}$$

and so we can assume that $baa = 0$ and that L_3 is generated by p^2b, or we can assume that L_3 is generated by baa and that $p^2b = 0$, baa or ωbaa.

40.18.1 $baa = 0$

If $baa = 0$ and L_3 is generated by p^2b then we can take $pa - ca = pc - ba = 0$, and we have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^{-1}b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2b' &= \alpha^{-1}p^2b, \\ c'b' &= \alpha^{-1}cb. \end{aligned}$$

So we have p algebras

$$\langle a, b, c \,|\, baa, cb - xp^2b,\ pa - ca,\ pc - ba, \text{ class } 3\rangle\ (0 \leq x < p).$$

40.18.2 $\quad p^2b = 0$

If L_3 is generated by baa and $p^2b = 0$ then adding a suitable scalar multiple of ca to b we can take $cb = 0$, adding a suitable scalar multiple of ba to c we can take $pa - ca = 0$, and adding a suitable scalar multiple of ca to c we can take $pc - ba = 0$. So we have 1 algebra

$$\langle a, b, c \,|\, p^2b, cb,\ pa - ca,\ pc - ba, \text{ class } 3\rangle.$$

40.18.3 $\quad p^2b = kbaa$ with $k = 1, \omega$

Let L_3 be generated by baa and $p^2b = kbaa$ with $k = 1$ or ω. Then, just as in the case when $p^2b = 0$ we get 1 algebra for each value of k.

$$\langle a, b, c \,|\, p^2b - baa, cb,\ pa - ca,\ pc - ba, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, p^2b - \omega baa, cb,\ pa - ca,\ pc - ba, \text{ class } 3\rangle.$$

40.19 Descendants of 6.103

$$\langle a, b, c \,|\, cb,\ pa - ba,\ pc - ca, \text{ class } 2\rangle$$

Algebra 6.103 has $p + 3$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.103 of order p^7 then L_3 has order p and is generated by bab and p^2b. If a', b', c' generate L and if $c'b'$, $pa' - b'a'$, $pc' - c'a' \in L_3$ then

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= b + \gamma c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= bab, \\ p^2b' &= p^2b \end{aligned}$$

so we can assume that $bab = 0$ and that L_3 is generated by p^2b, or that L_3 is generated by bab and that $p^2b = xbab$ for some x with $0 \leq x < p$.

40.19.1 $\quad bab = 0$

Let $bab = 0$ and let L_3 be generated by p^2b. We can take $pa - ba = pc - ca = 0$, and we have

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= b + \gamma c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} p^2 b' &= p^2 b, \\ c'b' &= \xi cb. \end{aligned}$$

So we have 2 algebras

$$\langle a, b, c \mid bab, cb, pa - ba, pc - ca, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid bab, cb - p^2b, pa - ba, pc - ca, \text{ class } 3\rangle.$$

40.19.2 $\quad p^2b = xbab$

Let L_3 be generated by bab and let $p^2b = xbab$ with $0 \le x < p$. Adding a suitable scalar multiple of ba to c we can take $cb = 0$, and adding a suitable scalar multiple of pb to b we can take $pa - ba = 0$. If $x \neq -1$ then adding a suitable scalar multiple of pb to c we can take $pc - ca = 0$. but if $x = -1$ then we have

$$\begin{aligned} p^2 b' &= p^2 b, \\ pc' - c'a' &= \xi(pc - ca) \end{aligned}$$

so we can take $pc - ca = 0$ or bab. So we have $p + 1$ algebras

$$\langle a, b, c \mid p^2b - xbab, cb, pa - ba, pc - ca, \text{ class } 3\rangle \; (0 \le x < p),$$
$$\langle a, b, c \mid p^2b + bab, cb, pa - ba, pc - ca - bab, \text{ class } 3\rangle.$$

40.20 Descendants of 6.104

$$\langle a, b, c \mid pa, pb, pc, \text{ class } 2\rangle$$

Algebra 6.104 has $5p + 24 + 3\gcd(p - 1, 3)$ descendants of order p^7 and p-class 3.

If L is a descendant of 6.104 of order p^7 then L_3 has order p and is generated by the commutators of weight 3 in a, b, c. The commutator structure of L must correspond to one of 7.89 ˘ 7.97 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we may assume that a, b, c satisfy one of the following 8 sets of commutator relations.

$$\begin{aligned}
bab &= bac = caa = cab = cac = cbb = cbc = 0, \\
baa &= bac = caa = cab = cbb = cbc = 0,\; cac = -bab, \\
baa &= bac = caa = cab = cbb = cbc = 0,\; cac = -\omega bab, \\
baa &= bac = cab = cac = cbb = cbc = 0,\; caa = bab, \\
bab &= bac = caa = cab = 0,\; cac = baa,\; cbb = baa,\; cbc = xbaa, \\
bab &= bac = caa = cab = 0,\; cac = baa,\; cbb = \omega baa,\; cbc = xbaa \,(p = 1 \bmod 3), \\
bab &= bac = caa = cab = 0,\; cac = baa,\; cbb = \omega^2 baa,\; cbc = xbaa \,(p = 1 \bmod 3), \\
bac &= cab = 0,\; baa = bab = caa = cbc,\; cac = cbb = -baa.
\end{aligned}$$

where $x = 0$ or x lies in a transversal for the cube roots of 1. In particular the descendants of algebra 6.104 of order p^7 and with characteristic p are:

$$\langle a, b, c \mid bab, bac, caa, cab, cac, cbb, cbc, pa, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac - bab, cbb, cbc, pa, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac - \omega bab, cbb, cbc, pa, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa - bab, cab, cac, cbb, cbc, pa, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cab, cac - baa, cbb - baa, cbc - xbaa, pa, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cab, cac-baa, cbb-\omega baa, cbc-xbaa, pa, pb, pc, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab, bac, caa, cab, cac-baa, cbb-\omega^2 baa, cbc-xbaa, pa, pb, pc, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab - baa, bac, caa - baa, cab, cac + baa, cbb + baa, cbc - baa, pa, pb, pc, \text{ class } 3\rangle.$$

40.20.1 $pL \neq \{0\}$

Let L be a descendant of algbra 6.104 or order p^7 and suppose that $pL \neq \{0\}$. Then we can pick generators a, b, c for L with $pa \neq 0$ and $pb = pc = 0$, and we consider the possible commutator relations that a, b, c might satisfy. If a', b', c' generate L and if $pb' = pc' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2. So $c'b'b'$ and $c'b'c'$ span the same space as cbb and cbc. Since L_3 has order p we can assume that either $cbb = cbc = 0$ or that $cbb \neq 0$, $cbc = 0$.

40.20.2 $cbb = cbc = 0$

Consider the case when $cbb = cbc = 0$. Then if a', b', c' are as above we have

$$\begin{aligned} b'a'b' &= \alpha\delta^2 bab + \alpha\delta\varepsilon bac + \alpha\delta\varepsilon cab + \alpha\varepsilon^2 cac, \\ b'a'c' &= \alpha\delta\nu bab + \alpha\delta\xi bac + \alpha\varepsilon\nu cab + \alpha\varepsilon\xi cac, \\ c'a'b' &= \alpha\delta\nu bab + \alpha\varepsilon\nu bac + \alpha\delta\xi cab + \alpha\varepsilon\xi cac, \\ c'a'c' &= \alpha\nu^2 bab + \alpha\nu\xi bac + \alpha\nu\xi cab + \alpha\xi^2 cac. \end{aligned}$$

One possibility is that $bab = bac = cab = cac = 0$, and another possibility is that $bab = cac = 0$, $cab = -bac \neq 0$. In every other case we can take $bab \neq 0$.

If $bab = bac = cab = cac = 0$ then L_3 is spanned by baa and caa and clearly we can take $caa = 0$. So we have

$$\langle a, b, c \mid bab, bac, caa, cab, cac, cbb, cbc, pa - baa, pb, pc, \text{ class } 3\rangle.$$

If $bab = cac = 0$, $cab = -bac \neq 0$ then we can take $baa = caa = 0$ giving

$$\langle a, b, c \mid baa, bab, caa, cab + bac, cac, cbb, cbc, pa - bac, pb, pc, \text{ class } 3\rangle.$$

If $bab \neq 0$, considering a', b', c' as above with $\varepsilon = 0$ we can choose ν so that $b'a'b' \neq 0$, $b'a'c' = 0$. So we can assume that $bab \neq 0$ and that $bac = 0$. Scaling, we can take $cab = 0$ or bab.

So consider the case when $bab \neq 0$, $bac = cab = 0$. Let $cac = xbab$. Then

$$\begin{aligned} b'a'b' &= (\alpha\delta^2 + \alpha\varepsilon^2 x)bab, \\ b'a'c' &= (\alpha\delta\nu + \alpha\varepsilon\xi x)bab, \\ c'a'b' &= (\alpha\delta\nu + \alpha\varepsilon\xi x)bab, \\ c'a'c' &= (\alpha\nu^2 + \alpha\xi^2 x)bab. \end{aligned}$$

Considering a', b', c' with $\varepsilon = \nu = 0$ we see that we can take $x = 0$ or -1 or $-\omega$, and it is not hard to see that these three possibilities are distinct. If $cac = kbab$ with $k = 1$ or ω then we can take $baa = caa = 0$. If we consider a', b', c' with $b'a'a' = c'a'a' = b'a'c' = c'a'b' = 0$, $c'a'c' = kb'a'b'$ then we have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm k\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and $b'a'b' = (\delta^2 - k\varepsilon^2)bab$. So we can take $pa = bab$ in both cases giving

$$\langle a, b, c \,|\, baa, bac, caa, cab, cac + bab, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cab, cac + \omega bab, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle.$$

On the other hand, if $cac = 0$ then we need $\nu = 0$ and we can take $baa = 0$, $caa = 0$ or bab. If $baa = caa = 0$ we have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and $b'a'b' = \alpha\delta^2 bab$ and so we have

$$\langle a, b, c \,|\, baa, bac, caa, cab, cac, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cab, cac, cbb, cbc, pa - \omega bab, pb, pc, \text{ class } 3\rangle.$$

If $baa = 0$, $caa = bab$ we have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b - \alpha^{-1}\beta\delta c, \\ c' &= \alpha^{-1}\delta^2 c \end{aligned}$$

modulo L_2 and $b'a'b' = \alpha\delta^2 bab$ and so we have

$$\langle a, b, c \,|\, baa, bac, caa - bab, cab, cac, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa - bab, cab, cac, cbb, cbc, pa - \omega bab, pb, pc, \text{ class } 3\rangle.$$

Next consider the case when $bab \neq 0$, $bac = 0$, $cab = bab$. We show that we can take $cac = xbab$ with $0 \leq x < p$. So let $cac = xbab$ and consider generators a', b', c' for L satisfying $b'a'b' \neq 0$, $b'a'c' = 0$, $c'a'b' = b'a'b'$, $c'a'c' = x'b'a'b'$. As above we consider a', b', c' of the form

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2, with

$$\begin{aligned} b'a'b' &= (\alpha\delta^2 + \alpha\delta\varepsilon + \alpha\varepsilon^2 x)bab, \\ b'a'c' &= (\alpha\delta\nu + \alpha\varepsilon\nu + \alpha\varepsilon\xi x)bab, \\ c'a'b' &= (\alpha\delta\nu + \alpha\delta\xi + \alpha\varepsilon\xi x)bab, \\ c'a'c' &= (\alpha\nu^2 + \alpha\nu\xi + \alpha\xi^2 x)bab. \end{aligned}$$

So we require

$$\begin{aligned} \delta^2 + \delta\varepsilon + \varepsilon^2 x &\neq 0, \\ \delta\nu + \varepsilon\nu + \varepsilon\xi x &= 0, \\ \delta^2 + \delta\varepsilon + \varepsilon^2 x &= \delta\nu + \delta\xi + \varepsilon\xi x. \end{aligned}$$

If $x = 0$ then we need $\nu = 0$ which gives $x' = 0$. So suppose $x \neq 0$. If $\delta + \varepsilon \neq 0$ we can use the equation $\delta\nu + \varepsilon\nu + \varepsilon\xi x = 0$ to solve for ν. But if $\delta + \varepsilon = 0$ we need $\varepsilon \neq 0$, $\xi = 0$, $\varepsilon^2 x = \delta\nu$ which gives

$$x' = \frac{\nu^2 + \nu\xi + \xi^2 x}{\delta^2 + \delta\varepsilon + \varepsilon^2 x} = \frac{\nu^2}{\varepsilon^2 x} = \frac{\nu}{\delta} = x \text{ (since } \delta = -\varepsilon).$$

If $\delta + \varepsilon \neq 0$ we have

$$\nu = -\frac{\varepsilon\xi x}{\delta + \varepsilon}$$

which gives

$$\begin{aligned} 0 &= (\delta^2 + \delta\varepsilon + \varepsilon^2 x) - (\delta\nu + \delta\xi + \varepsilon\xi x) \\ &= \frac{\delta^3 + 2\delta^2\varepsilon + \delta\varepsilon^2 + \varepsilon^2 x\delta + \varepsilon^3 x - \delta^2\xi - \delta\xi\varepsilon - \varepsilon^2\xi x}{\delta + \varepsilon}. \end{aligned}$$

So

$$\begin{aligned} 0 &= \delta^3 + 2\delta^2\varepsilon + \delta\varepsilon^2 + \varepsilon^2 x\delta + \varepsilon^3 x - \delta^2\xi - \delta\xi\varepsilon - \varepsilon^2\xi x \\ &= (\delta + \varepsilon - \xi)\left(\varepsilon^2 x + \delta\varepsilon + \delta^2\right). \end{aligned}$$

Now $\left(\varepsilon^2 x + \delta\varepsilon + \delta^2\right) \neq 0$ and so $\xi = \delta + \varepsilon$. (Note that we also had $\xi = \delta + \varepsilon$ in the case when $\delta + \varepsilon = 0$ and in the case when $x = 0$.) This gives

$$x' = \frac{\nu^2 + \nu\xi + \xi^2 x}{\delta^2 + \delta\varepsilon + \varepsilon^2 x} = x.$$

The argument above shows that we have p distinct possibilities

$$bab \neq 0,\ bac = 0,\ cab = bab,\ cac = xbab$$

with $0 \leq x < p$ and that if a', b', c' generate L and satisfy these relations then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= -\varepsilon x b + (\delta + \varepsilon)c \end{aligned}$$

with $\delta^2 + \delta\varepsilon + \varepsilon^2 x \neq 0$. If $x \neq -2$ we can take $baa = caa = 0$ and we then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b + \varepsilon c, \\ c' &= -\varepsilon x b + (\delta + \varepsilon)c \end{aligned}$$

with

$$\begin{aligned} pa' &= \alpha pa \\ b'a'b' &= \alpha(\delta^2 + \delta\varepsilon + \varepsilon^2 x)bab. \end{aligned}$$

Now

$$\delta^2 + \delta\varepsilon + \varepsilon^2 x = (\delta + \frac{\varepsilon}{2})^2 + (x - \frac{1}{4})\varepsilon^2$$

so if $x \neq \frac{1}{4}$ we can take $pa = bab$, and if $x = \frac{1}{4}$ we can take $pa = bab$ or ωbab. This gives p algebras

$\langle a, b, c \mid baa, bac, caa, cab - bab, cac - xbab, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle$ $(-1 \leq x \leq p-3)$,

$$\langle a, b, c \mid baa, bac, caa, cab - bab, cac - \frac{1}{4}bab, cbb, cbc, pa - \omega bab, pb, pc, \text{ class } 3\rangle.$$

Finally, consider the case when

$$bab \neq 0,\ bac = 0,\ cab = bab,\ cac = -2bab.$$

Considering a', b', c' as in the case immediately above with $x \neq -2$, but with $\varepsilon = 0$ we see that we can take $baa = 0$ and $caa = 0$ or bab. If $caa = 0$ then as above we have

$$\langle a, b, c \mid baa, bac, caa, cab - bab, cac + 2bab, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle.$$

If $baa = 0$ and $caa = bab$ then letting

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= 2\varepsilon b + (\delta + \varepsilon)c \end{aligned}$$

with

$$\delta^2 + \delta\varepsilon - 2\varepsilon^2 = (\delta + 2\varepsilon)(\delta - \varepsilon) \neq 0$$

we need

$$\begin{aligned} \alpha\varepsilon + (\beta - \gamma)(\delta + 2\varepsilon) &= 0, \\ \alpha(\delta + \varepsilon) + 2(\beta - \gamma)(\delta + 2\varepsilon) - (\delta + 2\varepsilon)(\delta - \varepsilon) &= 0. \end{aligned}$$

We can choose β, γ to satisfy these equations provided $\alpha = (\delta + 2\varepsilon)$. We then have

$$\begin{aligned} pa' &= (\delta + 2\varepsilon)\, pa \\ b'a'b' &= (\delta + 2\varepsilon)(\delta + 2\varepsilon)(\delta - \varepsilon)\, bab, \end{aligned}$$

and so we can take $pa = bab$. So we have

$$\langle a, b, c \mid baa, bac, caa - bab, cab - bab, cac + 2bab, cbb, cbc, pa - bab, pb, pc, \text{ class } 3\rangle.$$

40.20.3 $\quad cbc = 0,\ cbb \neq 0$

Consider the case when $cbc = 0$ and $cbb = 0$. We are now restricted to

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2. We then have

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ c'b'b' &= \delta^2\xi \end{aligned}$$

so we can take $cac = 0$ or $cbb = cac \neq 0$.

First consider the case when $cac = cbc = 0$, $cbb \neq 0$.

We can take $cab = 0$ though we then need $\beta = 0$. We then have

$$\begin{aligned} b'a'c' &= \alpha\delta\xi bac, \\ c'a'a' &= \alpha^2\xi caa, \\ c'b'b' &= \delta^2\xi cbb. \end{aligned}$$

So we can assume that in addition to $cab = cac = cbc = 0$, one of the following holds

$$\begin{aligned} bac &= caa = 0, \\ bac &= 0,\ cbb = -caa, \\ bac &= 0,\ cbb = -\omega caa, \\ cbb &= bac,\ caa = xbac \end{aligned}$$

with $0 \leq x < p$.

$bac = caa = 0$ If $bac = caa = cab = cac = cbc = 0$ we can take $bab = 0$, though we then need $\gamma = 0$, and we then have

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ c'b'b' &= \delta^2\xi cbb \end{aligned}$$

so we can take $baa = 0$ or cbb.

$$\langle a, b, c \mid baa, bab, bac, caa, cab, cac, cbc, pa - cbb, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cab, cac, cbb - baa, cbc, pa - baa, pb, pc,\ \text{class } 3\rangle.$$

If $bac = cab = cac = cbc = 0$, $cbb = -kcaa$ with $k = 1$ or ω we can take $baa = bab = 0$ though we then need $\gamma = \varepsilon = 0$.

$$\langle a, b, c \mid baa, bab, bac, cab, cac, cbb - caa, cbc, pa - caa, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, cab, cac, cbb - \omega caa, cbc, pa - caa, pb, pc,\ \text{class } 3\rangle.$$

If $cab = cac = cbc = 0$, $cbb = bac$, $caa = xbac$ then we can take $bab = 0$, and if $x \neq -2$ we can also take $baa = 0$. If $x = -2$ we can take $baa = 0$ or bac.

$$\langle a, b, c \mid baa, bab, caa - xbac, cab, cac, cbb - bac, cbc, pa - bac, pb, pc,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid bab, bac - baa, caa + 2baa, cab, cac, cbb - baa, cbc, pa - baa, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac - baa, caa + 2baa, cab, cac, cbb - baa, cbc, pa - \omega baa, pb, pc,\ \text{class } 3\rangle,$$

Next consider the case when $cbc = 0$, $cbb = cac \neq 0$. Adding a suitable scalar multiple of c to b we can take $bac = 0$, adding a suitable scalar multiple of b to a we can take

$cab = 0$, and adding a suitable scalar multiple of c to a we can take $caa = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b, \\ c' &= \alpha^{-1}\delta^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ b'a'b' &= \alpha\delta^2 bab, \\ c'a'c' &= \alpha^{-1}\delta^4 cac. \end{aligned}$$

We can assume that one of the following sets of commutator relations holds in additions to $bac = caa = cab = cbc = 0$, $cbb = cac \neq 0$:

$$\begin{aligned} baa &= bab = 0, \\ baa &= 0,\ cac = -bab, \\ baa &= 0,\ cac = -\omega bab, \\ cac &= baa,\ bab = xbaa, \\ cac &= \omega baa,\ bab = xbaa\ (p = 1 \bmod 3), \\ cac &= \omega^2 baa,\ bab = xbaa\ (p = 1 \bmod 3) \end{aligned}$$

where in the last three sets of relations $x = 0$, or x is in a transversal for the cube roots of unity.

$$\langle a, b, c \mid baa, bab, bac, caa, cab, cbb - cac, cbc, pa - cac, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, caa, cab, cbb - cac, cbc, pa - \omega cac, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac + bab, cbb + bab, cbc, pa - bab, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac + bab, cbb + bab, cbc, pa - \omega bab, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac + \omega bab, cbb + \omega bab, cbc, pa - bab, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, caa, cab, cac + \omega bab, cbb + \omega bab, cbc, pa - \omega bab, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - baa, cbb - baa, cbc, pa - baa, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - \omega baa, cbb - \omega baa, cbc, pa - baa, pb, pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - \omega^2 baa, cbb - \omega^2 baa, cbc, pa - baa, pb, pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - baa, cbb - baa, cbc, pa - \omega baa, pb, pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - \omega baa, cbb - \omega baa, cbc, pa - \omega baa, pb, pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab - xbaa, bac, caa, cab, cac - \omega^2 baa, cbb - \omega^2 baa, cbc, pa - \omega baa, pb, pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

with $x = 0$ or x in a transversal for the cube roots of unity.

$$\langle a, b, c \mid pa - cb,\ pb,\ pc,\ \text{class } 2\rangle$$

Algebra 6.105 has $10 + 6\gcd(p-1,3) + 2\gcd(p-1,5)$ descendants of order p^7 and p-class 3.

If L is a descendant of algebra 6.105 of order p^7 then L_3 has order p and is generated by baa, bab, bac, caa and cac. If a', b', c' generate L and if $pa' - c'b'$, pb', $pc' \in L_3$ then

$$\begin{aligned} a' &= (\delta\xi - \varepsilon\nu)a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa + \alpha\beta\delta bab + \alpha(\beta\varepsilon + \gamma\delta)bac + \alpha^2\varepsilon caa + \alpha\gamma\varepsilon cac, \\ b'a'b' &= \alpha\delta^2 bab + 2\alpha\delta\varepsilon bac + \alpha\varepsilon^2 cac, \\ b'a'c' &= \alpha\delta\nu bab + \alpha(\delta\xi + \varepsilon\nu)bac + \alpha\varepsilon\xi cac, \\ c'a'a' &= \alpha^2\nu baa + \alpha\beta\nu bab + \alpha(\beta\xi + \gamma\nu)bac + \alpha^2\xi caa + \alpha\gamma\xi cac, \\ c'a'c' &= \alpha\nu^2 bab + 2\alpha\nu\xi bac + \alpha\xi^2 cac \end{aligned}$$

with $\alpha = \delta\xi - \varepsilon\nu$.

First consider the case when $cac \neq 0$.

If $cac \neq 0$ then taking $\nu = 0$ we can choose ε so that $b'a'c' = 0$. So if $cac \neq 0$ we can assume that $bac = 0$. We can then have $bab = 0$ or $bab \neq 0$, with $cac = -xbab$ for some $x \neq 0$. Scaling we can take $x = 1$ or ω. So we can take $bab = 0$ or $cac = -xbab \neq 0$ with $x = 1$ or ω.

Let $bab = bac = 0$, $cac \neq 0$. Then we need $\varepsilon = 0$, and we have

$$\begin{aligned} b'a'a' &= \alpha^2\delta baa, \\ c'a'a' &= \alpha^2\nu baa + \alpha^2\xi caa + \alpha\gamma\xi cac, \\ c'a'c' &= \alpha\xi^2 cac \end{aligned}$$

We can take $caa = 0$ and we can take $baa = 0$ or $cac = baa \neq 0$.

Let $cac = -bab \neq 0$ and let $bac = 0$. Then we need

$$\begin{aligned} a' &= \pm(\delta^2 - \varepsilon^2)a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and we can choose β, γ so that $baa = caa = 0$.

Let $cac = -\omega bab \neq 0$, and let $bac = 0$. Then we need

$$\begin{aligned} a' &= \pm(\delta^2 - \omega\varepsilon^2)a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\omega\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and we can choose β, γ so that $baa = caa = 0$.

Next consiser the case when $cac = 0$. We can also assume that $bab = bac = 0$, or we can reduce to the case when $cac \neq 0$. Then we can take $baa = 0$, $caa \neq 0$.

So we have 5 possible commutator structures:

$$\begin{aligned}
baa &= bab = bac = cac = 0,\ caa \neq 0,\\
baa &= bab = bac = caa = 0,\ cac \neq 0,\\
bab &= bac = caa = 0,\ cac = baa \neq 0,\\
bab &= bac = caa = 0,\ cac = -bab \neq 0,\\
baa &= bac = caa = 0,\ cac = -\omega bab \neq 0.
\end{aligned}$$

40.21.1 Case 1

Let $baa = bab = bac = cac = 0$ and let L_3 be generated by caa. Then

$$\begin{aligned}
a' &= \delta\xi a + \beta b + \gamma c,\\
b' &= \delta b,\\
c' &= \nu b + \xi c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
c'a'a' &= \delta^2\xi^3 caa,\\
pa' - c'b' &= \delta\xi(pa - cb) + \beta pb + \gamma pc,\\
pb' &= \delta pb,\\
pc' &= \nu pb + \xi pc.
\end{aligned}$$

So we can take $pb = 0$ or caa and if $pb = caa$ we can take $pa - cb = pc = 0$. If $pb = 0$ we can take $pc = 0$, caa or ωcaa, and if $pc \neq 0$ we can take $pa - cb = 0$. Finally, if $pb = pc = 0$ then we can take $pa - cb = 0$ or caa. So we have 5 algebras.

$$\langle a, b, c \mid baa, bab, bac, cac, pa - cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\begin{aligned}
&\langle a, b, c \mid baa, bab, bac, cac, pa - cb - caa,\ pb,\ pc,\ \text{class } 3\rangle,\\
&\langle a, b, c \mid baa, bab, bac, cac, pa - cb,\ pb,\ pc - caa,\ \text{class } 3\rangle,\\
&\langle a, b, c \mid baa, bab, bac, cac, pa - cb,\ pb,\ pc - \omega caa,\ \text{class } 3\rangle,\\
&\langle a, b, c \mid baa, bab, bac, cac, pa - cb,\ pb - caa,\ pc,\ \text{class } 3\rangle.
\end{aligned}$$

40.21.2 Case 2

Let $baa = bab = bac = caa = 0$ and let L_3 be generated by cac. Adding a suitable scalar multiple of ca to b we can take $pa - cb = 0$. We then have

$$\begin{aligned}
a' &= \delta\xi a + \beta b,\\
b' &= \delta b,\\
c' &= \nu b + \xi c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
c'a'c' &= \delta\xi^3 cac,\\
pb' &= \delta pb,\\
pc' &= \nu pb + \xi pc.
\end{aligned}$$

So we can take $pb = 0$, cac or (if $p = 1 \bmod 3$) ωcac or $\omega^2 cac$ and if $pb \neq 0$ then we can take $pc = 0$. If $pb = 0$ we can take $pc = 0$ or cac. We have $2 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \mid baa, bab, bac, caa, pa - cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, caa, pa - cb,\ pb,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, caa, pa - cb,\ pb - cac,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, bab, bac, caa, pa - cb,\ pb - \omega cac,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baa, bab, bac, caa, pa - cb,\ pb - \omega^2 cac,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3).$$

40.21.3 Case 3

Let $bab = bac = caa = 0$, $cac = baa \neq 0$, so that L_3 is generated by baa. Adding a suitable scalar multiple of ca to b we can take $pa - cb = 0$. We then have

$$\begin{aligned} a' &= \delta^3 a + \beta b - \delta\nu c, \\ b' &= \delta b, \\ c' &= \nu b + \delta^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \delta^7 baa, \\ pb' &= \delta pb, \\ pc' &= \nu pb + \delta^2 pc. \end{aligned}$$

We can take $pb = 0$, baa, ωbaa or (if $p = 1 \bmod 3$) $\omega^2 baa$, $\omega^3 baa$, $\omega^4 baa$ or $\omega^5 baa$, and if $pb \neq 0$ we can take $pc = 0$ or baa or (if $p = 1 \bmod 5$) ωbaa, $\omega^2 baa$, $\omega^3 baa$ or $\omega^4 baa$. So we have $1 + 2\gcd(p-1, 3) + \gcd(p-1, 5)$ algebras

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb,\ pc - baa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb,\ pc - xbaa,\ \text{class } 3\rangle\ (p = 1 \bmod 5,\ x = \omega, \omega^2, \omega^3, \omega^4),$$

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, bac, caa, cac - baa, pa - cb,\ pb - xbaa,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3,\ x = \omega^2, \omega^3, \omega^4, \omega^5).$$

40.21.4 Case 4

Let $bab = bac = caa = 0$, $cac = -bab \neq 0$, so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $pa - cb = 0$. We then have

$$\begin{aligned} a' &= \pm(\delta^2 - \varepsilon^2) a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \pm(\delta^2 - \varepsilon^2)^2 bab, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \pm\varepsilon pb \pm \delta pc. \end{aligned}$$

So we can take $pb = 0$ and we can take $pc = 0$ or bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$, or we can take $pb = pc = bab$. So we have $2 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - \omega bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - \omega^2 bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb - bab,\ pc - bab,\ \text{class } 3\rangle.$$

40.21.5 Case 5

Let $baa = bac = caa = 0$, $cac = -\omega bab \neq 0$, so that L_3 is generated by bab. Adding a suitable scalar multiple of ba to c we can take $pa - cb = 0$. We then have

$$\begin{aligned} a' &= \pm(\delta^2 - \omega\varepsilon^2)a, \\ b' &= \delta b + \varepsilon c, \\ c' &= \pm\omega\varepsilon b \pm \delta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \pm(\delta^2 - \omega\varepsilon^2)^2 bab, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \pm\omega\varepsilon pb \pm \delta pc. \end{aligned}$$

So we can take $pb = 0$ and we can take $pc = 0$ or bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. So we have $1 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - \omega bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baa, bac, caa, cac + \omega bab, pa - cb,\ pb,\ pc - \omega^2 bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3).$$

40.22 Descendants of 6.106

$$\langle a, b, c \,|\, pa - ba,\ pb,\ pc,\ \text{class } 2\rangle$$

Algebra 6.106 has $p^2 + 10p + 32 + \gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.106 of order p^7. Then L_3 is generated by caa, cab, cac, cbb and cbc. If a', b', c' generate L and if $pa' - b'a'$, pb', $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= b, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned}
c'a'a' &= \alpha^2\xi caa + 2\alpha\beta\xi cab + \beta^2\xi cbb, \\
c'a'b' &= \alpha\xi cab + \beta\xi cbb, \\
c'a'c' &= \alpha\nu\xi cab + \alpha\xi^2 cac + \beta\nu\xi cbb + \beta\xi^2 cbc, \\
c'b'b' &= \xi cbb \\
c'b'c' &= \nu\xi cbb + \xi^2 cbc.
\end{aligned}$$

First consider the case when $cbb \neq 0$. Then we can take $cab = cbc = 0$, though we then need $\beta = \nu = 0$. This gives

$$\begin{aligned}
c'a'a' &= \alpha^2\xi caa, \\
c'a'c' &= \alpha\xi^2 cac, \\
c'b'b' &= \xi cbb
\end{aligned}$$

and so we can assume that one of the following holds (in addition to $cab = cbc = 0$)

$$\begin{aligned}
caa &= cac = 0, \\
caa &= 0,\ cbb = cac, \\
cbb &= caa,\ cac = 0, \\
cbb &= caa,\ cac = caa, \\
cbb &= \omega caa,\ cac = 0, \\
cbb &= \omega caa,\ cac = caa.
\end{aligned}$$

Next consider the case when $cbb = 0$, $cab \neq 0$. We can take $caa = 0$ though we then need $\beta = 0$, and we can then take $cac = 0$, though we then need $\nu = 0$. This gives

$$\begin{aligned}
c'a'b' &= \alpha\xi cab, \\
c'b'c' &= \xi^2 cbc
\end{aligned}$$

and so we can assume that one of the following holds (in addition to $cbb = caa = cac = 0$)

$$\begin{aligned}
cbc &= 0, \\
cbc &= cab.
\end{aligned}$$

And now consider the case when $cab = cbb = 0$, $cbc \neq 0$. We can take $cac = 0$ though we then need $\beta = 0$. We then have

$$\begin{aligned}
c'a'a' &= \alpha^2\xi caa, \\
c'b'c' &= \xi^2 cbc.
\end{aligned}$$

so we can assume that one of the following holds (in addition to $cab = cac = cbb = 0$)

$$\begin{aligned}
caa &= 0, \\
cbc &= caa.
\end{aligned}$$

So assume that $cab = cbb = cbc = 0$. We have

$$\begin{aligned}
c'a'a' &= \alpha^2\xi caa, \\
c'a'c' &= \alpha\xi^2 cac,
\end{aligned}$$

and so we can assume that one of the following holds (in addition to $cab = cbb = cbc$)

$$\begin{aligned} caa &= 0, \\ cac &= 0, \\ cac &= caa. \end{aligned}$$

So we can assume that one of the following 13 sets of commutator relations holds.

$$\begin{aligned} caa &= cab = cac = cbc = 0, \\ caa &= cab = cbc = 0,\ cbb = cac, \\ cbb &= caa,\ cab = cac = cbc = 0, \\ cbb &= caa,\ cac = caa,\ cab = cbc = 0, \\ cbb &= \omega caa,\ cab = cac = cbc = 0, \\ cbb &= \omega caa,\ cac = caa,\ cab = cbc = 0, \\ caa &= cac = cbb = cbc = 0, \\ caa &= cac = cbb = 0,\ cbc = cab, \\ caa &= cab = cac = cbb = 0, \\ cab &= cac = cbb = 0,\ cbc = caa, \\ caa &= cab = cbb = cbc = 0, \\ cac &= cab = cbb = cbc = 0, \\ cac &= caa,\ cab = cbb = cbc = 0. \end{aligned}$$

40.22.1 Case 1

Let $caa = cab = cac = cbc = 0$, and let L_3 be generated by cbb. Adding a suitable scalar multiple of cb to a we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'b'b' &= \xi cbb, \\ pb' &= pb, \\ pc' &= \xi pc \end{aligned}$$

and so we can take $pb = 0$ or cbb and we can take $pc = xcbb$ with $0 \leq x < p$. We have $2p$ algebras

$$\langle a, b, c \,|\, caa, cab, cac, cbc, pa - ba,\ pb,\ pc - xcbb,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, caa, cab, cac, cbc, pa - ba,\ pb - cbb,\ pc - xcbb,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.22.2 Case 2

Let $caa = cab = cbc = 0$, $cbb = cac$ so that L_3 is generated by cac. Adding a suitable scalar multiple of cb to a we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \alpha^{-1} c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha^{-1}cac, \\ pb' &= pb, \\ pc' &= \alpha^{-1}pc. \end{aligned}$$

So as in Case 1 we have $2p$ algebras

$$\langle a,b,c \,|\, caa, cab, cbb-cac, cbc, pa-ba,\ pb,\ pc-xcac,\ \text{class }3\rangle\ (0 \le x < p),$$

$$\langle a,b,c \,|\, caa, cab, cbb-cac, cbc, pa-ba,\ pb-cac,\ pc-xcac,\ \text{class }3\rangle\ (0 \le x < p).$$

40.22.3 Cases 3 & 5

Let $cbb = kcaa$ with $k = 1, \omega$, $cab = cac = cbc = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of cb to a we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \xi caa, \\ pb' &= pb, \\ pc' &= \xi pc. \end{aligned}$$

So again we have $2p$ algebras for each value of k:

$$\langle a,b,c \,|\, cab, cac, cbb-caa, cbc, pa-ba,\ pb,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p),$$

$$\langle a,b,c \,|\, cab, cac, cbb-caa, cbc, pa-ba,\ pb-caa,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p),$$

$$\langle a,b,c \,|\, cab, cac, cbb-\omega caa, cbc, pa-ba,\ pb,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p),$$

$$\langle a,b,c \,|\, cab, cac, cbb-\omega caa, cbc, pa-ba,\ pb-caa,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p).$$

40.22.4 Cases 4 & 6

Let $cbb = kcaa$ with $k = 1, \omega$, $cab = cbc = 0$, $cac = caa$ and let L_3 be generated by caa. Adding a suitable scalar multiple of cb to a we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \pm c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \pm caa, \\ pb' &= pb, \\ pc' &= \pm pc. \end{aligned}$$

So we have $p(p+1)/2$ algebras for each value of k.

$$\langle a,b,c \,|\, cab, cac-caa, cbb-caa, cbc, pa-ba,\ pb-ycaa,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p,\ 0 \le y \le (p-1)/2),$$

$$\langle a,b,c \,|\, cab, cac-caa, cbb-\omega caa, cbc, pa-ba,\ pb-ycaa,\ pc-xcaa,\ \text{class }3\rangle\ (0 \le x < p,\ 0 \le y \le (p-1)/2).$$

40.22.5 Case 7

Let $caa = cac = cbb = cbc = 0$, and let L_3 be generated by cab. Adding a suitable scalar multiple of cb to b we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'c' &= \alpha\xi bac, \\ pb' &= pb, \\ pc' &= \xi pc. \end{aligned}$$

So we have 4 algebras

$$\langle a, b, c \mid caa, cac, cbb, cbc, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc, pa - ba,\ pb,\ pc - cab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc, pa - ba,\ pb - cab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc, pa - ba,\ pb - cab,\ pc - cab,\ \text{class } 3\rangle.$$

40.22.6 Case 8

Let $caa = cac = cbb = 0$, $cbc = cab$ and let L_3 be generated by cab. Adding a suitable scalar multiple of cb to b we can take $pa - ba = 0$. Then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \alpha c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'b' &= \alpha^2 cab, \\ pb' &= pb, \\ pc' &= \alpha pc. \end{aligned}$$

We have $p + 3$ algebras

$$\langle a, b, c \mid caa, cac, cbb, cbc - cab, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc - cab, pa - ba,\ pb - cab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc - cab, pa - ba,\ pb - \omega cab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cac, cbb, cbc - cab, pa - ba,\ pb - xcab,\ pc - cab,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.22.7 Case 9

Let $caa = cab = cac = cbb = 0$ and let L_3 be generated by cbc. We have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \nu b + \alpha c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'b'c' &= \xi^2 cbc, \\ pa' - b'a' &= \alpha(pa - ba), \\ pb' &= pb, \\ pc' &= \nu pb + \xi pc. \end{aligned}$$

We have 8 algebras

$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba,\ pb,\ pc - cbc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba,\ pb - cbc,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba,\ pb - \omega cbc,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba - cbc,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba - cbc,\ pb,\ pc - cbc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba - cbc,\ pb - cbc,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid caa, cab, cac, cbb, pa - ba - cbc,\ pb - \omega cbc,\ pc,\ \text{class } 3\rangle.$$

40.22.8 Case 10

Let $cab = cac = cbb = 0$, $cbc = caa$ and let L_3 be generated by caa. Adding a suitable scalar multiple of ca to b we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \nu b + \alpha^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^4 caa, \\ pb' &= pb, \\ pc' &= \nu pb + \alpha^2 pc. \end{aligned}$$

We have $3 + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb,\ pc - caa,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb - caa,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb - \omega caa,\ pc,\ \text{class } 3\rangle,$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb - \omega^2 caa,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid cab, cac, cbb, cbc - caa, pa - ba,\ pb - \omega^3 caa,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 4).$$

40.22.9 Case 11

Let $caa = cab = cbb = cbc = 0$, and let L_3 be generated by cac. We have

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= b, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'c' &= \alpha\xi^2 cac, \\ pa' - b'a' &= \alpha(pa - ba) + \beta pb, \\ pb' &= pb, \\ pc' &= \nu pb + \xi pc. \end{aligned}$$

We have 7 algebras

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba - cac,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba - \omega cac,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba,\ pb,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba - cac,\ pb,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba - \omega cac,\ pb,\ pc - cac,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid caa, cab, cbb, cbc, pa - ba,\ pb - cac,\ pc,\ \text{class } 3\rangle.$$

40.22.10 Case 12

Let $cab = cac = cbb = cbc = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of ca to b we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= b, \\ c' &= \nu b + \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a' &= \alpha^2\xi caa, \\ pb' &= pb, \\ pc' &= \nu pb + \xi pc. \end{aligned}$$

We have 4 algebras

$$\langle a, b, c \mid cab, cac, cbb, cbc, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid cab, cac, cbb, cbc, pa - ba,\ pb,\ pc - caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid cab, cac, cbb, cbc, pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid cab, cac, cbb, cbc, pa - ba,\ pb - caa,\ pc,\ \text{class } 3\rangle.$$

40.22.11 Case 13

Let $cac = caa$, $cab = cbb = cbc = 0$, and let L_3 be generated by caa. Adding a suitable scalar multiple of ca to b we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= b, \\ c' &= \nu b + \alpha c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a' &= \alpha^3 caa, \\ pb' &= pb, \\ pc' &= \nu pb + \alpha pc. \end{aligned}$$

We have $3 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb,\ pc - caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb - caa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb - \omega caa,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cab, cac - caa, cbb, cbc, pa - ba,\ pb - \omega^2 caa,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3).$$

40.23 Descendants of 6.107

There is no algebra 6.107!

40.24 Descendants of 6.108

$$\langle a, b, c \,|\, pa - ca,\ pb - \mu cb,\ pc,\ \text{class } 2\rangle\ (\mu \neq 0,\ \mu, \mu^{-1} \text{ give isomorphic algebras})$$

If $\mu = 1$ we have 5 descendants of order p^7 and p-class 3, if $\mu \neq \pm 1, -2, -\frac{1}{2}$ we have $12 + \gcd(p-1, 3)$ descendants, and if $\mu = -2$ or $-\frac{1}{2}$ we have $13 + \gcd(p-1, 3)$ descendants, and if $\mu = -1$ we have $2 + \gcd(p-1, 3) + \gcd(p-1, 4)/2$. So the total number of descendants is $\frac{p-1}{2}(12 + \gcd(p-1, 3)) + \gcd(p-1, 4)/2$.

The automorphism group of algebra 6.108 depends on the value of μ, and we distinguish between the case $\mu = 1$ and $\mu \neq 1$.

40.24.1 $\mu = 1$

Let L be a descendant of algebra 6.108 (with $\mu = 1$) of order p^7. Then L_3 is generated by baa, bab and $bac = 2cab$. If a', b', c' generate L and if $pa' - c'a'$, $pb' - c'b'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha(\alpha\varepsilon-\beta\delta)baa+\beta(\alpha\varepsilon-\beta\delta)bab+\gamma(\alpha\varepsilon-\beta\delta)cab, \\ b'a'b' &= \delta(\alpha\varepsilon-\beta\delta)baa+\varepsilon(\alpha\varepsilon-\beta\delta)bab+\eta(\alpha\varepsilon-\beta\delta)cab, \\ c'a'b' &= (\alpha\varepsilon-\beta\delta)cab. \end{aligned}$$

So we can assume that $baa = bab = 0$ and that L_3 is generated by cab, or that $baa = cab = 0$ and that L_3 is generated by bab.

Case 1 Let $baa = bab = 0$ and let L_3 be generated by cab. Adding suitable scalar multiples of ba to a, b, c we can take $pa - ca = pb - cb = pc = 0$, so we have 1 algebra

$$\langle a, b, c \mid baa, bab, pa - ca,\ pb - cb,\ pc, \text{ class } 3\rangle.$$

Case 2 Let $baa = cab = 0$, and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to b we can take $pb - cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta a + \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ pa' - c'a' &= \alpha(pa - ca) + \gamma pc, \\ pc' &= pc. \end{aligned}$$

We can take $pc = 0$ or bab, and if $pc = bab$ we can take $pa - ca = 0$. If $pc = 0$ then we can take $pa - ca = 0$, bab or ωbab. So we have 4 algebras

$$\langle a, b, c \mid baa, bac, pa - ca,\ pb - cb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, pa - ca - bab,\ pb - cb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, pa - ca - \omega bab,\ pb - cb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid baa, bac, pa - ca,\ pb - cb,\ pc - bab, \text{ class } 3\rangle.$$

40.24.2 $\mu \neq \pm 1$

Let L be a descendant of algebra 6.108 (with $\mu \neq \pm 1$) of order p^7. Then L_3 is generated by baa, bab and bac, with $cab = \frac{\mu}{\mu+1} bac$. If a', b', c' generate L and if $pa' - c'a'$, $pb' - \mu c'b'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\varepsilon baa + \frac{\mu+2}{\mu+1}\alpha\gamma\varepsilon bac, \\ b'a'b' &= \alpha\varepsilon^2 bab + \frac{2\mu+1}{\mu+1}\alpha\varepsilon\eta bac, \\ b'a'c' &= \alpha\varepsilon bac. \end{aligned}$$

First consider the situation when $bac \neq 0$. If $\mu \neq -2, -\frac{1}{2}$ then we can take $baa = bab = 0$. If $\mu = -2$ then we can take $bab = 0$ and $baa = 0$ or bac, and if $\mu = -\frac{1}{2}$ we can take $baa = 0$ and $bab = 0$ or bac.

If $bac = 0$ then we can take $baa = 0$ or $bab = 0$ or $bab = baa$.

Case 1 Let $baa = bab = 0$ and let L_3 be generated by bac. Adding suitable scalar multiples of ba to a, b, c we can take $pa - ca = pb - \mu cb = pc = 0$, so we have 1 algebra

$\langle a, b, c \mid baa, bab, pa{-}ca,\ pb{-}\mu cb,\ pc,$ class $3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 2 Let $\mu = -2$ and let $bab = 0$, $bac = baa$ and let L_3 be generated by baa. Adding suitable scalar multiples of ba to a, b, c we can take $pa - ca = pb + 2cb = pc = 0$, so we have 1 algebra

$$\langle a, b, c \mid bab, bac - baa, pa - ca,\ pb + 2cb,\ pc, \text{ class } 3\rangle.$$

Case 3 Let $\mu = -\frac{1}{2}$ and let $baa = 0$, $bac = bab$ and let L_3 be generated by bab. Adding suitable scalar multiples of ba to a, b, c we can take $pa - ca = pb + \frac{1}{2}cb = pc = 0$, so we have 1 algebra, but $\mu = -\frac{1}{2}$ is paired with $\mu = -2$, so we do not need this algebra. However, for other reasons, we use this algebra, rather than the one from Case 2.

Case 4 Let $baa = bac = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $pb - \mu cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ pa' - c'a' &= \alpha(pa - ca) + \gamma pc, \\ pc' &= pc \end{aligned}$$

so we have 4 algebras

$\langle a, b, c \mid baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc,$ class $3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid baa, bac, pa{-}ca{-}bab,\ pb{-}\mu cb,\ pc,$ class $3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid baa, bac, pa{-}ca{-}\omega bab,\ pb{-}\mu cb,\ pc,$ class $3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc{-}bab,$ class $3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 5 Let $bab = bac = 0$ and let L_3 be generated by baa. Adding a suitable scalar multiple of ba to c we can take $pa - ca = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\varepsilon bab, \\ pb' - \mu c'b' &= \varepsilon(pb - \mu cb) + \eta pc, \\ pc' &= pc \end{aligned}$$

so we have 4 algebras

$\langle a, b, c \mid bab, bac, pa{-}ca,\ pb{-}\mu cb,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab, bac, pa{-}ca,\ pb{-}\mu cb{-}baa,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab, bac, pa{-}ca,\ pb{-}\mu cb{-}\omega baa,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab, bac, pa{-}ca,\ pb{-}\mu cb,\ pc{-}baa, \text{ class } 3\rangle$ ($\mu \neq 0{\pm}1,\ \mu, \mu^{-1}$ give isomorphic algebras).

Case 6 Let $bac = 0$, $bab = baa$, and let L_3 be generated by baa. Adding a suitable scalar multiple of ba to c we can take $pa - ca = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 bab, \\ pb' - \mu c'b' &= \alpha(pb - \mu cb) + \eta pc, \\ pc' &= pc \end{aligned}$$

so we have $3 + \gcd(p-1, 3)$ algebras

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb{-}baa,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb{-}\omega baa,\ pc, \text{ class } 3\rangle$ ($\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc{-}baa, \text{ class } 3\rangle$ ($\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc{-}\omega baa, \text{ class } 3\rangle$ ($p = 1 \bmod 3$, $\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras),

$\langle a, b, c \mid bab{-}baa, bac, pa{-}ca,\ pb{-}\mu cb,\ pc{-}\omega^2 baa, \text{ class } 3\rangle$ ($p = 1 \bmod 3$, $\mu \neq 0, \pm1,\ \mu, \mu^{-1}$ give isomorphic algebras)

40.24.3 $\mu = -1$

Let L be a descendant of algebra 6.108 (with $\mu = -1$) of order p^7. Then L_3 is generated by baa, bab and cab, with $bac = 0$. If a', b', c' generate L and if $pa' - c'a'$, $pb' + c'b'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^2\varepsilon baa - \alpha\gamma\varepsilon cab, \\ b'a'b' &= \alpha\varepsilon^2 bab + \alpha\varepsilon\eta cab, \\ c'a'b' &= \alpha\varepsilon cab, \end{aligned}$$

or

$$\begin{aligned} a' &= \beta b + \gamma c, \\ b' &= \delta a + \eta c, \\ c' &= -c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= -\beta^2\delta bab - \beta\gamma\delta cab, \\ b'a'b' &= -\beta\delta^2 baa + \beta\delta\eta cab, \\ c'a'b' &= -\beta\delta cab, \end{aligned}$$

So we can take $baa = bab = 0$ or $baa = cab = 0$ or $bab = baa$, $cab = 0$.

Case 1 Let $baa = bab = 0$ and let L_3 be generated by cab. Adding suitable scalar multiples of ba to a, b, c we can take $pa - ca = pb + cb = pc = 0$, so we have 1 algebra

$$\langle a, b, c \mid baa, bab, pa - ca,\ pb + cb,\ pc,\ \text{class } 3\rangle.$$

Case 2 Let $baa = cab = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $pb + cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \varepsilon b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha\varepsilon^2 bab, \\ pa' - c'a' &= \alpha(pa - ca) + \gamma pc, \\ pc' &= pc \end{aligned}$$

so we have 4 algebrasxxxxx

$$\langle a, b, c \mid baa, cab, pa - ca,\ pb + cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, cab, pa - ca - bab,\ pb + cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, cab, pa - ca - \omega bab,\ pb + cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid baa, cab, pa - ca,\ pb + cb,\ pc - bab,\ \text{class } 3\rangle.$$

Case 3 Let $bab = baa$, $cab = 0$ and let L_3 be generated by baa. Adding a suitable scalar multiple of ba to c we can take $pa - ca = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 baa, \\ pb' + c'b' &= \alpha(pb + cb) + \eta pc, \\ pc' &= pc \end{aligned}$$

or

$$\begin{aligned} a' &= \beta b + \gamma c, \\ b' &= \beta a + \eta c, \\ c' &= -c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= -\beta^3 baa, \\ pc' &= -pc. \end{aligned}$$

If $pc = 0$ and we ensure that $pa' - c'a' = 0$ we have

$$pb' + c'b' = \beta(pb + cb).$$

So we have $1 + \gcd(p-1, 3) + \gcd(p-1, 4)/2$ algebras

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb - baa,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb - \omega baa,\ pc, \text{ class } 3\rangle \ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb,\ pc - baa, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb,\ pc - \omega baa, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab - baa, cab, pa - ca,\ pb + cb,\ pc - \omega^2 baa, \text{ class } 3\rangle \ (p = 1 \bmod 3).$$

40.25 Descendants of 6.109

$$\langle a, b, c \mid pa - ca - cb,\ pb - cb,\ pc, \text{ class } 2\rangle$$

Algebra 6.109 has $7 + 2\gcd(p-1, 3)$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.109 of order p^7. Then L_3 has order p and is generated by baa, bab and caa with $bac = 2caa$. If a', b', c' generate L and if $pa' - c'a' - c'b'$, $pb' - c'b'$, $pc' \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha^3 baa + \alpha^2\beta bab + (3\alpha^2\gamma + \alpha^2\eta)caa, \\ b'a'b' &= \alpha^3 bab + 3\alpha^2\eta caa, \\ c'a'a' &= \alpha^2 caa. \end{aligned}$$

We can assume that one of the following 3 sets of commutator relations holds:

$$\begin{aligned} baa &= bab = 0,\ caa \neq 0, \\ baa &= caa = 0,\ bab \neq 0, \\ bab &= caa = 0,\ baa \neq 0. \end{aligned}$$

40.25.1 Case 1

Let $baa = bab = 0$ and let L_3 be generated by caa. Adding suitable scalar multiple of ba to a, b, c we can take $pa - ca - cb = pb - cb = pc = 0$ and so we have 1 algebra

$$\langle a, b, c \,|\, baa, bab, pa - ca - cb,\ pb - cb,\ pc,\ \text{class } 3\rangle.$$

40.25.2 Case 2

Let $baa = caa = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $pa - ca - cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^3 bab, \\ pb' - c'b' &= \alpha(pb - cb) + \eta pc, \\ pc' &= pc. \end{aligned}$$

We have $3 + \gcd(p-1, 3)$algebras

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb - bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb - \omega bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb,\ pc - bab,\ \text{class } 3\rangle,$$

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb,\ pc - \omega bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baa, caa, pa - ca - cb,\ pb - cb,\ pc - \omega^2 bab,\ \text{class } 3\rangle\ (p = 1 \bmod 3).$$

40.25.3 Case 3

Let $bab = caa = 0$ and let L_3 be generated by baa. Adding a suitable scalar multiple of ba to c we can take $pa - ca - cb = 0$. We then have

$$\begin{array}{rcl} a' & = & \alpha a + \beta b + \gamma c, \\ b' & = & \alpha b + \eta c, \\ c' & = & c \end{array}$$

modulo L_2 and

$$\begin{array}{rcl} b'a'a' & = & \alpha^3 baa, \\ pb' - c'b' & = & \alpha(pb - cb) + \eta pc, \\ pc' & = & pc. \end{array}$$

So as in Case 2 we have $3 + \gcd(p-1, 3)$algebras

$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb,\ pc, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb - baa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb - \omega baa,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb,\ pc - baa, \text{ class } 3\rangle,$$
$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb,\ pc - \omega baa, \text{ class } 3\rangle \ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, bab, caa, pa - ca - cb,\ pb - cb,\ pc - \omega^2 baa, \text{ class } 3\rangle \ (p = 1 \bmod 3).$$

40.26 Descendants of 6.110

$$\langle a, b, c \,|\, pa - \omega cb,\ pb - ca,\ pc, \text{ class } 2\rangle$$

Algebra 6.110 has $2 + \gcd(p-1, 4)/2 + \gcd(p-1, 3)$ descendants of order p^7 and p-class 3.

Let L be a descendant of algenra 6.110 of order p^7. Then L_3 has order p and is generated by baa, bab and cbb with $caa = -\omega cbb$. If a', b', c' generate L and if $pa - \omega cb$, $pb - ca$, $pc \in L_3$ then

$$\begin{array}{rcl} a' & = & \alpha a \pm \omega\beta b + \gamma c, \\ b' & = & \beta a \pm \alpha b + \eta c, \\ c' & = & \pm c \end{array}$$

modulo L_2 and

$$\begin{array}{rcl} b'a'a' & = & \pm\alpha(\alpha^2 - \omega\beta^2)baa + \omega\beta(\alpha^2 - \omega\beta^2)bab - \omega\eta(\alpha^2 - \omega\beta^2)cbb, \\ b'a'b' & = & \pm\beta(\alpha^2 - \omega\beta^2)baa + \alpha(\alpha^2 - \omega\beta^2)bab - \gamma(\alpha^2 - \omega\beta^2)cbb \\ c'b'b' & = & \pm\alpha(\alpha^2 - \omega\beta^2)cbb. \end{array}$$

We can assume that either $cbb \neq 0$ and that $baa = bab = 0$, or that $baa = cbb = 0$ and that $bab \neq 0$.

40.26.1 Case 1

Let $baa = bab = 0$, and let L_3 be generated by cbb. Adding suitable scalar multiples of ba to a, b, c we can take $pa - \omega cb = pb - ca = pc = 0$, and so we have 1 algebra

$$\langle a, b, c \mid baa, bab, pa - \omega cb,\ pb - ca,\ pc, \text{ class } 3\rangle.$$

40.26.2 Case 2

Let $baa = cbb = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $pa - \omega cb = 0$. Then we have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \pm\alpha b + \eta c, \\ c' &= \pm c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^3 bab, \\ pb' - c'a' &= \pm\alpha(pb - ca) + \eta pc, \\ pc' &= \pm pc. \end{aligned}$$

So we have $1 + \gcd(p-1, 4)/2 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca,\ pc - bab, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca,\ pc - \omega bab, \text{ class } 3\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca,\ pc - \omega^2 bab, \text{ class } 3\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca - bab,\ pc, \text{ class } 3\rangle,$$
$$\langle a, b, c \mid baa, cbb, pa - \omega cb,\ pb - ca - \omega bab,\ pc, \text{ class } 3\rangle\ (p = 1 \bmod 4).$$

40.27 Descendants of 6.111

$$\langle a, b, c \mid pa - \mu cb,\ pb - ca - cb,\ pc, \text{ class } 2\rangle\ (x^2 - x - \mu \text{ irreducible})$$

Algebra 6.111 has $4 + \gcd(p-1, 3)$ descendants of order p^7 and p-class 3 for each of the possible $(p-1)/2$ values of μ.

Let L be a descendant of algebra 6.111 of order p^7. The L_3 has order p and is generated by baa, bab and bac. If a', b', c' generate L and if $pa - \mu cb,\ pb - ca - cb,\ pc \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \mu\beta b + \gamma c, \\ b' &= \beta a + (\alpha + \beta) b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a' &= \alpha(\alpha^2 + \alpha\beta - \mu\beta^2) baa + \mu\beta(\alpha^2 + \alpha\beta - \mu\beta^2) bab + (\gamma + \mu\eta)(\alpha^2 + \alpha\beta - \mu\beta^2) bac, \\ b'a'b' &= \beta(\alpha^2 + \alpha\beta - \mu\beta^2) baa + (\alpha + \beta)(\alpha^2 + \alpha\beta - \mu\beta^2) bab + (\gamma + 2\eta)(\alpha^2 + \alpha\beta - \mu\beta^2) bac, \\ b'a'c' &= (\alpha^2 + \alpha\beta - \mu\beta^2) bac. \end{aligned}$$

So we can assume that $baa = bab = 0$ and that L_3 is generated by bac or that $baa = bac = 0$ and that L_3 is generated by bab.

40.27.1 Case 1

Let $baa = bab = 0$ and let L_3 be generated by bac. Adding suitable scalar multiples of ba to a, b, c we can take $pa - \mu cb = pb - ca - cb = pc = 0$, and so we have 1 algebra for each of the $(p-1)/2$ values of μ:

$$\langle a, b, c \,|\, baa, bab, pa - \mu cb,\ pb - ca - cb,\ pc, \text{ class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}).$$

40.27.2 Case 2

Let $baa = bac = 0$ and let L_3 be generated by bab. Adding a suitable scalar multiple of ba to c we can take $pa - \mu cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \eta c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha^3 bab, \\ pb' - c'a' - c'b' &= \alpha(pb - ca - cb) + \eta pc, \\ pc' &= pc \end{aligned}$$

so we have $3 + \gcd(p-1, 3)$ algebras for each of the $(p-1)/2$ values of μ.

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb,\ pc, \text{ class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb - bab,\ pc, \text{ class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb - \omega bab,\ pc, \text{ class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb,\ pc - bab, \text{ class } 2\rangle \ (x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb,\ pc - \omega bab, \text{ class } 2\rangle \ (p = 1 \bmod 3,\ x^2 - x - \mu \text{ irreducible}),$$

$$\langle a, b, c \,|\, baa, bac, pa - \mu cb,\ pb - ca - cb,\ pc - \omega^2 bab, \text{ class } 2\rangle \ (p = 1 \bmod 3,\ x^2 - x - \mu \text{ irreducible}).$$

40.28 Descendants of 6.112

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc, \text{ class } 2\rangle$$

Algebra 6.112 has $p^2 + 4p + 3 + 2\gcd(p-1, 3)$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.112 of order p^7. Then L_3 has order p and is generated by bab, cbb and cbc. If a', b', c' generate L and if $pa - ba,\ pb - ca,\ pc \in L_3$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \alpha^{-1} c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ c'b'b' &= \alpha^{-1} cbb, \\ c'b'c' &= \alpha^{-2} cbc. \end{aligned}$$

We can assume that one of the following sets of commutator relations holds:

$$\begin{aligned}
bab &= cbb = 0,\ cbc \neq 0, \\
bab &= cbc = 0,\ cbb \neq 0, \\
bab &= 0,\ cbb = cbc \neq 0, \\
cbb &= cbc = 0,\ bab \neq 0, \\
cbb &= 0,\ cbc = bab \neq 0, \\
cbb &= 0,\ cbc = \omega bab \neq 0\ (p = 1 \bmod 3), \\
cbb &= 0,\ cbc = \omega^2 bab \neq 0\ (p = 1 \bmod 3), \\
cbb &= bab \neq 0,\ cbc = xbab\ (0 \leq x \leq (p-1)/2), \\
cbb &= \omega bab \neq 0,\ cbc = xbab\ (0 \leq x \leq (p-1)/2).
\end{aligned}$$

40.28.1 Case 1

Let $bab = cbb = 0$ and let L_3 be generated by cbc. Adding a suitable scalar multiple of cb to a we can take $pb - ca = 0$. We then have

$$\begin{aligned}
a' &= \alpha a, \\
b' &= b, \\
c' &= \alpha^{-1} c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
c'b'c' &= \alpha^{-2} cbc, \\
pa' - b'a' &= \alpha(pa - ba), \\
pc' &= \alpha^{-1} c.
\end{aligned}$$

So we have $p + 1 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \mid bab, cbb, pa - ba,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, cbb, pa - ba - cbc,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, cbb, pa - ba - \omega cbc,\ pb - ca,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab, cbb, pa - ba - \omega^2 cbc,\ pb - ca,\ pc,\ \text{class } 3\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab, cbb, pa - ba - xcbc,\ pb - ca,\ pc - cbc,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.28.2 Case 2

Let $bab = cbc = 0$ and let L_3 be generated by cbb. Adding a suitable scalar multiple of cb to a we can take $pa - ba = 0$. We then have

$$\begin{aligned}
a' &= \alpha a, \\
b' &= b, \\
c' &= \alpha^{-1} c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
c'b'b' &= \alpha^{-1} cbb, \\
pb' - c'a' &= pb - ca, \\
pc' &= \alpha^{-1} c.
\end{aligned}$$

So we have $2p$ algebras

$$\langle a,b,c\,|\,bab, cbc, pa-ba,\ pb-ca,\ pc-xcbb,\ \text{class } 3\rangle\ (0\le x<p),$$

$$\langle a,b,c\,|\,bab, cbc, pa-ba,\ pb-ca-cbb,\ pc-xcbb,\ \text{class } 3\rangle\ (0\le x<p).$$

40.28.3 Case 3

Let $bab=0$ and let cbb generate L_3 with $cbc=cbb$. Adding a suitable scalar multiple of cb to a we can take $pa-ba=0$. We then have

$$\begin{aligned} a' &= a,\\ b' &= b,\\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'b'b' &= cbb,\\ pb'-c'a' &= pb-ca,\\ pc' &= c. \end{aligned}$$

We have p^2 algebras

$$\langle a,b,c\,|\,bab, cbc-cbb, pa-ba,\ pb-ca-xcbb,\ pc-ycbb,\ \text{class } 3\rangle\ (0\le x,y<p).$$

40.28.4 Case 4

Let $cbb=cbc=0$ and let L_3 be generated by bab. Adding suitable scalar multiple of ca to b,c we can take $pa-ba=pb-ca=0$, and adding a suitable scalar multiple of ba to c we can take $pc=0$. So we have 1 algebra

$$\langle a,b,c\,|\,cbb, cbc, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle.$$

40.28.5 Case 5,6 & 7

Let L_3 be generated by bab and let $cbb=0$, $cbc=kbab$ where $k=1$ or (when $p=1 \bmod 3$) ω or ω^2. Adding suitable scalar multiple of ca to b,c we can take $pa-ba=pb-ca=0$, and adding a suitable scalar multiple of ba to c we can take $pc=0$. So we have $\gcd(p-1,3)$ algebras

$$\langle a,b,c\,|\,cbb, cbc-bab, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle,$$

$$\langle a,b,c\,|\,cbb, cbc-\omega bab, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c\,|\,cbb, cbc-\omega^2 bab, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle\ (p=1 \bmod 3).$$

40.28.6 Cases 8 & 9

Let L_3 be generated by bab, let $cbb=kbab$ with $k=1,\omega$ and let $cbc=xbab$ with $0\le x\le (p-1)/2$. Adding suitable scalar multiple of ca to b,c we can take $pa-ba=pb-ca=0$, and adding a suitable scalar multiple of ba to c we can take $pc=0$. So we have $p+1$ algebras

$$\langle a,b,c\,|\,cbb-bab, cbc-xbab, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c\,|\,cbb-\omega bab, cbc-xbab, pa-ba,\ pb-ca,\ pc,\ \text{class } 3\rangle\ (0\le x\le (p-1)/2).$$

$$\langle a, b, c \mid pa - ba,\ pb - cb,\ pc,\ \text{class } 2\rangle$$

Algebra 6.113 has $5p + 4$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.113 of order p^7. Then L_3 has order p and is generated by bab, caa and cac. If a', b', c' generate L and if $pa - ba$, $pb - cb$, $pc \in L_3$ then

$$\begin{aligned} a' &= \alpha a + \beta b - \beta c, \\ b' &= b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \alpha bab, \\ c'a'a' &= -2\alpha\beta bab + \alpha^2 cac - \alpha\beta cac, \\ c'a'c' &= \alpha cac. \end{aligned}$$

We can assume that one of the following sets of commutator relations holds:

$$\begin{aligned} bab &= cac = 0,\ caa \neq 0, \\ bab &= caa = 0,\ cac \neq 0, \\ caa &= cac = 0,\ bab \neq 0, \\ cac &= xbab \neq 0,\ caa = 0, \\ cac &= -2bab \neq 0,\ caa = bab. \end{aligned}$$

40.29.1 Case 1

Let $bab = cac = 0$ and let L_3 be generated by caa. Adding a suitable scalar multiple of ca to b we can take $pa - ba = 0$. We then have

$$\begin{aligned} a' &= \alpha a + \beta b - \beta c, \\ b' &= b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a' &= \alpha^2 caa, \\ pb' - c'b' &= pb - cb, \\ pc' &= pc \end{aligned}$$

so we have $2p + 3$ algebras

$$\langle a, b, c \mid bab, cac, pa - ba,\ pb - cb,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, cac, pa - ba,\ pb - cb,\ pc - caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, cac, pa - ba,\ pb - cb,\ pc - \omega caa,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid bab, cac, pa - ba,\ pb - cb - caa,\ pc - xcaa,\ \text{class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid bab, cac, pa - ba,\ pb - cb - \omega caa,\ pc - xcaa,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.29.2 Case 2

Let $bab = caa = 0$ and let L_3 be generated by cac. Adding a suitable scalar multiple of ca to b we can take $pb - cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha cac, \\ pa' - b'a' &= \alpha(pb - ba), \\ pc' &= pc \end{aligned}$$

so we have $2p$ algebras

$$\langle a, b, c \,|\, bab, caa, pa - ba - xcac,\ pb - cb,\ pc, \text{ class } 3\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, bab, caa, pa - ba - xcac,\ pb - cb,\ pc - cac, \text{ class } 3\rangle\ (0 \leq x < p).$$

40.29.3 Case 3

Let $caa = cac = 0$ and let L_3 be generated by bab. Adding suitable scalar multiples of ca to a and ba to b and c we can take $pa - ba = pb - cb = pc = 0$, and so we have 1 algebra

$$\langle a, b, c \,|\, caa, cac, pa - ba,\ pb - cb,\ pc, \text{ class } 3\rangle.$$

40.29.4 Case 4

Let L_3 be generated by bab and let $caa = 0$, $cac = xbab$ with $0 < x < p$. Adding suitable scalar multiples of ca to a and ba to b and c we can take $pa - ba = pb - cb = pc = 0$, and so we have 1 algebra for each of the $p - 1$ values of x.

$$\langle a, b, c \,|\, caa, cac - xbab, pa - ba,\ pb - cb,\ pc, \text{ class } 3\rangle\ (0 < x < p).$$

40.29.5 Case 5

Let L_3 be generated by bab and let $cac = -2bab$, $caa = bab$. Adding suitable scalar multiples of ca to a and ba to b and c we can take $pa - ba = pb - cb = pc = 0$, and so we have 1 algebra

$$\langle a, b, c \,|\, caa - bab, cac + 2bab, pa - ba,\ pb - cb,\ pc, \text{ class } 3\rangle.$$

40.30 Descendants of 6.114

$$\langle a, b, c \,|\, pa - ba,\ pb - cb,\ pc - kba - ca, \text{ class } 2\rangle\ (k = 0, 1, \ldots, p - 1)$$

Over all p values of k, algebra 6.114 has $4p - 4$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.114 of order p^7. Then L_3 has order p and is generated by bab and bac. We consider possible generators a', b', c' for L which satisfy the same relations as a, b, c modulo L_3. We note that if $a' = \lambda a$ then $\lambda = 1$, $b' = b$, $c' = c$. Similarly if $b' = \lambda b$ then $\lambda = 1$, $a' = a$, $c' = c$.

It is straightforward to check that whatever dependance relation there is between bab and bac, we can add elements of L^2 to a, b, c so that

$$pa - ba = pb - cb = pc - kba - ca = 0,$$

except in the case when $k = -1$ and $bab = bac$. In the case when $k = -1$ and $bab = bac$ then we can take $pa - ba = pb - cb = 0$, and we can take $pc + ba - ca = xbab$ with $0 \leq x < p$. We have computed the automorphism group induced on L/L_2 in Appendix H, and have checked that when $x = -1$ and $bab = bac$ then the p values of x here give distinct algebras. So the number of descendants of algebra 6.114 depends on the orbits of the ratios $bab : bac$ under the automorphism group induced on L/L_2 by the automorphism group of 6.114. Proofs of the summary below are given in Appendix H. The matrices C, D, E and F referred to below can be found in Appendix H.

40.30.1 $k = -1$

If $x = -1$ then the automorphism group has order $2p$, the identity transformation øxes all $p + 1$ ratios, and everything øxes $(1, 1)$, and F with $\beta = 1$ øxes everything. So if we exclude the ratio $(1, 1)$, which is in an orbit on its own, a Burnside's Lemma count gives the number of other orbits as 1. In other words, we can take $bac = bab$ or $bac = 0$.

$$\langle a, b, c \,|\, bac, pa - ba,\ pb - cb,\ pc + ba - ca, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bac - bab, pa - ba,\ pb - cb,\ pc + ba - ca - xbab, \text{ class } 3\rangle \ (0 \leq x < p).$$

40.30.2 $k = 3$

The automorphism group has size p, consisting of C, D, and E with $\gamma \neq \pm 1$. The ratio $(1, -1)$ is øxed by everthing, and E with $\gamma = 0$ øxes everything. so we have two orbits: $(1, -1)$ and the rest. In other words we can take $bac = -bab$ or $bac = 0$.

$$\langle a, b, c \,|\, bac, pa - ba,\ pb - cb,\ pc - 3ba - ca, \text{ class } 3\rangle,$$

$$\langle a, b, c \,|\, bac + bab, pa - ba,\ pb - cb,\ pc - 3ba - ca, \text{ class } 3\rangle.$$

40.30.3 $\gamma k + \gamma^2 - \gamma + 1$ has no roots

The automorphism group has size $p+1$ and consists of C, D and E with $\gamma \neq -1$. Matrix C øxes nothing, matrix D øxes all $p+1$ ratios if $k = 1$, but otherwise øxes nothing, matrix E with $\gamma = 0$ øxes everything, and if $k \neq 1$ then E with $\gamma = -\frac{2}{k-1}$ øxes everything. So a Burnside's Lemma count gives 2 orbits. But we need to run a magma program to ønd the orbits. There are $(p-1)/2$ algebras here, each giving 2 algebras

$$\langle a, b, c \,|\, bac - xbab, pa - ba,\ pb - cb,\ pc - kba - ca, \text{ class } 3\rangle.$$

40.30.4 $\gamma k + \gamma^2 - \gamma + 1$ has two roots

The automorphism group has size $p - 1$ and consists of C, D and E with $\gamma \neq -1$ and $\gamma k + \gamma^2 - \gamma + 1 \neq 0$. The ratio $(1, z)$ is øxed by everything if $1 - z + zk + z^2 = 0$. So there are two ratios øxed by everything. Matrix C øxes nothing else and nor does D unless $k = 1$ when if øxes everthing. Matrix E with $\gamma = 0$ øxes everything, as does $\gamma = -\frac{2}{k-1}$ if $k \neq 1$. So a Burnside's Lemma count gives 4 orbits. Again, we need to run a magma program to ønd the orbits. There are $(p-3)/2$ values of k here, each giving 4 algebras

$$\langle a, b, c \,|\, bac - xbab, pa - ba,\ pb - cb,\ pc - kba - ca, \text{ class } 3\rangle.$$

$$\langle a, b, c \,|\, pa - ba,\ pb - ca,\ pc - cb,\ \text{class } 2\rangle$$

Algebra 6.115 has 3 descendants of order p^7 and p-class 3.

The automorphism group of algebra 6.115 is a little tricky! We need to ønd all generators a', b', c' for 6.115, where

$$pa' - b'a' = pb' - c'a' = pc' - c'b' = 0.$$

We note that one possibility is

$$\begin{aligned} a' &= c, \\ b' &= -b, \\ c' &= a \end{aligned} \tag{1}$$

and that there is no a', b', c' with $a' = \beta b$. Using these facts we see that it is suŒcient to ønd all a', b', c' with $a' = \alpha a + \beta b + \gamma c$ with $\alpha \neq 0$. We note that one possibility is

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha\lambda a + b, \\ c' &= \frac{1}{2}\alpha\lambda^2 a + \lambda b + \alpha^{-1} c. \end{aligned} \tag{2}$$

So it is suŒcient to ønd all a', b', c' with $a' = \alpha a + \beta b + \gamma c$ with $\alpha \neq 0$ and with b' a linear combination of b and c. It is straightforward to show that with these restrictions we have

$$\begin{aligned} a' &= \alpha a + \alpha\eta b + \frac{1}{2}\alpha\eta^2 c, \\ b' &= b + \eta c, \\ c' &= \alpha^{-1} c. \end{aligned} \tag{3}$$

Now let L be a descendant of algebra 6.115 of order p^7. Then L_3 has order p and is generated by bab, bac and cac. If we take a', b', c' of the form (1), (2), (3) as above we have

$$\begin{aligned} b'a'b' &= c'a'c', \\ b'a'c' &= -bac, \\ c'a'c' &= b'a'b', \end{aligned}$$

$$\begin{aligned} b'a'b' &= \alpha bab, \\ b'a'c' &= \alpha\lambda bab + bac, \\ c'a'c' &= \frac{1}{2}\alpha\lambda^2 bab + \lambda bac + \alpha^{-1} cac, \end{aligned}$$

$$\begin{aligned} b'a'b' &= \alpha bab + \alpha\eta bac + \frac{1}{2}\alpha\eta^2 cac, \\ b'a'c' &= bac + \eta cac, \\ c'a'c' &= \alpha^{-1} cac. \end{aligned}$$

We see that we can assume that L_3 is generated by bab, and considering a', b', c' of the form (2) we can take $bac = 0$. Let $cac = xbab$.

If $x = 0$ then for any a', b', c' of forms (1), (2) or (3) with $b'a'b' \neq 0$, $b'a'c' = 0$ we have $c'a'c' = 0$.

So let $cac = xbab$. Considering a', b', c' of the form (2) with $\lambda = 0$ we see that there are two orbits for x, squares and non-squares. Furthermore, transformations of form (1) and (3) which preserve the condition $b'a'c' = 0$ also preserve these orbits.

So we can assume that L_3 is generated by bab, that $bac = 0$ and that $cac = 0$, bab or ωbab. We can see that these three possibilities give non-isomorphic algebras as follows. If we pick a basis v for L_3 then we can write

$$\begin{aligned} bab &= \lambda v, \\ bac &= \mu v, \\ cac &= \nu v \end{aligned}$$

for some λ, μ, ν which are not all zero. (Note that since v is only determined up to a scale factor, the triple λ, μ, ν is only determined up to a scale factor.) If we pick generators a', b', c' for L then we obtain

$$\begin{aligned} b'a'b' &= \lambda' v, \\ b'a'c' &= \mu' v, \\ c'a'c' &= \nu' v. \end{aligned}$$

It is easy to check that $\mu'^2 - 2\lambda'\mu' = \mu^2 - 2\lambda\mu$ for a', b', c' of form (1), (2) or (3). So we get 3 orbits of triples λ, μ, ν corresponding to the case when $\mu^2 - 2\lambda\mu$ is zero, a square, or not a square.

40.31.1 Case 1

Let L_3 be generated by bab and let $bac = cac = 0$. Adding suitable scalar multiples of ca to b and c and a suitable scalar multiple of ba to c we can take $pa - ba = pb - ca = pc - cb = 0$, and so we have 1 algebra

$$\langle a, b, c \mid bac, cac, pa - ba,\ pb - ca,\ pc - cb, \text{ class } 3\rangle.$$

40.31.2 Cases 2 & 3

Let L_3 be generated by bab and let $bac = 0$, $cac = kbab$ where $k = 1$ or ω. Adding suitable scalar multiples of ca to b and c and a suitable scalar multiple of ba to c we can take $pa - ba = pb - ca = pc - cb = 0$, and so we have 2 algebras

$$\langle a, b, c \mid bac, cac - bab, pa - ba,\ pb - ca,\ pc - cb, \text{ class } 3\rangle,$$

$$\langle a, b, c \mid bac, cac - \omega bab, pa - ba,\ pb - ca,\ pc - cb, \text{ class } 3\rangle.$$

40.32 Descendants of 6.116

$$\langle a, b, c \mid pa - ba,\ pb - ca,\ pc + cb, \text{ class } 2\rangle$$

Algebra 6.116 has 3 descendants of order p^7 and p-class 3.

The automorphism group of algebra 6.116 is a little tricky! But if we consider its covering algebra M then we see that if u is a non-trivial linear combination of a and c

then Mu has order p, whereas if u is a linear combination of a, b and c which is not a linear combination of a and c then Mu has order p^2. It follows that if a', b', c' generate algebra 6.116 and satisfy the same relations as a, b, c then a' and c' are linear combinations of a and c. Using this fact it is straightforward to show that

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= b, \\ c' &= \mu a + \xi c \end{aligned}$$

modulo L_2, with $\alpha\xi - \gamma\mu = 1$.

Now let L be a descendant of algebra 6.116 of order p^7. Then L_3 has order p and is generated by bab and cac. If a', b', c' are as above then

$$\begin{aligned} b'a'b' &= \alpha bab - \gamma cac, \\ c'a'c' &= -\mu bab + \xi cac. \end{aligned}$$

So we can take $bab = 0$ and assume that L_3 is generated by cac. (We then need $\gamma = 0$ and $\xi = \alpha^{-1}$.) Adding suitable scalar multiples of ca to a and b we can take $pb - ca = pc + cb = 0$. We then have

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \mu a + \alpha^{-1} c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \alpha^{-1} cac, \\ pa' - b'a' &= \alpha(pa - ba) \end{aligned}$$

and so we have 3 algebras

$$\langle a, b, c \mid bab, pa - ba,\ pb - ca,\ pc + cb,\ \text{class } 3 \rangle,$$

$$\langle a, b, c \mid bab, pa - ba - cac,\ pb - ca,\ pc + cb,\ \text{class } 3 \rangle,$$

$$\langle a, b, c \mid bab, pa - ba - \omega cac,\ pb - ca,\ pc + cb,\ \text{class } 3 \rangle.$$

40.33 Descendants of 6.117

$$\langle a, b, c \mid pa - ba,\ pb - ca,\ pc - \omega ba + cb,\ \text{class } 2 \rangle$$

Algebra 6.117 has $p + 1$ descendants of order p^7 and p-class 3.

Let L be a descendant of algebra 6.117 of order p^7. Then L_3 has order p and is generated by bab and cac. If a', b', c' generate L and if $pa' - b'a'$, $pb' - c'a'$, $pc' - \omega b'a' + c'b' \in L_3$ then

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \lambda a \pm c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b' &= \pm bab, \\ c'a'c' &= -\lambda bab \pm cac. \end{aligned}$$

So we can assume that $bab = 0$ and that L_3 is generated by cac or that $cac = 0$ and that L_3 is generated by bab. [In establishing this form for a', b', c' we note that if M is the covering algebra of 6.117 then $M^2(\alpha a + \beta b + \gamma c)$ has order p^2 if $\beta \neq 0$ and order p if $\beta = 0$ and if $\alpha a + \beta b + \gamma c \neq 0$. So we can take a', c' to be linear combinations of a and c.]

40.33.1 $bab = 0$

Let $bab = 0$ and let L_3 be generated by cac. Adding suitable scalar multiples of ca, cb to a we can take $pb - ca = pc - \omega ba + cb = 0$. We then have

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \lambda a \pm c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c' &= \pm cac, \\ pa' - b'a' &= \pm(pa - ba) \end{aligned}$$

so we have p algebras

$$\langle a, b, c \,|\, bab, pa - ba - xcac,\ pb - ca,\ pc - \omega ba + cb,\ \text{class } 3\rangle\ (0 \leq x < p).$$

40.33.2 $cac = 0$

Let $cac = 0$ and let L_3 be generated by bab. Adding suitable scalar multiples of ca to b,c we can take $pa - ba = pb - ca = 0$, and adding a suitable scalar multiple of ba to a we can take $pc - \omega ba + cb = 0$ so we have 1 algebra

$$\langle a, b, c \,|\, cac, pa - ba,\ pb - ca,\ pc - \omega ba + cb,\ \text{class } 3\rangle.$$

40.34 Summary

Algebra 3.1 has $3p + 27$ immediate descendants of order p^6, with presentations 6.85 ˘ 6.117. All these algebras are capable, and between them they have

$$2p^2 + 63p + 362 + (p + 19)\gcd(p - 1, 3) + 5\gcd(p - 1, 4) + \gcd(p - 1, 5)$$

descendants of order p^7 and p-class 3. The table below gives the number of descendants of order p^7 of the algebras 6.85 ˘ 6.117. (There is no algebra 6.107.) Note that 5 of the entries in the table correspond to one parameter families of algebras. These are algebras 6.98, 6.101, 6.108, 6.111 and 6.114 (marked * below) with (respectively) $(p+1)/2$, $(p-1)/2$, $(p+1)/2$, $(p-1)/2$ and p algebras in each family.

In the family 6.98 two of the algebras each have 4 desendants of order p^7, and the remaining $(p-3)/2$ algebras each have 5 descendants of order p^7.

In the family 6.101 all $(p-1)/2$ algebras have 3 descendants of order p^7.

In the family 6.108 one of the algebras has 5 descendants of order p^7, one has $13 + \gcd(p-1, 3)$, one has $6 + \gcd(p-1, 3) + \gcd(p-1, 4)/2$, and the other $(p-5)/2$ algebras each have $12 + \gcd(p-1, 3)$ descendants of order p^7.

In the family 6.111 all $(p-1)/2$ algebras each have $4 + \gcd(p-1, 3)$ descendants of order p^7.

In the family 6.114 one of the algebras has $p + 1$ descendants of order p^7, $(p+1)/2$ of the algebras each have 2 descendants of order p^7, and $(p-3)/2$ of the algebras each have 4 descendants of order p^7.

In the table we indicate the numbers of descendants of diœerent algebras in the 5 families.

6.85	3
6.86	$p+17$
6.87	5
6.88	26
6.89	$p+6$
6.90	30
6.91	$2p+88+\gcd(p-1,4)$
6.92	$2p+13$
6.93	$p+15+2\gcd(p-1,3)+\gcd(p-1,4)$
6.94	$2p+15+3\gcd(p-1,3)+\gcd(p-1,4)$
6.95	$5p+10$
6.96	$2p+26$
6.97	$3p+18+\gcd(p-1,3)$
6.98*	$4, 4, \frac{p-3}{2}\times 5$
6.99	$p+4$
6.100	3
6.101*	$\frac{p-1}{2}\times 3$
6.102	$p+3$
6.103	$p+3$
6.104	$5p+24+3\gcd(p-1,3)$
6.105	$11+5\gcd(p-1,3)+\gcd(p-1,5)$
6.106	$p^2+10p+32+\gcd(p-1,3)+\gcd(p-1,4)$
6.108*	$5, 6+\gcd(p-1,3)+\frac{1}{2}\gcd(p-1,4), 13+\gcd(p-1,3), \frac{p-5}{2}\times(12+\gcd(p-1,3))$
6.109	$7+2\gcd(p-1,3)$
6.110	$2+\gcd(p-1,3)+\frac{1}{2}\gcd(p-1,4)$
6.111*	$\frac{p-1}{2}\times(4+\gcd(p-1,3))$
6.112	$p^2+4p+3+2\gcd(p-1,3)$
6.113	$5p+4$
6.114*	$p+1, \frac{p+1}{2}\times 2, \frac{p-3}{2}\times 4$
6.115	3
6.116	3
6.117	$p+1$

41 Grandchildren of algebra 24 (4.3)

Algebra 4.3 has $3p^2+13p+37+\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^6 and p-class 3. They have presentations labelled 6.118 ˘ 6.179. We next classify the descendants of these algebras of order p^7 and p-class 4.

41.1 Descendants of 6.118

Algebra 6.118 has $11+4\gcd(p-1,3)$ descendants of order p^7 and p-class 4.

Algebra 6.118 has presentation

$$\langle a,b,c \mid ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and so if L is a descendant of 6.118 of order p^7 then the commutator structure of L is the same as that of one of 7.98 ˘ 7.101 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we can assume that L satisøes one of the following four sets of commutator relations.

$$\begin{aligned} ca &= cb = baaa = baab = 0, \\ ca &= babb,\ cb = baaa = baab = 0, \\ ca &= cb = baab = 0,\ babb = -baaa, \\ ca &= cb = baab = 0,\ babb = -\omega baaa. \end{aligned}$$

41.1.1 Case 1

Let $ca = cb = baaa = baab = 0$. Then L_4 is generated by $babb$ and if a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta c, \\ b' &= \gamma a + \delta b + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$b'a'b'b' = \alpha\delta^3 babb.$$

So we can assume that $pc = 0$ or $babb$, and if $pc = babb$ we can assume that $pa = pb = 0$. If $pc = 0$ we can assume that $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and if $pa \neq 0$ we can assume that $pb = 0$. If $pa = pc = 0$ then we can assume that $pb = 0$ or $babb$. ($3 + \gcd(p-1, 3)$ algebras.)

$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, ca,\ cb,\ baaa,\ baab,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle.$$

41.1.2 Case 2

Let $ca - babb = cb = baaa = baab = 0$. Then L_4 is generated by $babb$ and if a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta c, \\ b' &= \gamma a + \delta b + \varepsilon c, \\ c' &= \delta^3 c \end{aligned}$$

modulo L_2 and

$$b'a'b'b' = \alpha\delta^3 babb.$$

So, as in Case 1 we get $3 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, ca - babb,\ cb,\ baaa,\ baab,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle.$$

41.1.3 Case 3

Let $ca = cb = baab = 0$, $babb = -kbaaa$ where $k = 1$ or ω. Then L_4 is generated by $baaa$ and if a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \pm k\beta a \pm \alpha b + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$b'a'a'a' = \pm(\alpha^2 - \omega\beta^2)^2 baaa.$$

So we can take $pc = 0$ or $baaa$, and if $pc \neq 0$ then we can take $pa = pb = 0$. If $pc = 0$ then one possibility is $pa = pb = pc = 0$. If $pc = 0$ and pa, pb are not both zero then we can assume that $pa = 0$, except in the case when $k = 1$ and $pa = \pm pb$. If $pa = 0$ and $pb \neq 0$ then we then need $\beta = 0$, and we can take $pb = baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. In the case when $k = 1$ and $pc = 0$, $pa = \pm pb \neq 0$, we can change the sign of b so that $pa = pb = \mu baaa$. Then

$$\begin{aligned} pa' &= pb' = (\alpha + \beta)\mu baaa, \\ b'a'a'a' &= (\alpha^2 - \beta^2)^2 baaa \end{aligned}$$

and so we can take $\mu = 1$.

So we have $3 + \gcd(p-1, 3)$ algebras when $k = 1$ and $2 + \gcd(p-1, 3)$ algebras when $k = \omega$.

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa - baaa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + \omega baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + \omega baaa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + \omega baaa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + \omega baaa,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ baab,\ babb + \omega baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

41.2 Descendants of 6.119

Algebra 6.119 has $(p-1)/2 + 3 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Algebra 6.119 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.119 of order p^7 then L_4 is generated by $baaa$ and ca, cb, $pa - bab$, pb, $pc \in L_4$. Adding a suitable scalar multiple of baa to c we can assume that $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= \pm\alpha^3 baaa, \\ c'b' &= \pm\lambda cb, \\ pa' - b'a'b' &= \alpha(pa - bab) + \beta pb + \gamma pc, \\ pb' &= \pm pb + \varepsilon pc, \\ pc' &= \lambda pc. \end{aligned}$$

So we can assume that $cb = 0$ or $baaa$. If $cb = 0$ then we can assume that $pc = 0$ or $baaa$, and if $cb = baaa$ then we can assume that $pc = xbaaa$ with $0 \leq x \leq (p-1)/2$. If $pc \neq 0$ then we can assume that $pa - bab = pb = 0$. If $pc = 0$ then we can assume that $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pc = 0$ and $pb \neq 0$ then we can assume that $pa - bab = 0$, but if $pb = pc = 0$ then we can assuem that $pa - bab = 0$, $baaa$ or (if $p = 1 \bmod 4$) $\omega baaa$.

So we have $(p-1)/2 + 3 + 2\gcd(p-1,3) + \gcd(p-1,4)$ algebras

$$\langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\begin{gathered} \langle a, b, c \mid ca,\ cb,\ pa - bab - baaa,\ pb,\ pc,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb,\ pa - bab - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4), \\ \langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb - baaa,\ pc,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \\ \langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \\ \langle a, b, c \mid ca,\ cb,\ pa - bab,\ pb,\ pc - baaa,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab,\ pb,\ pc,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab - baaa,\ pb,\ pc,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4), \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab,\ pb - baaa,\ pc,\ \text{class } 4\rangle, \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3), \\ \langle a, b, c \mid ca,\ cb - baaa,\ pa - bab,\ pb,\ pc - xbaaa,\ \text{class } 4\rangle\ (1 \leq x \leq (p-1)/2). \end{gathered}$$

41.3 Descendants of 6.120

Algebra 6.120 has $(p-1)/2 + 3 + 2\gcd(p-1,3) + \gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Algebra 6.120 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.119, and we have $(p-1)/2 + 3 + 2\gcd(p-1,3) + \gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Algebra 6.121 has $2p+4+2\gcd(p-1,3)$ descendants of order p^7 and p-class 4.

Algebra 6.121 has presentation

$$\langle a,b,c \mid ca,\ cb,\ pa-baa,\ pb,\ pc,\ \text{class } 3\rangle,$$

then L_4 is generated by $babb$ and $ca,\ cb,\ pa-baa,\ pb,\ pc \in L_4$. Adding a suitable scalar multiple of bab to c we can assume that $cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4 then

$$\begin{aligned} a' &= \alpha a+\gamma c,\\ b' &= \alpha^{-1}b+\varepsilon c,\\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \alpha^{-2}babb,\\ c'a' &= \alpha\lambda ca,\\ pa'-b'a'a' &= \alpha(pa-baa)+\gamma pc,\\ pb' &= \alpha^{-1}pb+\varepsilon pc,\\ pc' &= \lambda pc. \end{aligned}$$

So we can take $ca=0$ or $babb$, and we can (independantly) take $pc=0$ or $babb$. If $pc\neq 0$ then we can take $pa-baa=pb=0$. If $pc=0$ then we can take $pb=0$ or $babb$. If $pb=pc=0$ then we can take $pa-baa=0$ or $babb$ or (if $p=1\bmod 3$) $\omega babb$ or $\omega^2 babb$. And if $pb=babb$, $pc=0$ then we need $\alpha=1$, and so we can take $pa-baa=xbabb$ where $0\leq x<p$. So we have $2p+4+2\gcd(p-1,3)$ algebras

$$\langle a,b,c \mid ca,\ cb,\ pa-baa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid ca,\ cb,\ pa-baa-babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid ca,\ cb,\ pa-baa-\omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1\bmod 3),$$

$$\langle a,b,c \mid ca,\ cb,\ pa-baa-\omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1\bmod 3),$$

$$\langle a,b,c \mid ca,\ cb,\ pa-baa-xbabb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (0\leq x<p),$$

$$\langle a,b,c \mid ca,\ cb,\ pa-baa,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa-babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa-\omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1\bmod 3),$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa-\omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1\bmod 3),$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa-xbabb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (0\leq x<p),$$

$$\langle a,b,c \mid ca-babb,\ cb,\ pa-baa,\ pb,\ pc-babb,\ \text{class } 4\rangle.$$

41.5 Descendants of 6.122

Algebra 6.122 has $p+2$ descendants of order p^7 and p-class 4.

Algebra 6.122 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb - \lambda bab,\ pc,\ \text{class } 3\rangle,$$

with $\lambda \neq 0$ and λ, λ^{-1} giving isomorphic algebras ($(p+1)/2$ algebras), but the algebra is terminal unless $\lambda = -1$. If L is a descendant of order p^7 when $\lambda = -1$ then L_4 is generated by $baab$, and ca, cb, $pa - baa$, $pb + bab$, $pc \in L_4$. By adding suitable scalar multiples of baa and bab to c we can assume that $ca = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'b' &= baab, \\ pa' - b'a'a' &= \alpha(pa - baa) + \gamma pc, \\ pb' + b'a'b' &= \alpha^{-1}(pb + bab) + \varepsilon pc, \\ pc' &= \lambda pc, \end{aligned}$$

or

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \alpha^{-1} a + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'b' &= -baab, \\ pa' - b'a'a' &= \alpha(pb + bab) + \gamma pc, \\ pb' + b'a'b' &= \alpha^{-1}(pa - baa) + \varepsilon pc, \\ pc' &= \lambda pc. \end{aligned}$$

So we can take $pc = 0$ or $baab$, and if $pc \neq 0$ we can take $pa - baa = pb + bab = 0$. If $pc = 0$ then we can take $pa - baa = pb + bab = 0$ or we can take $pa - baa = 0$, $pb + bab = baab$, or we can take $pa - baa = baab$ and $pb + bab = xbaab$ with $0 < x < p$. So we have $p+2$ algebras

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb + bab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb + bab,\ pc - baab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa,\ pb + bab - baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baa - baab,\ pb + bab - xbaab,\ pc,\ \text{class } 4\rangle\ (0 < x < p).$$

41.6 Descendants of 6.123 and 6.124

These two algebras are terminal.

41.7 Descendants of 6.125

Algebra 6.125 has $p+1$ descendants of order p^7 and p-class 4.

Algebra 6.125 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega bab,\ pb - baa,\ pc, \text{ class } 3\rangle$$

but it is convenient to switch to an alternative presentation

$$\langle a, b, c \mid ca,\ cb,\ pa + bab,\ pb + \omega baa,\ pc, \text{ class } 3\rangle.$$

If L is a descendant of 6.125 of order p^7 then L_4 is generated by $baaa$ and ca, cb, $pa + bab$, $pb + \omega baa$, $pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we can assume that $ca = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \pm\omega\beta a \pm \alpha b + \varepsilon c, \\ c' &= \lambda c \end{aligned}$$

modulo L_2 with $\alpha^2 - \omega\beta^2 = 1$, and

$$\begin{aligned} b'a'a'a' &= \pm baaa, \\ pa' + b'a'b' &= \alpha(pa + bab) + \beta(pb + \omega bab) + \gamma pc, \\ pb' + \omega b'a'a' &= \pm\omega\beta(pa + bab) \pm \alpha(pb + \omega bab) + \varepsilon pc, \\ pc' &= \lambda pc. \end{aligned}$$

So we can assume that $pc = 0$ or $baaa$ and if $pc \neq 0$ we can assume that $pa + bab = pb + \omega baa = 0$, giving

$$\langle a, b, c \mid ca,\ cb,\ pa + bab,\ pb + \omega baa,\ pc - baaa, \text{ class } 4\rangle.$$

If $pc = 0$ then by the argument given in the analysis of 6.427 we have p algebras

$$\langle a, b, c \mid ca,\ cb,\ pa + bab - \lambda baaa,\ pb + \omega baa - \mu baaa,\ pc, \text{ class } 4\rangle\ (0 \leq \lambda, \mu < p),$$

with the isomorphism class depending only on the value of $\mu^2 - \omega\lambda^2$.

41.8 Descendants of 6.126

Algebra 6.126 is terminal.

41.9 Descendants of 6.127

Algebra 6.127 has $3p+4+6\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Algebra 6.127 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc - bab, \text{ class } 3\rangle,$$

and if L is a descendant of 6.127 of order p^7 then L_4 is generated by $baaa$, and ca, cb, pa, pb, $pc - bab \in L_4$. Adding a suitable scalar multiple of baa to c we may assume that $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= \gamma b, \\ c' &= \alpha\gamma^2 c \end{aligned}$$

modulo L_2, and

$$\begin{aligned}
b'a'a'a' &= \alpha^3\gamma baaa,\\
c'b' &= \alpha\gamma^3 cb,\\
pa' &= \alpha pa + \beta pb,\\
pb' &= \gamma pb,\\
pc' - b'a'b' &= \alpha\gamma^2(pc - bab).
\end{aligned}$$

So we can assume that $cb = 0$, $baaa$ or $\omega baaa$.

If $cb = 0$ then we can assume that $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$ and that $pc - bab = 0$ or $baaa$. If $pb \neq 0$ we can assume that $pa = 0$. If $pb = pc - bab = 0$ then we can assume that $pa = 0$ or $baaa$, and if $pb = 0$, $pc - bab = baaa$ we can assume that $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

If $cb = baaa$ or $\omega baaa$ then we need $\gamma = \pm\alpha$ and so we have

$$\begin{aligned}
b'a'a'a' &= \pm\alpha^4 baaa,\\
pa' &= \alpha pa + \beta pb,\\
pb' &= \pm\alpha pb,\\
pc' - b'a'b' &= \alpha^3(pc - bab).
\end{aligned}$$

So we can take $pc - bab = 0$ or $baaa$. If $pc - bab = 0$ then we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$ and if $pb \neq 0$ we can take $pa = 0$. If $pb = pc - bab = 0$ then we can take $pa = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pc - bab = baaa$ then we need $\alpha = \pm 1$ and $\gamma = 1$ so we have

$$\begin{aligned}
b'a'a'a' &= \pm baaa,\\
pa' &= \pm pa + \beta pb,\\
pb' &= pb,
\end{aligned}$$

and so we can take $pb = xbaaa$ where $0 \leq x < (p-1)/2$. If $pc - bab = baaa$ and $pb \neq 0$ we can take $pa = 0$, and if $pc - bab = baaa$ and $pb = 0$ we can take $pa = xbaaa$ where $0 \leq x < p$.

So we have

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc - bab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baaa,\ pb,\ pc - bab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - baaa,\ pc - bab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - \omega baaa,\ pc - bab,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - \omega^2 baaa,\ pc - bab,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - baaa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega baaa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega^2 baaa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid ca,\ cb,\ pa - \omega^3 baaa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - baaa,\ pc - bab - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb - \omega baaa,\ pc - bab - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid ca,\ cb,\ pa,\ pb-\omega^2 baaa,\ pc-bab-baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa,\ pb,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa-baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa-\omega baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa-\omega^2 baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa,\ pb-baaa,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa,\ pb-\omega baaa,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa,\ pb-\omega^2 baaa,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa,\ pb-xbaaa,\ pc-bab-baaa,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$
$$\langle a,b,c \mid ca,\ cb-baaa,\ pa-xbaaa,\ pb,\ pc-bab-baaa,\ \text{class } 4\rangle\ (0 < x < p),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa,\ pb,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa-baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa-\omega baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa-\omega^2 baaa,\ pb,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa,\ pb-baaa,\ pc-bab,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa,\ pb-\omega baaa,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa,\ pb-\omega^2 baaa,\ pc-bab,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa,\ pb-xbaaa,\ pc-bab-baaa,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$
$$\langle a,b,c \mid ca,\ cb-\omega baaa,\ pa-xbaaa,\ pb,\ pc-bab-baaa,\ \text{class } 4\rangle\ (0 < x < p).$$

41.10 Descendants of 6.128 ˘ 6.130

These algebras are terminal.

41.11 Descendants of 6.131

Algebra 6.131 has $15+(p+10)\gcd(p-1,3)+\gcd(p-1,4)+\gcd(p-1,7)$ descendants of order p^7 and p-class 4.

Algebra 6.131 has presentation

$$\langle a,b,c \mid ca-bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.131 of order p^7 then the commutator structure of L is the same as one of 7.102 ˘ 7.107 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we may assume that one of the following sets of commutator relations holds.

$$\begin{aligned}
ca &= bab,\ cb = baaa = baab = 0,\\
ca &= bab,\ cb = baab = babb = 0,\\
ca &= bab,\ cb = baaa,\ baab = babb = 0,\\
ca &= bab,\ cb = baab = 0,\ babb = baaa,\\
ca &= bab,\ cb = baab = 0,\ babb = \omega baaa,\\
ca &= bab,\ cb = baaa = babb = 0.
\end{aligned}$$

41.11.1 Case 1

Let $ca = bab$, $cb = baaa = baab = 0$, so that L_4 is generated by $babb$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta c, \\ b' &= \gamma b, \\ c' &= \gamma^2 c \end{aligned}$$

modulo L_2, and

$$b'a'b'b' = \alpha\gamma^3 babb.$$

So we can take pb and pc to be (independantly) equal to 0 or $babb$, and if $pc \neq 0$ we can take $pa = 0$. If $pc = 0$ we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. So we have $4 + 2\gcd(p-1, 3)$ algebras

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa,\ pb - babb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - babb,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - \omega babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baaa,\ baab,\ ca - bab,\ cb,\ pa - \omega^2 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3).$$

41.11.2 Case 2

Let $ca = bab$, $cb = baab = babb = 0$, so that L_4 is generated by $baaa$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \delta^2 c \end{aligned}$$

modulo L_2, and

$$b'a'a'a' = \alpha^3\delta baaa.$$

So we can take $pc = 0$ or $baaa$. If $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb \neq 0$ we can take $pa = 0$, and if $pb = pc = 0$ we can take $pa = 0$ or $baaa$. So we have $3 + \gcd(p-1, 3)$ algebras

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

41.11.3 Case 3

Let $ca = bab$, $cb = baaa$, $baab = babb = 0$, so that L_4 is generated by $baaa$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha^2 a + \beta b + \gamma c, \\ b' &= \alpha^3 b + \varepsilon c, \\ c' &= \alpha^6 c \end{aligned}$$

modulo L_2, and

$$b'a'a'a' = \alpha^9 baaa.$$

So we can assume that $pc = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pc \neq 0$ we can assume that $pa = pb = 0$. If $pc = 0$ we can take $pb = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 3$) $\omega^2 baaa$, $\omega^3 baaa$, $\omega^4 baaa$ or $\omega^5 baaa$. If $pb \neq 0$ then we can assume that $pa = 0$. And if $pb = pc = 0$ then we can assume that $pa = 0$ or $baaa$ or (if $p = 1 \bmod 7$) $\omega^k baaa$ for $1 \leq k \leq 6$. So we have $1 + 3\gcd(p-1,3) + \gcd(p-1,7)$ algebras.

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca{-}bab,\ cb{-}baaa,\ pa{-}\omega^k baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 7,\ 1 \leq k \leq 6),$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca{-}bab,\ cb{-}baaa,\ pa,\ pb{-}\omega^k baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3,\ k = 2, 3, 4, 5),$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baab,\ babb,\ ca - bab,\ cb - baaa,\ pa,\ pb,\ pc - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3).$$

41.11.4 Cases 4 and 5

Let $ca = bab$, $cb = baab = 0$, $babb = kbaaa$ where $k = 1$ or ω, so that L_4 is generated by $baaa$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \pm\alpha b + \varepsilon c, \\ c' &= \alpha^2 c \end{aligned}$$

modulo L_2, and

$$b'a'a'a' = \pm\alpha^4 baaa.$$

So we can take $pc = 0$, $baaa$ or (if $p = 1 \bmod 4$) $\omega baaa$, and if $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we can take $pa = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pa = pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. And if $pc = 0$ and $pa = kbaaa$ (with $k = 1, \omega$ or ω^2) then we can take $pb = xbaaa$ where $0 \leq x \leq (p-1)/2$. So we have $2 + (p+3)\gcd(p-1,3) + \gcd(p-1,4)$ algebras.

$$\langle a, b, c \,|\, baab,\ babb - baaa,\ ca - bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baab,\ babb - baaa,\ ca - bab,\ cb,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa-baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa-\omega baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3,\ 0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa-\omega^2 baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3,\ 0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa,\ pb,\ pc-baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baab,\ babb-baaa,\ ca-bab,\ cb,\ pa,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa-baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa-\omega baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3,\ 0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa-\omega^2 baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3,\ 0 \le x \le (p-1)/2),$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb,\ pc-baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baab,\ babb-\omega baaa,\ ca-bab,\ cb,\ pa,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p=1 \bmod 4).$$

41.11.5 Case 6

Let $ca = bab$, $cb = baaa = babb = 0$, so that L_4 is generated by $baab$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \delta^2 c \end{aligned}$$

modulo L_2, and

$$b'a'a'b' = \alpha^2\delta^2 baab.$$

So we can assume that $pc = 0$, $baab$ or $\omega baab$, and if $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we can take pa to be 0 or $baab$. If $pa = pc = 0$ we can take $pb = 0$ or $baab$, and if $pa = baab$, $pc = 0$ we can take $pb = 0$ or $baab$ or (if $p = 1 \bmod 3$) $\omega baab$ or $\omega^2 baab$. So we have $5 + \gcd(p-1, 3)$ algebras

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa,\ pb-baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa-baab,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa-baab,\ pb-baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa-baab,\ pb-\omega baab,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa-baab,\ pb-\omega^2 baab,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa,\ pb,\ pc-baab,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baaa,\ babb,\ ca-bab,\ cb,\ pa,\ pb,\ pc-\omega baab,\ \text{class } 4\rangle.$$

Algebra 6.132 has $(p+1+3(p+1)\gcd(p-1,3)+\gcd(p-1,4))/2$ descendants of order p^7 and p-class 4.

Algebra 6.132 has presentation

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb,\ pc,\ \text{class } 3\rangle$$

then L_4 is generated by $baaa$ and $ca-bab,\ cb,\ pa-bab,\ pb,\ pc \in L_4$. Adding a suitable scalar multiple of baa to c we can assume that $ca = bab$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= \pm b+\varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a'a' &= \pm\alpha^3 baaa, \\ c'b' &= \pm cb, \\ pa'-b'a'b' &= \alpha(pa-bab)+\beta pb+\gamma pc, \\ pb' &= \pm pb+\varepsilon pc, \\ pc' &= pc. \end{aligned}$$

So we can assume that $cb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

If $cb=0$ we can take $pc=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$ and if $pc\neq 0$ we can take $pa-bab=pb=0$. If $cb=pc=0$ we can take $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$ and if $pb \neq 0$ we can take $pa-bab=0$. If $cb=pb=pc=0$ then we can take $pa-bab=0$, $baaa$ or (if $p=1 \bmod 4$) $\omega baaa$.

If $cb=kbaaa$ (with $k=1,\omega$ or ω^2) then we need $\alpha^3=1$ and so we can take $pc=xbaaa$ where $0\leq x\leq (p-1)/2$. Again, if $pc\neq 0$ we can take $pa-bab=pb=0$. If $pc=0$ then we can take $pb=xbaaa$ with $0\leq x<p$. If $pb\neq 0$ we can take $pa-bab=0$, and if $pb=0$ then we can take $pa-bab=xbaaa$ where $x=0$ or x lies in a transversal for the sixth roots of unity.

So we have $(p+1+3(p+1)\gcd(p-1,3)+\gcd(p-1,4))/2$ algebras.

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab-baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab-\omega baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb-baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb,\ pc-baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \,|\, ca-bab,\ cb,\ pa-bab,\ pb,\ pc-\omega^2 baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \,|\, ca-bab,\ cb-baaa,\ pa-bab,\ pb,\ pc-xbaaa,\ \text{class } 4\rangle\ (0\leq x\leq (p-1)/2),$$

$\langle a, b, c \mid ca - bab,\ cb - baaa,\ pa - bab,\ pb - xbaaa,\ pc,\ \text{class } 4\rangle$ $(0 < x < p)$,
$\langle a, b, c \mid ca - bab,\ cb - baaa,\ pa - bab - xbaaa,\ pb,\ pc,\ \text{class } 4\rangle$ (x in a transversal for the 6^{th} roots of 1),
$\langle a, b, c \mid ca - bab,\ cb - \omega baaa,\ pa - bab,\ pb,\ pc - xbaaa,\ \text{class } 4\rangle$ $(p = 1 \bmod 3, 0 \leq x \leq (p-1)/2)$,
$\langle a, b, c \mid ca - bab,\ cb - \omega baaa,\ pa - bab,\ pb - xbaaa,\ pc,\ \text{class } 4\rangle$ $(p = 1 \bmod 3, 0 < x < p)$,
$\langle a, b, c \mid ca - bab,\ cb - \omega baaa,\ pa - bab - xbaaa,\ pb,\ pc,\ \text{class } 4\rangle$ ($p = 1 \bmod 3, x$ in a transversal for the 6^{th} roots of 1),
$\langle a, b, c \mid ca - bab,\ cb - \omega^2 baaa,\ pa - bab,\ pb,\ pc - xbaaa,\ \text{class } 4\rangle$ $(p = 1 \bmod 3, 0 \leq x \leq (p-1)/2)$,
$\langle a, b, c \mid ca - bab,\ cb - \omega^2 baaa,\ pa - bab,\ pb - xbaaa,\ pc,\ \text{class } 4\rangle$ $(p = 1 \bmod 3, 0 < x < p)$,
$\langle a, b, c \mid ca - bab,\ cb - \omega^2 baaa,\ pa - bab - xbaaa,\ pb,\ pc,\ \text{class } 4\rangle$ ($p = 1 \bmod 3, x$ in a transversal for the 6^{th} roots of 1).

41.13 Descendants of 6.133

Algebra 6.133 has $(p + 1 + 3(p + 1)\gcd(p - 1, 3) + \gcd(p - 1, 4))/2$ descendants of order p^7 and p-class 4.

Algebra 6.133 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to 6.132, and there are

$$(p + 1 + 3(p + 1)\gcd(p - 1, 3) + \gcd(p - 1, 4))/2$$

descendants of order p^7 and p-class 4.

41.14 Descendants of 6.134

Algebra 6.134 has $3p - 1 + \gcd(p - 1, 3)$ descendants of order p^7 and p-class 4.

Algebra 6.134 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle$$

and so if L is a descendant of 6.134 of order p^7 then L_4 is generated by $babb$ and $ca - bab$, cb, $pa - baa$, pb, $pc \in L_4$. Adding a suitable scalar multiple of bab to c we may assume that $cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= \alpha^{-2} c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'b'b' &= \alpha^{-2} babb, \\ c'a' - b'a'b' &= \alpha^{-1}(ca - bab) - 2\varepsilon babb, \\ pa' - b'a'a' &= \alpha(pa - bab) + \gamma pc - \gamma babb, \\ pb' &= \alpha^{-1} pb + \varepsilon pc, \\ pc' &= \alpha^{-2} pc. \end{aligned}$$

So we can take $ca - bab = 0$, though we then need to take $\varepsilon = 0$. We can take $pc = xbabb$ with $0 \leq x < p$ and we can take $pb = 0$ or $babb$. If $x \neq 1$ we can take $pa - baa = 0$. If $pb = 0$ and $pc = babb$ then we can take $pa - baa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pb = pc = babb$ then we can take $pa - baa = xbabb$ with $0 \leq x < p$. So we have $3p - 1 + \gcd(p - 1, 3)$ algebras.

Algebra 6.135 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \alpha baa,\ pb - bab,\ pc,\ \text{class } 3\rangle \ (0 \leq \alpha < p),$$

but this algebra is terminal unless $\alpha = 0$ or $\alpha = -1$. If $\alpha = 0$ it has $p^2 + 2p - 1 + \gcd(p - 1, 3) + \gcd(p - 1, 4) + \gcd(p - 1, 5)$ descendants of order p^7 and p-class 4, and if $\alpha = -1$ it has 4.

41.15.1 Case $\alpha = 0$

If $\alpha = 0$ we have

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - bab,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.135 of order p^7 then L_4 is generated by $baaa$ and $ca - bab$, cb, pa, $pb - bab$, $pc \in L_4$. Adding a suitable scalar multiple of baa to c we may assume that $ca = bab$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= \alpha^{-2} c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a'a' &= \alpha^2 baaa, \\ c'b' &= \alpha^{-3} cb, \\ pa' &= \alpha pa + \gamma pc, \\ pb' - b'a'b' &= \alpha^{-1}(pb - bab) + \varepsilon pc, \\ pc' &= \alpha^{-2} pc. \end{aligned}$$

So we can assume that $pc = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$. If $pc = 0$ then we can take $cb = 0$, $baaa$ or (if $p = 1 \bmod 5$) $\omega^k baaa$ for $k = 1, 2, 3, 4$, and if $pc \neq 0$ then we can take $cb = 0$ or $xbaaa$ where x lies in a transversal for the fourth roots of unity.

If $pc \neq 0$ we can take $pa = pb - bab = 0$.

If $cb = pc = 0$ then we can take $pa = 0$ or $baaa$. If $pa = 0$ we can take $pb - bab = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pa = baaa$ then we can take $pb - bab = xbaaa$ with $0 \leq x < p$.

If $cb = \omega^k baaa$ where $k = 0, 1, 2, 3$ or 4 and $pc = 0$ then we need $\alpha^5 = 1$, and so we can take $pa = 0$ or $xbaaa$ where x lies in a transversal for the øfth roots of unity. If $pa = 0$ then we can take $pb - bab = 0$ or $pb - bab = xbaaa$ where x lies in a transversal for the øfth roots of unity, and if $pa \neq 0$ then we can take $pb - bab = ybaaa$ with $0 \leq y < p$.

So there are $p^2 + 2p - 1 + \gcd(p - 1, 3) + \gcd(p - 1, 4) + \gcd(p - 1, 5)$ algebras.

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - bab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - bab - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - bab - \omega baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - bab - \omega^2 baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3),$$

$\langle a, b, c \mid ca - bab,\, cb,\, pa - baaa,\, pb - bab - xbaaa,\, pc, \text{ class } 4\rangle\ (0 \leq x < p)$,

$\langle a, b, c \mid ca - bab,\, cb - baaa,\, pa,\, pb - bab,\, pc, \text{ class } 4\rangle$,

$\langle a, b, c \mid ca - bab,\, cb - baaa,\, pa,\, pb - bab - xbaaa,\, pc, \text{ class } 4\rangle$ (x in a transversal for 5^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\, cb - baaa,\, pa - xbaaa,\, pb - bab - ybaaa,\, pc, \text{ class } 4\rangle$ (x in a transversal for 5^{th} roots of $1, 0 \leq y < p$),

and if $p = 1 \bmod 5$ then we have

$$\langle a, b, c \mid ca - bab,\, cb - \omega^k baaa,\, pa,\, pb - bab,\, pc, \text{ class } 4\rangle,$$

$\langle a, b, c \mid ca - bab,\, cb - \omega^k baaa,\, pa,\, pb - bab - xbaaa,\, pc, \text{ class } 4\rangle$ (x in a transversal for 5^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\, cb - \omega^k baaa,\, pa - xbaaa,\, pb - bab - ybaaa,\, pc, \text{ class } 4\rangle$ (x in a transversal for 5^{th} roots of $1, 0 \leq y < p$)

for $k = 1, 2, 3, 4$,

$$\langle a, b, c \mid ca - bab,\, cb,\, pa,\, pb - bab,\, pc - baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa,\, pb - bab,\, pc - \omega baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa,\, pb - bab,\, pc - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa,\, pb - bab,\, pc - \omega^3 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$\langle a, b, c \mid ca - bab,\, cb - xbaaa,\, pa,\, pb - bab,\, pc - baaa, \text{ class } 4\rangle$ (x in a transversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\, cb - xbaaa,\, pa,\, pb - bab,\, pc - \omega baaa, \text{ class } 4\rangle$ (x in a transversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\, cb - xbaaa,\, pa,\, pb - bab,\, pc - \omega^2 baaa, \text{ class } 4\rangle$ ($p = 1 \bmod 4$, x in a transversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\, cb - xbaaa,\, pa,\, pb - bab,\, pc - \omega^3 baaa, \text{ class } 4\rangle$ ($p = 1 \bmod 4$, x in a transversal for 4^{th} roots of 1).

41.15.2 Case $\alpha = -1$

If $\alpha = -1$ we have

$$\langle a, b, c \mid ca - bab,\, cb,\, pa + baa,\, pb - bab,\, pc, \text{ class } 3\rangle,$$

and if L is a descendant of 6.135 of order p^7 then L_4 is generated by $baab$ and $ca - bab$, cb, $pa + baa$, $pb - bab$, $pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we can assume that $ca - bab = cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= \alpha^{-2} c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'a'b' &= baab, \\ pa' + b'a'a' &= \alpha(pa + baa) + \gamma pc + \alpha^2 \varepsilon baab, \\ pb' - b'a'b' &= \alpha^{-1}(pb - bab) + \varepsilon pc, \\ pc' &= \alpha^{-2} pc. \end{aligned}$$

So we can take $pc = 0$, $baab$ or $\omega baab$. If $pc \neq 0$ we can take $pa + baa = pb - bab = 0$, and if $pc = 0$ we can take $pa + baa = 0$ and $pb - bab = 0$ or $baab$. So we have 4 algebras

$$\langle a, b, c \mid ca - bab,\, cb,\, pa + baa,\, pb - bab,\, pc, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa + baa,\, pb - bab - baab,\, pc, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa + baa,\, pb - bab,\, pc - baab, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid ca - bab,\, cb,\, pa + baa,\, pb - bab,\, pc - \omega baab, \text{ class } 4\rangle.$$

41.16 Descendants of 6.136 and 137

Algebras 6.136 and 6.137 are terminal.

41.17 Descendants of 6.138

Algebra 6.138 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa - \beta bab,\ pb - baa,\ pc,\ \text{class } 3\rangle \ (0 \leq \beta < p),$$

but this algebra is terminal unless $\beta = 0$. When $\beta = 0$ it has $p(p+1)/2$ descendants of order p^7 and p-class 4.

When $\beta = 0$ we have

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $babb$ and $ca - bab$, cb, $pa - baa$, $pb - baa$, $pc \in L_4$. Adding a suitable scalar multiple of bab to c we may assume that $cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'b'b' &= babb, \\ c'a' - b'a'b' &= \pm(ca - bab) - 2\varepsilon babb \\ pa' - b'a'a' &= \pm(pa - baa) + \gamma pc - \gamma babb, \\ pb' - b'a'a' &= \pm(pb - baa) + \varepsilon pc - \gamma babb, \\ pc' &= pc. \end{aligned}$$

So we can take $ca = bab$, though we then need $\varepsilon = 0$ and we can take $pb = baa$ though we then need $\gamma = 0$. We can take $pc = xbabb$ with $0 \leq x < p$ and we can take $pa - baa = ybabb$ with $0 \leq y \leq (p-1)/2$. So we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baa - ybabb,\ pb - baa,\ pc - xbabb,\ \text{class } 3\rangle$$

with $0 \leq x < p$, $0 \leq y \leq (p-1)/2$.

41.18 Descendants of 6.139

Algebra 6.139 has $(p+3)/2 + \gcd(p-1, 3)$ descendants of order p^7 and p-class 4.

Algebra 6.139 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.139 of order p^7 then L_4 is generated by $babb$ and $ca - bab$, cb, pa, $pb - baa$, $pc \in L_4$. Adding a suitable scalar multiple of bab to c we may assume that $cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \delta^2 c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} b'a'b'b' &= \pm\delta^3 babb, \\ c'a' - b'a'b' &= \pm\delta^2(ca - bab) \mp 2\delta\varepsilon babb \\ pa' &= \pm pa + \gamma pc, \\ pb' - b'a'a' &= \delta(pb - baa) + \varepsilon pc \mp \gamma\delta babb, \\ pc' &= \delta^2 pc. \end{aligned}$$

We can assume that $ca = bab$ and $pb = baa$, though we then need $\gamma = \varepsilon = 0$. We can take $pc = 0$ or $babb$. If $pc = 0$ we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and if $pc = babb$ we can take $pa = xbabb$ with $0 \le x \le (p-1)/2$. So we have $(p+3)/2 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - babb,\ pb - baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega babb,\ pb - baa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega^2 babb,\ pb - baa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - xbabb,\ pb - baa,\ pc - babb,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2).$$

41.19 Descendants of 6.140

Algebras 6.140 and 6.141 between them have $(3p+1)/2$ descendants of order p^7 and p-class 4.

Algebra 6.140 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.140 of order p^7 then L_4 is generated by $baaa$ (with $babb = -baaa$) and $ca - bab,\ cb,\ pa - bab,\ pb - baa,\ pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we may assume that $ca - bab = cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2, with α, δ independently equal to ± 1, and

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pa - bab) + \gamma pc + 2\alpha\delta\varepsilon baaa, \\ pb' - b'a'a' &= \delta(pb - baa) + \varepsilon pc + \alpha\gamma\delta baaa, \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ where $0 \le x \le (p-1)/2$, and we can take $pa - bab = pb - baa = 0$ provided $x^2 \ne 2$. If $x^2 = 2$ we can take $pa - bab = 0$, $pb - baa = ybaaa$ where $0 \le y \le (p-1)/2$. So we have $(p+1)/2$ algebras

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$

and a further $(p-1)/2$ algebras if 2 is a square:

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - baa - xbaaa,\ pc - \sqrt{2}baaa,\ \text{class } 4\rangle\ (0 < x \le (p-1)/2).$$

41.20 Descendants of 6.141

Algebra 6.141 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega bab,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

so this case is almost identical to 6.140. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2, with α, δ independently equal to ± 1, and

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pa - \omega bab) + \gamma pc + 2\alpha\delta\varepsilon baaa, \\ pb' - b'a'a' &= \delta(pb - baa) + \varepsilon pc + \omega^{-1}\alpha\gamma\delta baaa, \\ pc' &= pc. \end{aligned}$$

We have $(p+1)/2$ descendants of order p^7 and p-class 4.

$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega bab,\ pb - baa,\ pc - xbaaa,\ \text{class } 4\rangle$ $(0 \leq x \leq (p-1)/2)$,

and a further $(p-1)/2$ algebras if $2\omega^{-1}$ is a square:

$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega bab,\ pb - baa - xbaaa,\ pc - \sqrt{2\omega^{-1}}baaa,\ \text{class } 4\rangle$ $(0 < x \leq (p-1)/2)$.

41.21 Descendants of 6.142

Algebra 6.142 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa - \beta bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle \ (0 \leq \beta < p),$$

and is terminal unless $\beta = 0$. If $\beta = 0$ it has $p(p+1)/2$ descendants of order p^7 and p-class 4.

If $\beta = 0$ we have

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

which is almost identical to 6.138 (when $\beta = 0$), and so we have $p(p+1)/2$ algebras

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - baa - ybabb,\ pb - \omega baa,\ pc - xbabb,\ \text{class } 3\rangle$$

with $0 \leq x < p$, $0 \leq y \leq (p-1)/2$.

41.22 Descendants of 6.143

Algebra 6.143 has descendants $(p+3)/2 + \gcd(p-1, 3)$ of order p^7 and p-class 4.

Algebra 6.143 has presentation

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to 6.139 so we have $(p+3)/2 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \mid ca - bab,\ cb,\ pa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - babb,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega babb,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - \omega^2 babb,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - bab,\ cb,\ pa - xbabb,\ pb - \omega baa,\ pc - babb,\ \text{class } 3\rangle \ (0 \leq x \leq (p-1)/2).$$

41.23 Descendants of 6.144

Algebras 6.144 and 6.145 between them have $(3p+1)/2$ descendants of order p^7 and p-class 4.

Algebra 6.144 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.140 and 6.141 and we have $(p+1)/2$ descendants of order p^7 and p-class 4.

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - bab,\ pb - \omega baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

and a further $(p-1)/2$ algebras if 2ω is a square:

$$\langle a, b, c \,|\, ca-bab,\ cb,\ pa-bab,\ pb-\omega baa-xbaaa,\ pc-\sqrt{2\omega^3}baaa,\ \text{class } 4\rangle\ (0 < x \leq (p-1)/2).$$

41.24 Descendants of 6.145

Algebra 6.145 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.140, 6.141 and 6.144 and we have $(p+1)/2$ descendants of order p^7 and p-class 4.

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \omega bab,\ pb - \omega baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

and a further $(p-1)/2$ algebras if 2 is a square:

$$\langle a, b, c \,|\, ca-bab,\ cb,\ pa-\omega bab,\ pb-\omega baa-xbaaa,\ pc-\sqrt{2\omega^2}baaa,\ \text{class } 4\rangle\ (0 < x \leq (p-1)/2).$$

41.25 Descendants of 6.146

Algebra 6.146 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - \lambda baa,\ pc - bab,\ \text{class } 3\rangle\ (0 \leq \lambda < p),$$

but the algebra is terminal unless $\lambda = 0$. If $\lambda = 0$ we have $2p^2 + 3p$ descendants of order p^7 and p-class 4.

If $\lambda = 0$ we have

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb,\ pc - bab,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and $ca - bab$, cb, pa, pb, $pc - bab \in L_4$. Adding a suitable scalar multiple of baa to c we can assume that $ca = bab$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= a + \beta b, \\ b' &= \delta b, \\ c' &= \delta^2 c \end{aligned}$$

modulo L_2, with α, δ independently equal to ± 1, and

$$\begin{aligned} b'a'a'a' &= \delta baaa, \\ c'b' &= \delta^3 cb \\ pa' &= pa + \beta pb, \\ pb' &= \delta pb, \\ pc' - b'a'b' &= \delta^2 (pc - bab). \end{aligned}$$

We can assume that $cb = 0$, $baaa$ or $\omega baaa$ and that $pb = xbaaa$ with $0 \le x < p$. If $pb \ne 0$ we can assume that $pa = 0$.

If $cb = 0$ then we can take $pc - bab = 0$ or $baaa$. If $cb = pb = pc - bab = 0$ we can take $pa = 0$ or $baaa$. If $cb = pb = 0$, $pc - bab = baaa$ then we can take $pa = xbaaa$ with $0 \le x < p$.

If $cb = baaa$ or $\omega baaa$ then we need $\delta^2 = 1$, and so we can take $pc - bab = ybaaa$ with $0 \le y \le (p-1)/2$. If $pb = pc - bab = 0$ we can take $pa = zbaaa$ with $0 \le z \le (p-1)/2$, and if $pb = 0$, $pc - bab \ne 0$ then we can take $pa = zbaaa$ with $0 \le z < p$.

So we have $2p^2 + 3p$ algebras.

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - xbaaa,\ pc - bab,\ \text{class } 4\rangle\ (0 \le x < p),$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb - xbaaa,\ pc - bab - baaa,\ \text{class } 4\rangle\ (0 \le x < p),$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - baaa,\ pb,\ pc - bab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - xbaaa,\ pb,\ pc - bab - baaa,\ \text{class } 4\rangle\ (0 < x < p),$$

$$\langle a, b, c \,|\, ca-bab,\ cb-baaa,\ pa,\ pb-xbaaa,\ pc-bab-ybaaa,\ \text{class } 4\rangle\ (0 \le x < p,\ 0 \le y \le (p-1)/2),$$

$$\langle a, b, c \,|\, ca - bab,\ cb - baaa,\ pa - zbaaa,\ pb,\ pc - bab,\ \text{class } 4\rangle\ (0 < z \le (p-1)/2),$$

$$\langle a, b, c \,|\, ca-bab,\ cb-baaa,\ pa-zbaaa,\ pb,\ pc-bab-ybaaa,\ \text{class } 4\rangle\ (0 < z < p,\ 0 < y \le (p-1)/2),$$

$$\langle a, b, c \,|\, ca-bab,\ cb-\omega baaa,\ pa,\ pb-xbaaa,\ pc-bab-ybaaa,\ \text{class } 4\rangle\ (0 \le x < p,\ 0 \le y \le (p-1)/2),$$

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baaa,\ pa - zbaaa,\ pb,\ pc - bab,\ \text{class } 4\rangle\ (0 < z \le (p-1)/2),$$

$$\langle a, b, c \,|\, ca-bab,\ cb-\omega baaa,\ pa-zbaaa,\ pb,\ pc-bab-ybaaa,\ \text{class } 4\rangle\ (0 < z < p,\ 0 < y \le (p-1)/2).$$

41.26 Descendants of 6.147

Algebra 6.147 is terminal.

41.27 Descendants of 6.148

Algebra 6.148 has presentation

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa - \alpha bab,\ pb,\ pc - baa,\ \text{class } 3\rangle$$

where $\alpha = 0$, 1, ω or (when $p = 1 \bmod 4$) ω^2 or ω^3, but the algebra is terminal unless $\alpha = 0$. If $\alpha = 0$ we have $p^3 + p^2 + p + (p+2)\gcd(p-1, 3) + \gcd(p-1, 5)$ descendants of order p^7 and p-class 4.

When $\alpha = 0$ we have

$$\langle a, b, c \,|\, ca - bab,\ cb,\ pa,\ pb,\ pc - baa,\ \text{class } 3\rangle$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $babb$ and $ca - bab$, cb, pa, pb, $pc - baa \in L_4$. Adding a suitable scalar multiple of bab to c we can assume that $cb = 0$. If a', b', c' generate L and satisfy the same commutator relations as a, b, c then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^2 b, \\ c' &= \alpha^4 c \end{aligned}$$

modulo L_2, with α, δ independently equal to ± 1, and

$$\begin{aligned} b'a'b'b' &= \alpha^7 babb, \\ c'a' - b'a'b' &= \alpha^5(ca - bab), \\ pa' &= \alpha pa, \\ pb' &= \alpha^2 pb, \\ pc' - b'a'a' &= \alpha^4(pc - baa). \end{aligned}$$

We can take $ca - bab = 0$, $babb$ or $\omega babb$.

If $ca - bab = 0$ we can take $pa = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 3$) $\omega^2 babb$, $\omega^3 babb$, $\omega^4 babb$ or $\omega^5 babb$, and if $ca - bab = babb$ or $\omega babb$ then we can take $pa = xbabb$ with $0 \leq x < p$.

If $ca - bab = pa = 0$ we can take $pb = 0$, $babb$ or (if $p = 1 \bmod 5$) $\omega babb$, $\omega^2 babb$, $\omega^3 babb$ or $\omega^4 babb$. If $pb = 0$ we can take $pc - baa = 0$ or $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and if $pb \neq 0$ we can take $pc = 0$ or $pc = xbabb$ where x lies in a transversal for the øfth roots of unity.

If $ca - bab = 0$ and $pa = kbabb$ where $k = 1, \omega, \omega^2, \omega^3, \omega^4$ or ω^5 then we need $\alpha^6 = 1$ so we can take $pb = 0$ or $pb = xbabb$ where x lies in a transversal for the sixth roots of unity. If $pb = 0$ then we can take $pc - baa = xbabb$ where $0 \leq x \leq (p-1)/2$, and if $pb \neq 0$ we can take $pc - baa = ybabb$ with $0 \leq y < p$.

If $ca - bab = babb$ or $\omega babb$ then we need $\alpha^2 = 1$. We have $pa = xbabb$ and we can take $pb = ybabb$ where $0 \leq y \leq (p-1)/2$. If $y = 0$ we can take $pc - baa = zbabb$ where $0 \leq z \leq (p-1)/2$, and if $y \neq 0$ we can take $pc = zbabb$ where $0 \leq z < p$.

So we have $p^3 + p^2 + p + (p+2)\gcd(p-1,3) + \gcd(p-1,5)$ algebras.

41.28 Descendants of 6.149

This algebra is terminal.

41.29 Descendants of 6.150

The descendants of 6.150 of order p^7 and p-class 4 dvide up into 8 cases, and the number of descendants in each case is given as follows:

$4 + 2\gcd(p-1,3)$
$p + 2 + \gcd(p-1,3) + \gcd(p-1,4)$
$(p + 1 + (p+3)\gcd(p-1,3) + \gcd(p-1,4))/2$
$3 + \gcd(p-1,3)(p + 3 + \gcd(p-1,4))/4$
$2 + \gcd(p-1,3)(p + 7 - \gcd(p-1,4))/4$
$3 + \gcd(p-1,3)$
$3 + (p+1)\gcd(p-1,3)$
$(5p - 7 + (p^2-5)\gcd(p-1,3) - \gcd(p-1,4))/2$

The grand total of descendants is

$$4p+14+(\frac{1}{2}p^2+2p+\frac{13}{2})\gcd(p-1,3)+\gcd(p-1,4).$$

Algebra 6.150 has presentation

$$\langle a,b,c \mid ca-baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 3\rangle$$

and so if L is a descendant of 6.150 of order p^7 then the commutator structure of L is the same as one of 7.108 ˘ 7.115 from the list of nilpotent Lie algbebras over $\mathbb{Z}_p$. So we can assume that one of the following sets of commutator relations holds.

$$\begin{aligned}
baaa &= baab = ca-baa = cb = 0,\\
baaa &= baab = ca-baa-babb = cb = 0,\\
baaa &= baab = babb,\ ca-baa = cb = 0,\\
babb &= -baaa,\ baab = ca-baa = cb = 0,\\
babb &= -\omega baaa,\ baab = ca-baa = cb = 0,\\
baaa &= babb = ca-baa = cb = 0,\\
babb &= baab,\ baaa = ca-baa = cb = 0,\\
baaa &= baab,\ babb = xbaaa,\ ca-baa = cb = 0\ (2 \le x < p).
\end{aligned}$$

For each of these cases we obtain a generator d for L_4 (d equals one of $baaa$, $baab$, $babb$) and we write

$$\begin{pmatrix} pa \\ pb \\ pc \end{pmatrix} = Ad$$

where A is a 3×1 matrix over $\mathbb{Z}_p$. In each case the isomorphism classes of algebras are given by the orbits of the matrices A under the action speciøed below.

41.29.1 Case 1

Let $baaa = baab = ca-baa = cb = 0$. Then L_4 is generated by $babb$ and the action on A is

$$A \to \alpha^{-1}\delta^{-3}\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \alpha\delta \end{pmatrix} A.$$

We can assume that $pc = 0$, $babb$ or $\omega babb$, and if $pc \neq 0$ we can assume that $pa = pb = 0$. If $pc = 0$ then we can assume that $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$ and that $pb = 0$ or $babb$.

So we have $4+2\gcd(p-1,3)$ algebras.

$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-babb,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-\omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-\omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-babb,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-\omega babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa-\omega^2 babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baaa,\ baab,\ ca-baa,\ cb,\ pa,\ pb,\ pc-\omega babb,\ \text{class } 4\rangle.$$

41.29.2 Case 2

Let $baaa = baab = ca - baa - babb = cb = 0$. Then L_4 is generated by $babb$ and the action on A is

$$A \to \delta^{-5} \begin{pmatrix} \delta^2 & 0 & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \delta^3 \end{pmatrix} A.$$

We can assume that $pc = 0$, $babb$ or $\omega babb$, and if $pc \neq 0$ we can assume that $pa = pb = 0$. If $pc = 0$ then we can assume that $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pa = 0$ we can take $pb = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 4$) $\omega babb$ or $\omega^2 babb$ and if $pa \neq 0$ we can take $pb = 0$ or $pb = xbabb$ where x lies in a transversal for the cube roots of unity.

So we have $p + 2 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb - \omega babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb - \omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb - \omega^3 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaa,\ baab,\ ca{-}baa{-}babb,\ cb,\ pa{-}babb,\ pb{-}xbabb,\ pc,\ \text{class } 4\rangle\ (x \text{ in a transversal for } 3^{th} \text{ roots of } 1),$$

$$\langle a, b, c \mid baaa,\ baab,\ ca{-}baa{-}babb,\ cb,\ pa{-}\omega babb,\ pb{-}xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for } 3^{th} \text{ roots}$$

$$\langle a, b, c \mid baaa,\ baab,\ ca{-}baa{-}babb,\ cb,\ pa{-}\omega^2 babb,\ pb{-}xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for } 3^{th} \text{ root}$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baaa,\ baab,\ ca - baa - babb,\ cb,\ pa,\ pb,\ pc - \omega babb,\ \text{class } 4\rangle.$$

41.29.3 Case 3

If $baaa = baab = babb$, $ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & -\gamma \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to -\alpha^{-4} \begin{pmatrix} 0 & \alpha & \gamma \\ \alpha & 0 & -\gamma \\ 0 & 0 & \alpha^2 \end{pmatrix} A.$$

We can take $pc = 0$ or $baaa$ or (if $p = 1 \bmod 4$) $\omega baaa$.

If $pc = 0$ and $pa = xbaaa$, $pb = ybaaa$ then we have

$$(x, y) \sim (\alpha^{-3}x, \alpha^{-3}y) \sim (-\alpha^{-3}y, -\alpha^{-3}x).$$

So if $p = 1 \bmod 3$ we can take $(x, y) = (0, 0)$, $(0, 1)$, $(0, \omega)$, $(0, \omega^2)$, $(1, y^3)$ with $(1, y^3) \sim (1, y^{-3})$, $(\omega, \omega y^3)$ with $(\omega, \omega y^3) \sim (\omega, \omega y^{-3})$, $(\omega^2, \omega^2 y^3)$ with $(\omega^2, \omega^2 y^3) \sim (\omega^2, \omega^2 y^{-3})$,

$(1, \omega y^3)$, $(1, \omega^2 y^3)$, $(\omega, \omega^2 y^3)$. ($\frac{1}{2}(p-1)+p+6$ algebras.) And if $p = 2 \bmod 3$ we can take $(x, y) = (0,0)$, $(0,1)$, or $(1, y)$ with $(1,y) \sim (1, y^{-1})$. ($\frac{1}{2}(p+1)+2$ algebras.)

If $p = 1 \bmod 4$ and $pc = baaa$ or $\omega baaa$ then we are restricted to

$$A \to \begin{pmatrix} \pm 1 & 0 & \gamma \\ 0 & \pm 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} A$$

and

$$A \to \begin{pmatrix} 0 & \alpha & \gamma \\ \alpha & 0 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} A \text{ where } \alpha^2 = -1.$$

Using transformations of the ørst kind we can take $pa = 0$, $pb = xbaaa$ where $0 \leq x \leq (p-1)/2$. If we let $a' = \alpha b + \gamma c$, $b' = \alpha a - \gamma c$ where $\alpha^2 = -1$, and if we choose γ so that $pa' = 0$ then $pb' = \alpha x b'a'a'a'$. If follows that we can take $pa = 0$ and $pb = 0$ or $pb = xbaaa$ where x lies in a transversal for the fourth roots of unity. ($(p+3)/2$ algebras.)

If $p = 3 \bmod 4$ and $pc = baaa$ then we are restricted to

$$A \to \begin{pmatrix} \pm 1 & 0 & \gamma \\ 0 & \pm 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix} A$$

so we cand take $pa = 0$, $pb = xbaaa$ where $0 \leq x \leq (p-1)/2$. ($(p+1)/2$ algebras.)

So we have $(p + 1 + (p+3)\gcd(p-1,3) + \gcd(p-1,4))/2$ algebras.

$\langle a, b, c \mid baab - baaa,\ babb - baaa,\ ca - baa,\ cb,\ pa - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle$ (x, y speciøed above),

$\langle a, b, c \mid baab - baaa,\ babb - baaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle$,

$\langle a, b, c \mid baab - baaa,\ babb - baaa,\ ca - baa,\ cb,\ pa,\ pb - xbaaa,\ pc - baaa,\ \text{class } 4\rangle$ (x in a transversal for 4^{th} roots of 1),

$\langle a, b, c \mid baab - baaa,\ babb - baaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle$ ($p = 1 \bmod 4$),

$\langle a, b, c \mid baab - baaa,\ babb - baaa,\ ca - baa,\ cb,\ pa,\ pb - xbaaa,\ pc - \omega baaa,\ \text{class } 4\rangle$ ($p = 1 \bmod 4, x$ in a transversal for 4

41.29.4 Case 4

If $babb + baaa = baab = ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \pm\alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to \alpha^{-4} \begin{pmatrix} 0 & \pm\alpha & \gamma \\ \alpha & 0 & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A.$$

We can take $pc = 0$, $baaa$ or $\omega baaa$, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ and $p = 2 \bmod 3$ then we can take $pa = xbaaa$, $pb = ybaaa$ where $(x, y) = (0,0)$, $(0,1)$ or $(1, y)$ where $y \neq 0$ and $y \sim \pm y$, $\pm y^{-1}$ ($(p - 1 + \gcd(p-1, 4))/4$ algebras).

If $pc = 0$ and $p = 1 \bmod 3$ then we can take $(x, y) = (0,0)$, $(0,1)$, $(0, \omega)$, $(0, \omega^2)$, $(1, z)$, $(\omega, \omega z)$, $(\omega^2, \omega^2 z)$, $(1, \omega t)$, $(1, \omega^2 t)$, $(\omega, \omega^2 t)$ where z runs over a set of representatives for the equivaleces of the set $\{y^3 \mid y \neq 0\}$ under the equivalence relation with equivalence classes $\{y^3, -y^3, y^{-3}, -y^{-3}\}$ and t runs over a set of representatives for the equivalence classes of the set $\{y^3 \mid y \neq 0\}$ under the equivalence relation with equivalence classes $\{y^3, -y^3\}$.

So we have $3 + \gcd(p-1,3)(p+3+\gcd(p-1,4))/4$ algebras.

$$\langle a,b,c \mid babb + baaa,\ baab,\ ca - baa,\ cb,\ pa - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle,$$

with x, y as speciøed above, and

$$\langle a,b,c \mid babb + baaa,\ baab,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid babb + baaa,\ baab,\ ca - baa,\ cb,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle.$$

41.29.5 Case 5

If $babb + \omega baaa = baab = ca - baa = cb = 0$ then L_4 is generated by $baaa$ and the action on A is

$$A \to \alpha^{-4}\begin{pmatrix} \pm\alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to \omega^{-2}\alpha^{-4}\begin{pmatrix} 0 & \pm\alpha & \gamma \\ \omega\alpha & 0 & \varepsilon \\ 0 & 0 & \omega\alpha^2 \end{pmatrix} A.$$

We can take $pc = 0$ or $baaa$, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ and $p = 1 \bmod 3$ we can take $pa = xbaaa$, $pb = ybaaa$ where $(x,y) = (0,0)$, $(0,1)$, $(0,\omega)$, $(0,\omega^2)$, or $(1,z)$, $(1,\omega z)$, $(\omega,\omega z)$ where z lies in a set of representatives for the equivalence classes $\{\pm y^3\}$ with $y \neq 0$, or $(1,\omega^2 t)$ where t lies in a set of representatives for the equivalence classes $\{\pm y^3, \pm\frac{1}{\omega^3 y^3}\}$ with $y \neq 0$, or (ω,t), $(\omega^2,\omega t)$ where t lies in a set of representatives for the equivalence classes $\{\pm y^3, \pm\frac{\omega^3}{y^3}\}$ with $y \neq 0$. ($4 + 3\frac{p-1}{6} + (p+11-3\gcd(p-1,4))/4$ pairs (x,y).)

And if $pc = 0$ and $p = 2 \bmod 3$ we can take $pa = xbaaa$, $pb = ybaaa$ where $(x,y) = (0,0)$, $(0,1)$ or $(1,y)$ where y lies in a set of representatives for the equivalence classes for non-zero elements of $\mathbb{Z}_p$ consisting of the sets $\{\pm y, \pm\frac{\omega}{y}\}$. (The number of these equivalence classes is $(p+3-\gcd(p-1,4))/4$.)

So we have $2 + \gcd(p-1,3)(p+7-\gcd(p-1,4))/4$ algebras

$$\langle a,b,c \mid babb + \omega baaa,\ baab,\ ca - baa,\ cb,\ pa - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle,$$

with x, y as speciøed above, and

$$\langle a,b,c \mid babb + \omega baaa,\ baab,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

Combining the number of algebras in Case 4 and Case 5 we have $5+\gcd(p-1,3)(p+5)/2$ algebras.

41.29.6 Case 6

If $baaa = babb = ca - baa = cb = 0$ then L_4 is generated by $baab$ and the action on A is

$$A \to \alpha^{-2}\delta^{-2}\begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \alpha\delta \end{pmatrix} A$$

and

$$A \to \alpha^{-2}\delta^{-2} \begin{pmatrix} 0 & \alpha & \gamma \\ \delta & 0 & \varepsilon \\ 0 & 0 & -\alpha\delta \end{pmatrix} A.$$

We can take $pc = 0$ or $baab$, and if $pc = baab$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pa = xbaab$, $pb = ybaab$ where $(x, y) = (0, 0)$, $(0, 1)$, $(1, 1)$ or (if $p = 1 \bmod 3$) $(1, \omega)$ or $(1, \omega^2)$.

So there are $3 + \gcd(p - 1, 3)$ algebras

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa,\ pb - baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa - baab,\ pb - baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa - baab,\ pb - \omega baab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa - baab,\ pb - \omega^2 baab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, baaa,\ babb,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baab,\ \text{class } 4\rangle.$$

41.29.7 Case 7

If $babb = baab$, $baaa = ca - baa = cb = 0$ then L_4 is generated by $baab$ and the action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A.$$

We can take $pc = 0$, $baab$ or $\omega baab$, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pa = 0$, $baab$ or (if $p = 1 \bmod 3$) $\omega baab$ or $\omega^2 baab$. If $pa = pc = 0$ we can take $pb = 0$, $baab$ or (if $p = 1 \bmod 3$) $\omega baab$ or $\omega^2 baab$, and if $pa = kbaab$ (with $k = 1, \omega$ or ω^2) then we can take $pb = xbaab$ where $0 \leq x < p$.

So we have $3 + (p + 1)\gcd(p - 1, 3)$ algebras.

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb - baab,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb - \omega baab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb - \omega^2 baab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa - baab,\ pb - xbaab,\ pc,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, babb-baab,\ baaa,\ ca-baa,\ cb,\ pa-\omega baab,\ pb-xbaab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, babb-baab,\ baaa,\ ca-baa,\ cb,\ pa-\omega^2 baab,\ pb-xbaab,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baab,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, babb - baab,\ baaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc - \omega baab,\ \text{class } 4\rangle.$$

41.29.8 Case 8

If $baaa = baab$, $babb = xbaaa$, $ca - baa = cb = 0$ where $2 \leq x < p$ then L_4 is generated by $baaa$. (If we set $x = 0$ we have Case 7, and if we set $x = 1$ we have Case 3.) The action on A is

$$A \to \alpha^{-4} \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ 0 & 0 & \alpha^2 \end{pmatrix} A$$

and

$$A \to x^{-2}\alpha^{-4} \begin{pmatrix} 0 & \alpha & \gamma \\ x\alpha & 0 & \varepsilon \\ 0 & 0 & -x\alpha^2 \end{pmatrix} A.$$

We can take $pc = 0$, $baaa$ or (if $-x$ is a square) $\omega baaa$, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $p = 2 \bmod 3$ and $pc = 0$ we can take $pa = ybaaa$, $pb = zbaaa$ where $(y, z) = (0, 0)$, $(0, 1)$ or $(1, z)$ where z lies in a set of representatives for the equivalence classes $\{z, \frac{x}{z}\}$ of non-zero elements in $\mathbb{Z}_p$. (The number of equivalence classes is $(p - 1)/2$ if x is not a square, and $(p + 1)/2$ if x is a square.)

If $p = 1 \bmod 3$ and $pc = 0$ we can take $pa = ybaaa$, $pb = zbaaa$ where $(y, z) = (0, 0)$, $(0, 1)$, $(0, \omega)$, $(0, \omega^2)$, or (y, z) where (y, z) lies in a set of representatives for the pairs of non-zero elements in $\mathbb{Z}_p$ under the equivalence relation given by $(y, z) \sim (\frac{y}{\alpha^3}, \frac{z}{\alpha^3}) \sim (\frac{z}{x^2\alpha^3}, \frac{y}{x\alpha^3})$.

So the number of algebras with $pc = 0$ is $1 + (p + 1)\gcd(p - 1, 3)/2$ algebras if x is not a square, and $1 + (p + 3)\gcd(p - 1, 3)/2$ if x is a square. To get the total number of algebras (including $pc \neq 0$) we have to add 1 if $-x$ is not a square, and add 2 if $-x$ is a square.

The total number of algebras over all allowable values of x is $(5p - 7 + (p^2 - 5)\gcd(p - 1, 3) - \gcd(p - 1, 4))/2$.

$\langle a, b, c \mid baab{-}baaa,\ babb{-}xbaaa,\ ca{-}baa,\ cb,\ pa{-}ybaaa,\ pb{-}zbaaa,\ pc,\ \text{class } 4\rangle$ $(2 \leq x < p)$

with y, z as specified above,

$\langle a, b, c \mid baab - baaa,\ babb - xbaaa,\ ca - baa,\ cb,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle$ $(2 \leq x < p)$,

$\langle a, b, c \mid baab{-}baaa,\ babb{-}xbaaa,\ ca{-}baa,\ cb,\ pa,\ pb,\ pc{-}\omega baaa,\ \text{class } 4\rangle$ $(2 \leq x < p,\ -x$ a square$)$.

41.30 Descendants of 6.151

Algebra 6.151 has $p^2 + p + 2 + (p + 1)\gcd(p - 1, 3)$ descendants of order p^7 and p-class 4.

Algebra 6.151 has presentation

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.151 of order p^7 then L_4 is generated by $babb$ and $ca - baa$, cb, $pa - baa$, pb, $pc \in L_4$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \alpha^{-2}babb, \\ c'a' - b'a'a' &= \alpha(ca - baa), \\ pa' - b'a'a' &= \alpha(pa - baa) + \gamma pc, \\ pb' &= \alpha^{-1}pb + \varepsilon pc, \\ pc' &= pc. \end{aligned}$$

We can take $pc = 0$, $babb$ or $\omega babb$.

If $pc = 0$ we can take $pb = 0$ or $babb$.

If $pb = pc = 0$ then we can take $ca - baa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pb = pc = ca - baa = 0$ we can take $pa - baa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and if $pb = pc = 0$, $ca - baa = kbabb$ (with $k = 1, \omega$ or ω^2) then we can take $pa - baa = xbabb$ with $(0 \leq x < p)$. $(1 + (p+1)\gcd(p-1, 3)$ algebras.)

If $pb = babb$ and $pc = 0$ then we can take $ca - baa = xbabb$, $pa - baa = ybabb$ with $0 \leq x, y < p$. (p^2 algebras.)

If $pc = babb$ or $\omega babb$ then we can take $pa - baa = pb = 0$, $ca - baa = xbabb$ with $0 \leq x \leq (p-1)/2$. ($p + 1$ algebras.

So we have a total of $p^2 + p + 2 + (p+1)\gcd(p-1, 3)$ algebras.

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa - babb,\ cb,\ pa - baa - xbabb,\ pb,\ pc,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa - \omega babb,\ cb,\ pa - baa - xbabb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa - \omega^2 babb,\ cb,\ pa - baa - xbabb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa - xbabb,\ cb,\ pa - baa - ybabb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (0 \leq x, y < p),$$

$$\langle a, b, c \mid ca - baa - xbabb,\ cb,\ pa - baa,\ pb,\ pc - babb,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid ca - baa - xbabb,\ cb,\ pa - baa,\ pb,\ pc - \omega babb,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

41.31 Descendants of 6.152

Algebra 6.152 has $\frac{1}{2}p^2 + p + \frac{3}{2} + \gcd(p-1, 3) + \frac{(p+1)}{2}\gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Algebra 6.152 has presentation

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.152 of order p^7 then L_4 is generated by $baaa$ and $ca - baa$, cb, $pa - bab$, pb, $pc \in L_4$. Adding a suitable scalar multiple of baa to c we may take $ca - baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= \pm \alpha c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= \pm\alpha^3 baaa, \\ c'b' &= \alpha cb, \\ pa' - b'a'b' &= \alpha(pa - bab) + \gamma pc, \\ pb' &= \pm pb + \varepsilon pc, \\ pc' &= \pm\alpha pc. \end{aligned}$$

We can take $pc = 0$, $baaa$ or $\omega baaa$.

If $pc = 0$ we can take $cb = 0$, $baaa$ or (if $p = 1 \bmod 4$) $\omega baaa$.

If $cb = pc = 0$ we can take $pa - bab = 0$, $baaa$ or (if $p = 1 \bmod 4$) $\omega baaa$. If $cb = pa - bab = pc = 0$ then we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $cb = pc = 0$ and $pa - bab \neq 0$ then we can take $pb = 0$ or $pb = xbaaa$ where x is in a transversal for the fourth roots of unity. ($1 + \gcd(p-1,3) + (p - 1 + \gcd(p-1,4))/2$ algebras.)

If $cb = baaa$ or $\omega baaa$ and $pc = 0$ then we can take $pa - bab = xbaaa$ and $pb = 0$ or $pb = ybaaa$ with y in a transversal for the fourth roots of unity. ($p(p-1)/2 + p\gcd(p-1,4)/2$ algebras.)

If $pc = baaa$ or $\omega baaa$ then we can take $pa - baa = pb = 0$ and $cb = xbaaa$ where $0 \leq x \leq (p-1)/2$. ($p + 1$ algebras.)

So we have $\frac{1}{2}p^2 + p + \frac{3}{2} + \gcd(p-1,3) + \frac{(p+1)}{2}\gcd(p-1,4)$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab - baaa,\ pb - xbaaa,\ pc,\ \text{class } 4\rangle\ (x = 0 \text{ or } x \text{ in a transversal for } 4^{th} \text{ roots of } 1),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - bab - \omega baaa,\ pb - xbaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4, x = 0 \text{ or } x \text{ in a transversal for } 4^{th} \text{ roots of}$$

$$\langle a, b, c \mid ca - baa,\ cb - baaa,\ pa - bab - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle$$

with $0 \leq x < p$ and $y = 0$ or y in a transversal for the fourth roots of unity,

$$\langle a, b, c \mid ca - baa,\ cb - \omega baaa,\ pa - bab - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4)$$

with $0 \leq x < p$ and $y = 0$ or y in a transversal for the fourth roots of unity,

$$\langle a, b, c \mid ca - baa,\ cb - xbaaa,\ pa - bab,\ pb,\ pc - baaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid ca - baa,\ cb - xbaaa,\ pa - bab,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

41.32 Descendants of 6.153

Algebra 6.153 has $p(p+1)/2$ descendants of order p^7 and p-class 4.

Algebra 6.153 has presentation

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.153 of order p^7 then L_4 is generated by $baaa$ and $ca - baa$, cb, $pa - baa - bab$, pb, $pc \in L_4$. Adding a suitable scalar multiple of baa to c, we can take $cb = 0$, and adding a suitable scalar multiple of c to a we can take $ca = baa$. If a', b', c'

generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'a' - b'a'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \gamma c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - b'a'b' &= \pm(pa - baa - bab) + \gamma pc + 2\gamma baaa, \\ pb' &= \pm pb + \gamma pc, \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ with $0 \leq x < p$, and if $pc \neq 0$ we can take $pb = 0$ and $pa - baa - bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. If $pc = 0$ we can take $pa - baa - bab = 0$ and $pb = ybaaa$ with $0 \leq y \leq (p-1)/2$.

So we have $p(p+1)/2$ algebras.

$$\langle a, b, c \,|\, ca-baa,\ cb,\ pa-baa-bab-ybaaa,\ pb,\ pc-xbaaa,\ \text{class } 4\rangle\ (0 < x < p,\ 0 \leq y \leq (p-1)/2),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle\ (0 \leq y \leq (p-1)/2).$$

41.33 Descendants of 6.154

Algebra 6.154 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

which similarly to 6.152 gives $\frac{1}{2}p^2 + p + \frac{3}{2} + \gcd(p-1, 3) + \frac{(p+1)}{2}\gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, ca-baa,\ cb,\ pa-\omega bab-baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (x = 0 \text{ or } x \text{ in a transversal for } 4^{th} \text{ roots of } 1),$$

$$\langle a, b, c \,|\, ca-baa,\ cb,\ pa-\omega bab-\omega baaa,\ pb-xbaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4, x = 0 \text{ or } x \text{ in a transversal for } 4^{th} \text{ roots o}$$

$$\langle a, b, c \,|\, ca - baa,\ cb - baaa,\ pa - \omega bab - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle$$

with $0 \leq x < p$ and $y = 0$ or y in a transversal for the fourth roots of unity,

$$\langle a, b, c \,|\, ca - baa,\ cb - \omega baaa,\ pa - \omega bab - xbaaa,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4)$$

with $0 \leq x < p$ and $y = 0$ or y in a transversal for the fourth roots of unity,

$$\langle a, b, c \,|\, ca - baa,\ cb - xbaaa,\ pa - \omega bab,\ pb,\ pc - baaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, ca - baa,\ cb - xbaaa,\ pa - \omega bab,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

41.34 Descendants of 6.155

Algebra 6.155 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \omega bab,\ pb,\ pc,\ \text{class } 3\rangle,$$

which is similar to 6.153. So we obtain $p(p+1)/2$ algebras.

$$\langle a, b, c \,|\, ca{-}baa,\ cb,\ pa{-}baa{-}\omega bab{-}ybaaa,\ pb,\ pc{-}xbaaa,\ \text{class } 4\rangle\ (0 < x < p,\ 0 \leq y \leq (p{-}1)/2),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle\ (0 \leq y \leq (p-1)/2).$$

41.35 Descendants of 6.156

Algebra 6.156 has p descendants of order p^7 and p-class 4.

Algebra 6.156 (using the alternative formulation from p6.tex) has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - bab,\ pb + baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and $ca - baa,\ cb,\ pa - bab,\ pb + baa,\ pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we can take $ca - baa = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 and if in addition $c'a' - b'a'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 with $\alpha, \delta = \pm 1$ and

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pa - bab) + \gamma pc, \\ pb' + b'a'a' &= \delta(pb + baa) + \varepsilon pc + \varepsilon baaa, \\ pc' &= \alpha\delta pc \end{aligned}$$

or

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \delta a + \varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 with $\alpha, \delta = \pm 1$ and

$$\begin{aligned} b'a'a'a' &= -\alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pb + baa) + \gamma pc + \gamma baaa, \\ pb' + b'a'a' &= \delta(pa - bab) + \varepsilon pc, \\ pc' &= \alpha\delta pc + \alpha\delta baaa \end{aligned}$$

We can take $pc = xbaaa$ where x lies in a set of representatives for the equivalence classes $\{x, -1-x\}$ of elements of $\mathbb{Z}_p$. This restricts us to transformations of the ørst type. If $x \neq 0, -1$ we can take $pa - bab = pb - baa = 0$. We pick $x = 0$ from the equivalence class $\{0, -1\}$, and if $x = 0$ we can take $pb - baa = 0$ and $pa - bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. (We can take the representatives for the equivalence classes $\{x, -1-x\}$ to be $0, 1, 2, \ldots, (p-1)/2$.) So we have p algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - bab,\ pb + baa,\ pc - xbaaa,\ \text{class } 4\rangle\ (1 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - bab - xbaaa,\ pb + baa,\ pc,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

Algebra 6.157 has $2p-1$ descendants of order p^7 and p-class 4.

Algebra 6.157 has presentation

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-bab,\ pb+\omega baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.157 of order p^7 then L_4 is generated by $baaa$ and $ca-baa$, cb, $pa-bab$, $pb+\omega baa$, $pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we can take $ca-baa=cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4 and if in addition $c'a'-b'a'a'=c'b'=0$ then

$$\begin{aligned} a' &= \alpha a+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 with $\alpha,\delta=\pm 1$ and

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa'-b'a'b' &= \alpha(pa-bab)+\gamma pc, \\ pb'+\omega b'a'a' &= \delta(pb+\omega baa)+\varepsilon pc+\omega\varepsilon baaa, \\ pc' &= \alpha\delta pc. \end{aligned}$$

We can take $pc=xbaaa$ with $0\le x<p$, and if $x\ne 0,-\omega$ we can take $pa-bab=pb+\omega baa=0$. If $x=0$ we cna take $pa-bab=ybaaa$ with $0\le y\le (p-1)/2$, $pb+\omega bab=0$, and if $x=-\omega$ we can take $pa-bab=0$, $pb+\omega bab=ybaaa$ with $0\le y\le (p-1)/2$.

So we have $2p-1$ algebras

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-bab,\ pb+\omega baa,\ pc-xbaaa,\ \text{class } 3\rangle \ (0\le x<p),$$

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-bab-xbaaa,\ pb+\omega baa,\ pc,\ \text{class } 3\rangle \ (0<x\le (p-1)/2),$$

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-bab,\ pb+\omega baa-xbaaa,\ pc+\omega baaa,\ \text{class } 3\rangle \ (0<x\le (p-1)/2).$$

41.37 Descendants of 6.158

Algebra 6.158 has p descendants of order p^7 and p-class 4.

Algebra 6.158 has presentation

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-\omega bab,\ pb+\omega baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and $ca-baa$, cb, $pa-bab$, $pb+baa$, $pc\in L_4$. Adding suitable scalar multiples of baa and bab to c we can take $ca-baa=cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4 and if in addition $c'a'-b'a'a'=c'b'=0$ then

$$\begin{aligned} a' &= \alpha a+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 with $\alpha,\delta=\pm 1$ and

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa'-b'a'b' &= \alpha(pa-bab)+\gamma pc, \\ pb'+b'a'a' &= \delta(pb+baa)+\varepsilon pc+\omega\varepsilon baaa, \\ pc' &= \alpha\delta pc \end{aligned}$$

or

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \delta a + \varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

modulo L_2 with $\alpha, \delta = \pm 1$ and

$$\begin{aligned} b'a'a'a' &= -\alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pb + baa) + \gamma pc + \omega\gamma baaa, \\ pb' + b'a'a' &= \delta(pa - bab) + \varepsilon pc, \\ pc' &= \alpha\delta pc + \omega\alpha\delta baaa. \end{aligned}$$

We can take $pc = xbaaa$ where x lies in a set of representatives for the equivalence classes $\{x, -\omega - x\}$ of elements of $\mathbb{Z}_p$. This restricts us to transformations of the ørst type. If $x \neq 0, -\omega$ we can take $pa - bab = pb - baa = 0$. We pick $x = 0$ from the equivalence class $\{0, -\omega\}$, and if $x = 0$ we can take $pb - baa = 0$ and $pa - bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. So we have p algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab,\ pb + \omega baa,\ pc - xbaaa,\ \text{class } 3\rangle\ (1 \leq x < p,\ x \leq (-\omega - x) \bmod p),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \omega bab - xbaaa,\ pb + \omega baa,\ pc,\ \text{class } 3\rangle\ (0 \leq x \leq (p-1)/2).$$

41.38 Descendants 6.159

Algebra 6.159 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab,\ pb - baa,\ pc,\ \text{class } 3\rangle\ (0 \leq \mu < p),$$

but this algebra is terminal unless $\mu = 0$. If $\mu = 0$ we have $(p^3 + p^2)/2$ descendants of order p^7 and p-class 4.

When $\mu = 0$ we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa,\ pb - baa,\ pc,\ \text{class } 3\rangle$$

and if L is a descendant of this agebra of order p^7 then L_4 is generated by $babb$ and $ca - baa$, cb, $pa - baa$, $pb - baa$, $pc \in L_4$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 and if in addition $c'b' = 0$ then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= babb, \\ c'a' - b'a'a' &= \pm(ca - baa), \\ pa' - b'a'a' &= \pm(pa - baa) + \gamma pc, \\ pb' - b'a'a' &= \pm(pb - baa) + \varepsilon pc, \\ pc' &= pc. \end{aligned}$$

We can take $ca - baa = xbabb$ with $0 \leq x \leq (p-1)/2$ and we can take $pc = ybabb$ with $0 \leq y < p$. If $pc \neq 0$ we can take $pa - baa = pb - baa = 0$. If $pc = 0$ then we can take $pa - baa = ybabb$, $pb = baa - zbabb$ with (x, y, z) giving an algebra isomorphic to $(-x, -y, -z)$. So we have $(p^3 + p^2)/2$ algebras.

$$\langle a, b, c \,|\, ca{-}baa{-}xbabb,\ cb,\ pa{-}baa,\ pb{-}baa,\ pc{-}ybabb,\ \text{class } 4\rangle\ (0 \leq x \leq (p{-}1)/2,\ 0 < y < p),$$

$$\langle a, b, c \,|\, ca - baa - xbabb,\ cb,\ pa - baa - ybabb,\ pb - baa - zbabb,\ pc,\ \text{class } 4\rangle$$

where $0 \leq x, y, z < p$, and either $0 < x \leq (p-1)/2$ or $x = 0$ and $0 < y \leq (p-1)/2$ or $x = y = 0$ and $0 \leq z \leq (p-1)/2$, or $x = y = z = 0$.

41.39 Descendants of 6.160

Algebra 6.160 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle\ (0 \leq \mu < p)$$

but this algebra is terminal unless $\mu = 0$. If $\mu = 0$ we have $(p^3 + p^2)/2$ descendants of order p^7 and p-class 4.

When $\mu = 0$ we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle$$

which is almost identical to the case $\mu = 0$ in algebra 6.159. As in the descendants of 6.159 we have $(p^3 + p^2)/2$ algebras.

$$\langle a, b, c \,|\, ca{-}baa{-}xbabb,\ cb,\ pa{-}baa,\ pb{-}\omega baa,\ pc{-}ybabb,\ \text{class } 4\rangle\ (0 \leq x \leq (p{-}1)/2,\ 0 < y < p),$$

$$\langle a, b, c \,|\, ca - baa - xbabb,\ cb,\ pa - baa - ybabb,\ pb - \omega baa - zbabb,\ pc,\ \text{class } 4\rangle$$

where $0 \leq x, y, z < p$, and either $0 < x \leq (p-1)/2$ or $x = 0$ and $0 < y \leq (p-1)/2$ or $x = y = 0$ and $0 \leq z \leq (p-1)/2$, or $x = y = z = 0$.

41.40 Descendants of 6.160A

Algebra 6.160A has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa,\ pb - \xi bab,\ pc,\ \text{class } 3\rangle\ (\xi \neq 0,\ \xi \sim \xi^{-1})$$

but this algebra is terminal unless $\xi = -1$. If $\xi = -1$ it has $(p+3)/2$ descendants of order p^7 and p-class 4.

When $\xi = -1$ we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa,\ pb + bab,\ pc,\ \text{class } 3\rangle$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $baab$ and $ca - baa$, cb, $pa - baa$, $pb + bab$, $pc \in L_4$. Adding suitable scalar multiples of baa and bab to c we can take $ca - baa = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 and if in addition $c'a' - b'a'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'b' &= baab, \\ pa' - b'a'a' &= \alpha(pa - baa) + \gamma pc - \gamma baab, \\ pb' + b'a'b' &= \alpha^{-1}(pb + bab) + \varepsilon pc + 2\varepsilon baab, \\ pc' &= pc \end{aligned}$$

or

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \alpha^{-1} a + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= -baab, \\ pa' - b'a'a' &= \alpha(pb + bab) + \gamma pc + 2\gamma baab, \\ pb' + b'a'b' &= \alpha^{-1}(pa - baa) + \varepsilon pc - \varepsilon baab, \\ pc' &= pc + baab. \end{aligned}$$

We can take $pc = xbaab$ where x lies in a set of representatives for the equivalence classes $\{x, -1-x\}$ of elements of $\mathbb{Z}_p$. (We can take the representatives to be $0, 1, \ldots (p-1)/2$.) If $x \neq 1$ we can take $pa - baa = pb + bab = 0$, and if $x = 1$ we can take $pb + bab = 0$, $pa - baa = 0$ or $baab$. So we have $(p+3)/2$ algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa,\ pb + bab,\ pc - xbaab,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - baab,\ pb + bab,\ pc - baab,\ \text{class } 4\rangle.$$

41.41 Descendants of 6.161 and 6.162

These two algebras have presentations

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb - \xi bab,\ pc,\ \text{class } 3\rangle\ (\xi \neq 0),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \xi bab,\ pc,\ \text{class } 3\rangle\ (\xi \neq 0),$$

but both algebras are terminal unless $\xi = -1$. When $\xi = -1$ they each have $2p - 1$ descendants of order p^7 and p-class 4.

When $\xi = -1$ we have

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + bab,\ pc,\ \text{class } 3\rangle,$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb + bab,\ pc,\ \text{class } 3\rangle.$$

If L is a descendant of order p^7 of either of these algebras then L_4 is generated by $baaa$ and $ca - baa$, cb, $pa - baa - kbab$, $pb + bab$, $pc \in L_4$ ($k = 1$ or ω). Adding suitable scalar multiples of baa and bab to c we can take $ca - baa = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 and if in addition $c'a' - b'a'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned}
b'a'a'a' &= baaa, \\
pa' - b'a'a' - kbab &= \pm(pa - baa - kbab) + \gamma pc + (k^{-1}\gamma + \varepsilon)baaa, \\
pb' + b'a'b' &= \pm(pb + bab) + \varepsilon pc - 2k^{-1}\varepsilon baaa, \\
pc' &= pc.
\end{aligned}$$

We can take $pc = xbaaa$ with $0 \leq x < p$, and if $x \neq -k^{-1}, 2k^{-1}$ we can take $pa - baa - kbab = pb + bab = 0$. If $x = -k^{-1}$ or $2k^{-1}$ we can take $pa - baa - kbab = 0$, $pb + bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. So we have $2p - 1$ algbras for each value of k ($4p - 2$ algebras in all)

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + bab,\ pc - xbaaa,\ \text{class } 4\rangle \ (0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + bab - xbaaa,\ pc + baaa,\ \text{class } 4\rangle \ (0 < x < (p-1)/2),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + bab - xbaaa,\ pc - 2baaa,\ \text{class } 4\rangle \ (0 < x \leq (p-1)/2),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb + bab,\ pc - xbaaa,\ \text{class } 4\rangle \ (0 \leq x < p),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb + bab - xbaaa,\ pc + \omega^{-1}baaa,\ \text{class } 4\rangle \ (0 < x < (p-1)/2),$$

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \omega bab,\ pb + bab - xbaaa,\ pc - 2\omega^{-1}baaa,\ \text{class } 4\rangle \ (0 < x \leq (p-1)/2).$$

41.42 Descendants of 6.163 ˘ 6.167

Algebras 6.163 ˘ 6.167 give a classiøcation of algebra of order p^6 with presentations

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - \lambda baa - \mu bab,\ pb + \nu baa + \xi bab,\ pc,\ \text{class } 3\rangle$$

with $\lambda, \mu, \nu, \xi \neq 0$. Most of these algebras are terminal, and we need a slightly diœerent classiøcation of these algebras from that given in p6.tex, so as to classify the capable ones. It turns out that $\frac{5}{2}p - \frac{9}{2} + \frac{1}{2}\gcd(p-1, 4)$ of these algebras are capable, and that they have a total of $\frac{1}{2}p^3 + 2p^2 - 5p + \frac{1}{2} + \frac{p}{2}\gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Let L have the presentation above, and suppose that a', b', c' generate L and satisfy similar relations, but with (possibly) diœerent λ, μ, ν, ξ. Then

$$\begin{aligned}
a' &= \alpha a + \gamma c, \\
b' &= \delta b + \varepsilon c, \\
c' &= \alpha\delta c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned}
pa' &= \frac{\lambda}{\alpha\delta}b'a'a' + \frac{\mu}{\delta^2}b'a'b', \\
pb' &= -\frac{\nu}{\alpha^2}b'a'a' - \frac{\xi}{\alpha\delta}b'a'b'
\end{aligned}$$

or

$$\begin{aligned}
a' &= \alpha b + \gamma c, \\
b' &= \delta a + \varepsilon c, \\
c' &= \alpha\delta c
\end{aligned}$$

modulo L_2 and

$$\begin{aligned} pa' &= \frac{\xi}{\alpha\delta}b'a'a' + \frac{\nu}{\delta^2}b'a'b', \\ pb' &= -\frac{\mu}{\alpha^2}b'a'a' - \frac{\lambda}{\alpha\delta}b'a'b'. \end{aligned}$$

So we can take $\lambda = 1$ and $\mu = 1$ or ω. Given these values of λ, μ it turns out that the algebra is terminal unless $\xi = 1$ or $\xi = \mu\nu$.

First consider the case when $\lambda = \xi = 1$, and consider generators a', b', c' which also have $\lambda = \xi = 1$. Then we have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1}b + \varepsilon c, \\ c' &= c \end{aligned} \tag{(*)}$$

modulo L_2 and

$$\begin{aligned} pa' &= b'a'a' + \alpha^2\mu b'a'b', \\ pb' &= -\frac{\nu}{\alpha^2}b'a'a' - b'a'b' \end{aligned}$$

or

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \alpha^{-1}a + \varepsilon c, \\ c' &= c \end{aligned} \tag{(**)}$$

modulo L_2 and

$$\begin{aligned} pa' &= b'a'a' + \alpha^2\nu b'a'b', \\ pb' &= -\frac{\mu}{\alpha^2}b'a'a' - b'a'b'. \end{aligned}$$

Taking $\mu = 1$ or ω and expressing ν as x^2 or $x^2\omega$ we see that we can take (μ, ν) to be $(1, x^2)$, $(1, x^2\omega)$, (ω, x^2) or $(\omega, x^2\omega)$. Transfomations of type (*) with $\alpha = \pm 1$ preserve all these values of (μ, ν). Transformations of type (**) with $\alpha = \pm x^{-1}$ preserve $(1, x^2)$ and $(\omega, x^2\omega)$ and interchange $(1, x^2\omega)$, (ω, x^2). So we get a complete set of isomorphism classes of algebras with $\lambda = \xi = 1$ by taking (μ, ν) to be $(1, x^2)$, $(\omega, x^2\omega)$ or $(1, x^2\omega)$. In each case we can allow transformations of type (*) with $\alpha = \pm 1$, and in the ørst two cases we can also allow transformations of type (**) with $\alpha = \pm x^{-1}$. So we have $3(p-1)/2$ algebras of this type.

41.42.1 Case 1

Now let L be a descendant of order p^7 of

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2baa + bab,\ pc,\ \text{class } 3\rangle.$$

Then L_4 is generated by $baaa$ and $ca - baa$, cb, $pa - baa - bab$, $pb + x^2baa + bab$, $pc \in L_4$. If $x^2 \neq 1$ we can add suitable scalar multiples of baa and bab to c so that $ca - baa = cb = 0$. So suppose for the moment that $x^2 \neq 1$. Then under transformations of type (*) with $\alpha = \pm 1$ we have

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - b'a'b' &= \alpha(pa - baa - bab) + \gamma pc + (\gamma + \varepsilon)baaa, \\ pb' + x^2b'a'a' + b'a'b' &= \alpha(pb + x^2baa + bab) + \varepsilon pc + (x^2\varepsilon - 2\varepsilon - x^2\gamma)baaa, \\ pc' &= pc \end{aligned}$$

and under transformations of type (**) with $\alpha = \pm x^{-1}$ we have

$$\begin{aligned}
b'a'a'a' &= -baaa, \\
pa' - b'a'a' - b'a'b' &= \alpha(pb + x^2baa + bab) + \gamma pc + (x^2\gamma - 2\gamma - \varepsilon)baaa, \\
pb' + x^2b'a'a' + b'a'b' &= \alpha^{-1}(pa - baa - bab) + \varepsilon pc + (x^2\gamma + \varepsilon)baaa, \\
pc' &= pc + (x^2 - 1)baaa.
\end{aligned}$$

We can take $pc = ybaaa$ where y lies in a set of representatives for the equivalence classes $\{y, 1 - y - x^2\}$ of elements of $\mathbb{Z}_p$. If $y^2 + (x^2 - 1)y + 2(x^2 - 1) \neq 0$ we can take $pa - baa - bab = pb + x^2baa + bab = 0$. And if $y^2 + (x^2 - 1)y + 2(x^2 - 1) = 0$ we can take $pa - baa - bab = 0$ and $pb + x^2baa + bab = zbaaa$ where $0 \leq z \leq (p - 1)/2$. Note that if $x^2 = 9$ then $y^2 + (x^2 - 1)y + 2(x^2 - 1)$ has only one root -4, and that for other values of x^2 there are either 0 or 2 roots. But in each equivalence class, either both elements are roots, or neither is a root (and in the equivalence class $\{\frac{1-x^2}{2}\}$, the element $\frac{1-x^2}{2}$ is not a root unless $x^2 = 9$, in which case $\frac{1-x^2}{2} = -4$ and the roots are $-4, -4$). So there are $(p+1)/2$ equivalence classes, and the number of descendants is $(p+1)/2$ if $y^2 + (x^2 - 1)y + 2(x^2 - 1)$ has no roots, and p if it has roots.

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2baa + bab,\ pc - ybaaa,\ \text{class } 4\rangle\ (x^2 \neq 1)$$

where y is a set of representatives for the equivalence classes $\{y, 1 - y - x^2\}$, and if α is a root of $y^2 + (x^2 - 1)y + 2(x^2 - 1)$

$$\langle a, b, c \mid ca{-}baa,\ cb,\ pa{-}baa{-}bab,\ pb{+}x^2baa{+}bab{-}zbaaa,\ pc{-}\alpha baaa,\ \text{class } 4\rangle\ (x^2 \neq 1,\ 0 \leq z \leq (p{-}1)/2)$$

Now consider the case when $x^2 = 1$. Let L be a descendant of order p^7 of

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + baa + bab,\ pc,\ \text{class } 3\rangle.$$

Then L_4 is generated by $baaa$ and $ca - baa, cb, pa - baa - bab, pb + baa + bab, pc \in L_4$. Adding a suitable scalar multiple of Then under transformations of type (*) with $\alpha = \pm 1$ we have

$$\begin{aligned}
b'a'a'a' &= baaa, \\
c'a' - b'a'a' &= \alpha(ca - baa) + (\gamma - \varepsilon)baaa \\
pa' - b'a'a' - b'a'b' &= \alpha(pa - baa - bab) + \gamma pc + (\gamma + \varepsilon)baaa, \\
pb' + b'a'a' + b'a'b' &= \alpha(pb + baa + bab) + \varepsilon pc - (\gamma + \varepsilon)baaa, \\
pc' &= pc
\end{aligned}$$

so we can take $ca - baa = 0$ as well as $cb = 0$. If we restrict to transforamtions of type (**) with $\alpha = \pm 1$ and $c'a' - b'a'a' = c'b' = 0$ then we need $\gamma = \varepsilon$ and we have

$$\begin{aligned}
b'a'a'a' &= -baaa, \\
pa' - b'a'a' - b'a'b' &= \alpha(pb + baa + bab) + \gamma pc + -2\gamma baaa, \\
pb' + b'a'a' + b'a'b' &= \alpha(pa - baa - bab) + \gamma pc + 2\gamma baaa, \\
pc' &= pc.
\end{aligned}$$

So we can take $pc = xbaaa$ where $0 \leq x \leq (p - 1)/2$ and we can take $pa - baa - bab = 0$, $pb + baa + bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. So the number of descendants is $(p+1)^2/4$.

$$\langle a, b, c \mid ca{-}baa,\ cb,\ pa{-}baa{-}bab,\ pb{+}baa{+}bab{-}ybaaa,\ pc{-}xbaaa,\ \text{class } 4\rangle\ (0 \leq x, y \leq (p{-}1)/2).$$

41.42.2 Case 2

Next let L be a descendant of order p^7 of

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-baa-\omega bab,\ pb+x^2\omega baa+bab,\ pc,\ \text{class }3\rangle.$$

If $\omega^2x^2\neq 1$ we can add suitable scalar multiples of baa and bab to c so that $ca-baa=cb=0$. So suppose for the moment that $\omega^2x^2\neq 1$. Then under transformations of type (*) with $\alpha=\pm 1$ we have

$$\begin{aligned}
b'a'a'a' &= baaa,\\
pa'-b'a'a'-\omega b'a'b' &= \alpha(pa-baa-\omega bab)+\gamma pc+(\omega^{-1}\gamma+\varepsilon)baaa,\\
pb'+x^2\omega b'a'a'+b'a'b' &= \alpha(pb+x^2\omega baa+bab)+\varepsilon pc+(\omega x^2\varepsilon-2\omega^{-1}\varepsilon-x^2\gamma)baaa,\\
pc' &= pc
\end{aligned}$$

and under transformations of type (**) with $\alpha=\pm x^{-1}$ we have

$$\begin{aligned}
b'a'a'a' &= -baaa,\\
pa'-b'a'a'-\omega b'a'b' &= \alpha(pb+x^2\omega baa+bab)+\gamma pc+(\omega x^2\gamma-2\omega^{-1}\gamma-\varepsilon)baaa,\\
pb'+x^2\omega b'a'a'+b'a'b' &= \alpha^{-1}(pa-baa-\omega bab)+\varepsilon pc+(x^2\gamma+\omega^{-1}\varepsilon)baaa,\\
pc' &= pc+(\omega x^2-\omega^{-1})baaa.
\end{aligned}$$

We can take $pc=ybaaa$ where y lies in a set of representatives for the equivalence classes $\{y,\omega^{-1}-y-\omega x^2\}$ of elements of $\mathbb{Z}_p$. If $y^2+\frac{\omega^2x^2-1}{\omega}y+2\frac{\omega^2x^2-1}{\omega^2}\neq 0$ we can take $pa-baa-\omega bab=pb+x^2\omega baa+bab=0$. And if $y^2+\frac{\omega^2x^2-1}{\omega}y+2\frac{\omega^2x^2-1}{\omega^2}=0$ we can take $pa-baa-\omega bab=0$ and $pb+x^2\omega baa+bab=zbaaa$ where $0\leq z\leq(p-1)/2$. Note that this quadratic in y has roots if and only if $y^2+(x^2-1)y+2(x^2-1)$ has roots. So, as in the case above, the number of descendants is $(p+1)/2$ if $y^2+(x^2-1)y+2(x^2-1)$ has no roots, and p if it has roots.

$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-baa-\omega bab,\ pb+x^2\omega baa+bab,\ pc-ybaaa,\ \text{class }4\rangle$ $(\omega^2x^2\neq 1)$

where y is a set of representatives for the equivalence classes $\{y,\omega^{-1}-y-\omega x^2\}$, and if α is a root of $y^2+\frac{\omega^2x^2-1}{\omega}y+2\frac{\omega^2x^2-1}{\omega^2}$

$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-baa-\omega bab,\ pb+x^2\omega baa+bab-zbaaa,\ pc-\alpha baaa,\ \text{class }4\rangle$ $(\omega^2x^2\neq 1,\ 0\leq z\leq(p-1)/2)$

Now consider the case when $\omega^2x^2=1$. We have

$$\langle a,b,c \,|\, ca-baa,\ cb,\ pa-baa-\omega bab,\ pb+\omega^{-1}baa+bab,\ pc,\ \text{class }3\rangle.$$

If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$, and as above we can take $ca-baa=cb=0$ provided we restrict ourselves to transformations of type (*) above with $\gamma=\omega\varepsilon$ which give

$$\begin{aligned}
b'a'a'a' &= baaa,\\
pa'-b'a'a'-\omega bab &= \pm(pa-baa-\omega bab)+\omega\varepsilon pc+2\varepsilon baaa,\\
pb'+\omega^{-1}b'a'a'+b'a'b' &= \pm(pb+\omega^{-1}baa+bab)+\varepsilon pc-2\omega^{-1}\varepsilon baaa,\\
pc' &= pc
\end{aligned}$$

and to transformations of type (**) above with $\gamma=\omega\varepsilon$ which give

$$\begin{aligned}
b'a'a'a' &= -baaa,\\
pa'-b'a'a'-\omega bab &= \pm\omega(pb+\omega^{-1}baa+bab)+\omega\varepsilon pc-2\varepsilon baaa,\\
pb'+\omega^{-1}b'a'a'+b'a'b' &= \pm\omega^{-1}(pa-baa-\omega bab)+\varepsilon pc+2\omega^{-1}\varepsilon baaa,\\
pc' &= pc.
\end{aligned}$$

So we can take $pc = xbaaa$ where x is a set of representatives for the equivalence classes $\{x, -x\}$, and provided we do not take $x = -2\omega^{-1}$ as a representative, we can take $pa - baa - bab = 0$, $pb + baa + bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. So the number of descendants is $(p+1)^2/4$.

$$\langle a, b, c \mid ca-baa,\ cb,\ pa-baa-bab,\ pb+baa+bab-ybaaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (0 \leq y \leq (p-1)/2),$$

with $0 \leq x \leq (p-1)/2$ except that we replace $x = -2\omega^{-1}$ by $x = 2\omega^{-1}$ if necessary.

41.42.3 Case 3

Next consider descendants of

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2\omega baa + bab,\ pc,\ \text{class } 3\rangle.$$

(Recall that in solving the isomorphism problem we restrict ourselves to transformations of type (*) with $\alpha = \pm 1$.) If L is a descendant of this algebra then L_4 is generated by $baaa$ and $ca - baa$, cb, $pa - baa - bab$, $pb + x^2\omega baa + bab$, $pc \in L_4$. Adding suitable scalar multiple of baa and bab to c we may take $ca - baa = cb = 0$. Then applying a transformation of type (*), with $\alpha = \pm 1$ we get

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - b'a'b' &= \pm(pa - baa - bab) + \gamma pc + (\gamma + \varepsilon)baaa, \\ pb' + x^2\omega b'a'a' + b'a'b' &= \pm(pb + x^2\omega baa + bab) + \varepsilon pc - (x^2\omega\gamma - x^2\omega\varepsilon + 2\varepsilon)baaa, \\ pc' &= pc. \end{aligned}$$

We can take $pc = ybaaa$, and if $y^2 + (x^2\omega - 1)y + 2(x^2\omega - 1) \neq 0$ we can take $pa - baa - bab = pb + x^2\omega baa + bab = 0$. If $y^2 + (x^2\omega - 1)y + 2(x^2\omega - 1) = 0$ then we can take $pa - baa = bab = 0$ and $pb + x^2\omega baa + bab = zbaaa$ with $0 \leq z \leq (p-1)/2$. So we have p descendants

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2\omega baa + bab,\ pc - ybaaa,\ \text{class } 4\rangle$$

with $0 \leq y < p$ for each non-zero x^2, and a further $(p-1)/2$ descendants

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2\omega baa + bab - zbaaa,\ pc - ybaaa,\ \text{class } 4\rangle$$

with $0 < z \leq (p-1)/2$ for each value of y which is a root of $y^2 + (x^2\omega - 1)y + 2(x^2\omega - 1)$.

Note that this polynomial has a root if $(x^2\omega - 1)(x^2\omega - 9)$ is a square, and in cases 1 and 2 above the extra $(p-1)/2$ algebras arose when $(x^2\omega - 1)(x^2\omega - 9)$ is a square. Note also that $y^2 + (x^2\omega - 1)y + 2(x^2\omega - 1)$ cannot have repeated roots. So we get $(p-1)$ extra algebras in Case 3 when $(x^2\omega - 1)(x^2\omega - 9)$ is a square, and $(p-1)/2$ extra algebras in each of cases 1 and 2 when $(x^2\omega - 1)(x^2\omega - 9)$ is a square. Now, as we run over all possible $x \neq 0$, $(x-1)(x-9)$ is a square for $(p-1)/2$ values of x, so the total number of non-zero values of x^2 for which $(x^2\omega - 1)(x^2\omega - 9)$ is a square, added to to the total number of non-zero values of $x^2\omega$ for which $(x^2\omega - 1)(x^2\omega - 9)$ is a square, is $(p-1)/2$. Note that this includes the value $x^2 = 1$, which we dealt with separately in Case 1, as was the equivalent value $\omega^2 x^2 = 1$ in Case 2. So the total number of times that extra algebras arise is $(p-3)/2$.

Thus the total number of algebras in cases 1,2,3 is $2p^2 - \frac{5}{2}p + \frac{1}{2}$. (See orb6.163.)

And now consider algebras

$$\langle a, b, c \mid ca - baa,\ cb,\ pa - baa - \mu bab,\ pb + \nu baa + \mu\nu bab,\ pc,\ \text{class } 3\rangle$$

with $\mu = 1, \omega$, and $\mu\nu \neq 1$. We can take (μ, ν) to be one of the following (all with $x \neq 0$): $(1, x^2)$ with $x^2 \neq 1$, $(1, \omega x^2)$, (ω, x^2) or $(\omega, \omega^{-1}x^2)$ with $x^2 \neq 1$. These are all left unchanged by transformations of type (*) with $\alpha = \pm 1$. Transformations of type

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \delta a + \varepsilon c, \\ c' &= \alpha\delta c \end{aligned}$$

map $(1, x^2)$ to $(1, x^{-2})$, map $(1, \omega x^2)$ to (ω, x^2), and map $(\omega, \omega^{-1}x^2)$ to $(\omega, \omega^{-1}x^{-2})$. So we consider

- pairs $(1, x^2)$ where x^2 lies in a set of representatives for the equivalence classes $\{x^2, x^{-2}\}$, though we exclude $x^2 = 1$, and we need to deal with $x^2 = -1$ separately (if it arises).
- pairs $(\omega, \omega^{-1}x^2)$ where x^2 lies in a set of representatives for the equivalence classes $\{x^2, x^{-2}\}$, though we exclude $x^2 = 1$, and we need to deal with $x^2 = -1$ separately (if it arises).
- pairs $(1, \omega x^2)$.

We restrict ourselves to transformations of type (*) with $\alpha = \pm 1$ in all these cases, except for the cases of the ørst two types where $x^2 = -1$.

The number of algebras here is $p - 3 + \gcd(p - 1, 4)/2$.

41.42.4 Case 4

Consider algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2baa + x^2bab,\ pc, \text{ class } 3\rangle$$

where $x^2 \neq 0, \pm 1$ (and x^2 is taken from a set of representatives for the equivalence classes $\{x^2, x^{-2}\}$. If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$, and we can also take $ca - baa = 0$ if we take $\gamma = \varepsilon$ in (*). We then have

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - b'a'b' &= \pm(pa - baa - bab) + \gamma pc + 2\gamma baaa, \\ pb' + x^2b'a'a' + x^2b'a'b' &= \pm(pb + x^2baa + x^2bab) + \gamma pc - 2x^2\gamma baaa, \\ pc' &= pc. \end{aligned}$$

So we can take $pc = ybaaa$ with $0 \leq y < p$. If $y \neq -2$ we can take $pa - baa - bab = 0$ and $pb + x^2baa + x^2bab = zbaaa$ with $0 \leq z \leq (p-1)/2$. If $y = -2$ we can take $pa - baa - bab = zbaaa$ with $0 \leq z \leq (p-1)/2$ and $pb + x^2baa + x^2bab = 0$. So for each $x^2 \neq \pm 1$ we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb + x^2baa + x^2bab - zbaaa,\ pc - ybaaa, \text{ class } 4\rangle\ (y \neq -2),$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab - zbaaa,\ pb + x^2baa + x^2bab,\ pc + 2baaa, \text{ class } 4\rangle,$$

with $0 \leq y < p$, $0 \leq z \leq (p-1)/2$.

41.42.5 Case 5 ($p = 1 \bmod 4$)

Consider algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb - baa - bab,\ pc,\ \text{class } 3\rangle.$$

If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$. As in Case 4 we can take $ca - baa = cb = 0$. We can have transformations of type (*) with $\alpha = \pm$ and $\gamma = \varepsilon$ which give

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - b'a'b' &= \pm(pa - baa - bab) + \gamma pc + 2\gamma baaa, \\ pb' - b'a'a' - b'a'b' &= \pm(pb - baa - bab) + \gamma pc + 2\gamma baaa, \\ pc' &= pc, \end{aligned}$$

and transformations of the form

$$\begin{aligned} a' &= \alpha b + \gamma c, \\ b' &= \alpha a + \gamma c, \\ c' &= -c \end{aligned}$$

with $\alpha^2 = -1$ (so that $-\alpha^{-1} = \alpha$) give

$$\begin{aligned} b'a'a'a' &= -baaa, \\ pa' - b'a'a' - b'a'b' &= \alpha(pb - baa - bab) + \gamma pc + 2\gamma baaa, \\ pb' - b'a'a' - b'a'b' &= \alpha(pa - baa - bab) + \gamma pc + 2\gamma baaa, \\ pc' &= -pc, \end{aligned}$$

So we can take $pc = ybaaa$ with $0 \leq y < p$. If $y \neq -2$ we can take $pa - baa - bab = 0$ and $pb - baa - bab = zbaaa$ with $z = 0$ or z in a transversal for the fourth roots of unity

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb - baa - bab - zbaaa,\ pc - ybaaa,\ \text{class } 4\rangle\ (y \neq -2),$$

and if $y = -2$ we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab - zbaaa,\ pb - baa - bab - tbaaa,\ pc + 2baaa,\ \text{class } 4\rangle,$$

where $z = 0$ and $0 \leq t \leq (p-1)/2$, or where z, t are non-zero and lie in a set of representatives for the equivalence classes $\{(\pm z, \pm t), (\pm\alpha z, \pm\alpha t)\}$ where $\alpha^2 = -1$ ($(p^2 + 3)/4$ representatives.)

The total number of descendants is $\frac{1}{2}p^2 + \frac{1}{2}p$.

41.42.6 Case 6

Consider algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \omega bab,\ pb + \omega^{-1}x^2 baa + x^2 bab,\ pc,\ \text{class } 3\rangle$$

where $x^2 \neq 0, \pm 1$ (and x^2 is taken from a set of representatives for the equivalence classes $\{x^2, x^{-2}\}$. If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$, and we can also take $ca - baa = 0$ if we take $\gamma = \omega\varepsilon$ in (*). We then have

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - b'a'a' - \omega b'a'b' &= \pm(pa - baa - \omega bab) + \omega\varepsilon pc + 2\varepsilon baaa, \\ pb' + \omega^{-1}x^2 b'a'a' + x^2 b'a'b' &= \pm(pb + \omega^{-1}x^2 baa + x^2 bab) + \varepsilon pc - 2x^2\omega^{-1}\varepsilon baaa, \\ pc' &= pc. \end{aligned}$$

We can take $pc = ybaaa$ with $0 \leq y < p$. Provided $y \neq -2\omega^{-1}$ we can take $pa - baa - \omega bab = 0$ and $pb + \omega^{-1}x^2baa + x^2bab = zbaaa$ with $0 \leq z \leq (p-1)/2$. And if $y = -2\omega^{-1}$ we can take $pb + \omega^{-1}x^2baa + x^2bab = 0$ and $pa - baa - \omega bab = zbaaa$ with $0 \leq z \leq (p-1)/2$.

So we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, ca{-}baa,\ cb,\ pa{-}baa{-}\omega bab,\ pb{+}\omega^{-1}x^2baa{+}x^2bab{-}zbaaa,\ pc{-}ybaaa,\ \text{class } 4\rangle\ (y \neq -2\omega^{-1}),$$

$$\langle a, b, c \,|\, ca{-}baa,\ cb,\ pa{-}baa{-}\omega bab{-}zbaaa,\ pb{+}\omega^{-1}x^2baa{+}x^2bab,\ pc{+}2\omega^{-1}baaa,\ \text{class } 4\rangle,$$

with $0 \leq y < p$, $0 \leq z \leq (p-1)/2$.

41.42.7 Case 7 ($p = 1 \bmod 4$)

Consider algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \omega bab,\ pb - \omega^{-1}baa - bab,\ pc,\ \text{class } 3\rangle.$$

As above, if L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and we can take $ca - baa = cb = 0$. Under transformations of type (*) we have

$$\begin{aligned}
b'a'a'a' &= baaa,\\
pa' - b'a'a' - \omega b'a'b' &= \pm(pa - baa - \omega bab) + \omega\varepsilon pc + 2\varepsilon baaa,\\
pb' - \omega^{-1}b'a'a' - b'a'b' &= \pm(pb - \omega^{-1}baa - bab) + \varepsilon pc + 2\omega^{-1}\varepsilon baaa,\\
pc' &= pc,
\end{aligned}$$

and under transformations of type

$$\begin{aligned}
a' &= \omega\alpha b + \omega\varepsilon c,\\
b' &= \omega^{-1}\alpha a + \varepsilon c,\\
c' &= -c
\end{aligned}$$

with $\alpha^2 = -1$ give

$$\begin{aligned}
b'a'a'a' &= -baaa,\\
pa' - b'a'a' - \omega b'a'b' &= \omega\alpha(pb - \omega^{-1}baa - bab) + \omega\varepsilon pc + 2\varepsilon baaa,\\
pb' - \omega^{-1}b'a'a' - b'a'b' &= \omega^{-1}\alpha(pa - baa - \omega bab) + \varepsilon pc + 2\omega^{-1}\varepsilon baaa,\\
pc' &= -pc.
\end{aligned}$$

We can take $pc = xbaaa$ where $0 \leq x < p$. If $x \neq -2\omega^{-1}$ we can take $pa{-}baa{-}\omega bab = 0$ and $pb - \omega^{-1}baa - bab = ybaaa$ where $y = 0$ or y lies in a transversal for the fourth roots of unity.

$$\langle a, b, c \,|\, ca{-}baa,\ cb,\ pa{-}baa{-}\omega bab,\ pb{-}\omega^{-1}baa{-}bab{-}ybaaa,\ pc{-}xbaaa,\ \text{class } 4\rangle\ (x \neq -2\omega^{-1}).$$

If $x = -2\omega^{-1}$ then we have

$$\langle a, b, c \,|\, ca{-}baa,\ cb,\ pa{-}baa{-}\omega bab{-}ybaaa,\ pb{-}\omega^{-1}baa{-}bab{-}zbaaa,\ pc{+}2\omega^{-1}baaa,\ \text{class } 4\rangle$$

where (y, z) lie in a set of representatives for the equivalence classes $\{(\pm y, \pm z), (\pm\omega\alpha z, \pm\omega^{-1}\alpha y)\}$ where $\alpha^2 = -1$ ($(p^2+3)/4$ representatives.)

The total number of descendants is $\frac{1}{2}p^2 + \frac{1}{2}p$.

41.42.8 Case 8

Finally consider the descendants of

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb + \omega x^2 baa + \omega x^2 bab,\ pc,\ \text{class } 3\rangle.$$

If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and as in the other cases we can take $ca - baa = cb = 0$. Under transformations of the form (*) with $\gamma = \varepsilon$ we have

$$\begin{aligned}
b'a'a'a' &= baaa, \\
pa' - b'a'a' - b'a'b' &= \pm(pa - baa - bab) + \varepsilon pc + 2\varepsilon baaa, \\
pb' + \omega x^2 b'a'a' + \omega x^2 b'a'b' &= \pm(pb + \omega x^2 baa + \omega x^2 bab) + \varepsilon pc - 2\omega x^2 \varepsilon baaa, \\
pc' &= pc.
\end{aligned}$$

We can take $pc = ybaaa$ with $0 \leq y < p$. If $y \neq -2$ we can take $pa - baa - bab = 0$ and $pb + \omega x^2 baa + \omega x^2 bab = zbaaa$ where $0 \leq z \leq (p-1)/2$.

$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb + \omega x^2 baa + \omega x^2 bab - zbaaa,\ pc - ybaaa,\ \text{class } 4\rangle$ $(y \neq -2)$.

If $y = -2$ and $\omega x^2 \neq -1$ we have

$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab - zbaaa,\ pb + \omega x^2 baa + \omega x^2 bab,\ pc + 2baaa,\ \text{class } 4\rangle$

with $0 \leq z \leq (p-1)/2$.

So if $\omega x^2 \neq -1$ we have $p(p+1)/2$ descendants.

41.42.9 Case 9 ($p = 3 \bmod 4$)

The case $\omega x^2 = -1$ can only occur when $p = 3 \bmod 4$ in which case we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb - baa - bab,\ pc,\ \text{class } 3\rangle.$$

As above, if L is a descendant of this algebra (with the restriction to transformations of type (*) as in Case 8) we have the following possbilities for L.

$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab,\ pb - baa - bab - zbaaa,\ pc - ybaaa,\ \text{class } 4\rangle$ $(y \neq -2)$,

with $0 \leq z \leq (p-1)/2$.

$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - bab - ybaaa,\ pb - baa - bab - zbaaa,\ pc + 2baaa,\ \text{class } 4\rangle$,

where $0 \leq y, z < p$ and $(y, z) \sim (-y, -z)$.

So we have a total of p^2 algebras here.

The total number of algebras in cases 4 ˘ 9 is $\frac{1}{2}p^3 - \frac{5}{2}p + p\gcd(p-1, 4)/2$. (See orb6.164.)

If we are looking for descendants of algebras

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - wbab,\ pb + xbaa + wxbab,\ pc,\ \text{class } 3\rangle$$

(where $wx \neq 1$) of the form

$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - wbab - ybaaa,\ pb + xbaa + wxbab - zbaaa,\ pc - tbaaa,\ \text{class } 3\rangle$

then under transformations of type (*) we see that

$$(y, z, t) \to (2\varepsilon + \alpha y + w\varepsilon t, -2x\varepsilon + \alpha z + \varepsilon t, t),$$

with $\alpha = \pm 1$.

And under the transformation

$$\begin{aligned} a' &= w\alpha b + w\varepsilon c, \\ b' &= -x\alpha a + \varepsilon c, \\ c' &= -c, \end{aligned}$$

modulo L_2 with $\alpha^2 = -1$, which only applies in the case when $wx = -1$ and $p = 1 \bmod 4$, we have

$$(y, z, t) \to \left(-2\varepsilon - z\alpha w - w\varepsilon t, -\frac{1}{w}\left(2\varepsilon + \alpha y + w\varepsilon t\right), t\right),$$

with $\alpha^2 = -1$.

41.43 Descendants of 6.168

Algebra 6.168 has $p^3 + 2p^2 + 2p + 2 + \gcd(p-1, 3)$ descendants of order p^7 and p-class 4.

Algebra 6.168 has presentation

$$\langle a, b, c \mid ca - baa,\, cb,\, pa,\, pb,\, pc - baa,\, \text{class } 3\rangle,$$

and if L is a descendant of 6.168 of order p^7 then L_4 is generated by $babb$ and $ca - baa$, cb, pa, pb, $pc - baa \in L_4$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4 then

$$\begin{aligned} a' &= a, \\ b' &= \beta b, \\ c' &= \beta c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \beta^3 babb, \\ c'a' - b'a'a' &= \beta(ca - baa), \\ pa' &= pa, \\ pb' &= \beta pb, \\ pc' - b'a'a' &= \beta(pc - baa). \end{aligned}$$

We can take $ca - baa = pb = pc - baa = 0$, $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. Or we can take $ca - baa = pb = 0$, $pc - baa = babb$ or $\omega babb$, $pa = xbabb$ with $0 \leq x \leq (p-1)/2$. Or we can take $ca - baa = 0$, $pb = babb$ or $\omega babb$, $pc = xbabb$ $(0 \leq x < p)$, $pa = ybabb$ $(0 \leq y \leq (p-1)/2)$. Or we can take $ca - baa = babb$ or $\omega babb$, $pb = xbabb$ $(0 \leq x < p)$, $pc - baa = ybabb$ $(0 \leq y < p)$, $pa = zbabb$ $(0 \leq z \leq (p-1)/2)$.

$$\langle a, b, c \mid ca - baa,\, cb,\, pa,\, pb,\, pc - baa,\, \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\, cb,\, pa - babb,\, pb,\, pc - baa,\, \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - baa,\, cb,\, pa - \omega babb,\, pb,\, pc - baa,\, \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa,\, cb,\, pa - \omega^2 babb,\, pb,\, pc - baa,\, \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid ca - baa,\, cb,\, pa - xbabb,\, pb,\, pc - baa - babb,\, \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid ca - baa,\, cb,\, pa - xbabb,\, pb,\, pc - baa - \omega babb,\, \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid ca-baa,\, cb,\, pa-ybabb,\, pb-babb,\, pc-baa-xbabb,\, \text{class } 4\rangle\ (0 \leq x < p,\, 0 \leq y \leq (p-1)/2),$$

$$\langle a, b, c \mid ca-baa,\, cb,\, pa-ybabb,\, pb-\omega babb,\, pc-baa-xbabb,\, \text{class } 4\rangle\ (0 \leq x < p,\, 0 \leq y \leq (p-1)/2),$$

$$\langle a, b, c \mid ca-baa-babb,\, cb,\, pa-zbabb,\, pb-xbabb,\, pc-baa-ybabb,\, \text{class } 4\rangle\ (0 \leq x, y < p,\, 0 \leq z \leq (p-1)/2),$$

$$\langle a, b, c \mid ca-baa-\omega babb,\, cb,\, pa-zbabb,\, pb-xbabb,\, pc-baa-ybabb,\, \text{class } 4\rangle\ (0 \leq x, y < p,\, 0 \leq z \leq (p-1)/2).$$

41.44 Descendants of 6.169 ˘ 6.171

These algebras are terminal.

41.45 Descendants of 6.172

Algebra 6.172 has presentation

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \nu bab,\ pb - \xi bab,\ pc - baa - bab,\ \text{class } 3\rangle\ (0 \leq \nu \leq \xi < p),$$

but this algebra is terminal unless $\nu = \xi = 0$. If $\nu = \xi = 0$ then we have $(p^4 + p^2)/2$ descendants of order p^7 and p-class 4.

If $\nu = \xi = 0$ then we have

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa,\ pb,\ pc - baa - bab,\ \text{class } 3\rangle$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$ and $ca - baa$, cb, pa, pb, $pc - baa - bab \in L_4$. Adding a suitable scalar multiple of baa to c we may assume that $ca - baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'a' = 0$, then $a' = a$, $b' = b$, $c' = c$ modulo L_2 and

$$\begin{aligned}
b'a'a'a' &= baaa, \\
c'b' &= cb, \\
pa' &= pa, \\
pb' &= pb \\
pc' - b'a'a' - b'a'b' &= pc - baa - bab.
\end{aligned}$$

or $a' = -b$, $b' = -a$, $c' = c$ and

$$\begin{aligned}
b'a'a'a' &= -baaa, \\
c'b' &= -cb, \\
pa' &= -pb, \\
pb' &= -pa \\
pc' - b'a'a' - b'a'b' &= pc - baa - bab.
\end{aligned}$$

So we can take $pc = xbaaa$ where $0 \leq x \leq (p-1)/2$. If $pc \neq 0$ we can take $cb = ybaaa$, $pa = zbaaa$, $pb = tbaaa$ with $0 \leq y, z, t < p$, and if $pc = 0$ we can take $cb = ybaaa$, $pa = zbaaa$, $pb = tbaaa$ with $0 \leq y < p$, $0 \leq z \leq t < p$.

We have $\frac{1}{2}p^4 + \frac{1}{2}p^2$ descendants of order p^7 and p-class 4.

$$\langle a, b, c \,|\, ca - baa,\ cb - ybaaa,\ pa - zbaaa,\ pb - tbaaa,\ pc - baa - bab - xbaaa,\ \text{class } 4\rangle$$

with $1 \leq x \leq (p-1)/2$, $0 \leq y, z, t < p$,

$$\langle a, b, c \,|\, ca - baa,\ cb - ybaaa,\ pa - zbaaa,\ pb - tbaaa,\ pc - baa - bab,\ \text{class } 4\rangle$$

with $0 \leq y < p$, $0 \leq z \leq t < p$.

Algebra 6.173 has
$$3p+3+\frac{1}{2}(p^2+2p+3)\gcd(p-1,3)$$
descendants of order p^7 and p-class 4.

Algebra 6.173 has presentation
$$\langle a,b,c \mid ca-bab,\ cb-\omega baa,\ pa,\ pb,\ pc,\ \text{class }3\rangle,$$
and so if L is a descendant of 6.173 of order p^7 then the commutator structure of L is the same as that of one of the $p+2$ algebras with presentations 7.106 and 7.107 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we can assume that L has the following commutator relations
$$ca=bab,\ cb=\omega baa,\ baab=\lambda baaa,\ babb=\mu baaa$$
where (λ,μ) is one of the following:

1. $(0,0)$,
2. $(0,-\omega)$,
3. (λ,ω) where $\lambda^2-\omega$ is a square (all these algebras are isomorphic),
4. (λ,ω) where $\lambda^2-\omega$ is not a square (all these algebras are isomorphic),
5. $(\lambda,0)$ where $1\leq\lambda\leq(p-1)/2$ (giving $(p-1)/2$ diœerent algebras),
6. (λ,μ) where $\lambda^2-\mu$ is not a square, $\mu\neq\omega$, $\lambda\neq 0$ if $\mu=-\omega$; these parameters give $(p-3)/2$ diœerent algebras with two pairs (λ,μ), (λ',μ') giving isomorphic algebras if $(\lambda,\mu)=(\lambda',\mu')$ or if
$$(\lambda',\mu')=\left(\frac{r^2\lambda+r(\omega+\mu)+\omega\lambda}{r^2+2r\lambda+\mu},\frac{r^2\mu+2r\omega\lambda+\omega^2}{r^2+2r\lambda+\mu}\right)$$
for some $r\in\mathbb{Z}_p$.

In each case we give the most general possible expression modulo L_2 for generators a',b',c' of L which satisfy the same commutator relations as a,b,c, and give the value of $b'a'a'a'$. We then compute the orbits of pa,pb,pc.

41.46.1 Case 1

Let $\lambda=\mu=0$. Then
$$\begin{aligned} a' &= \alpha a,\\ b' &= \pm\alpha b+\varepsilon c,\\ c' &= \alpha^2 c,\end{aligned}$$
with $b'a'a'a'=\pm\alpha^4 baaa$.

We can take $pc=0$, $baaa$ or (if $p=1 \bmod 4$) $\omega baaa$. If $pc\neq 0$ we can take $pb=0$ and $pa=0$ or $xbaaa$ where x lies in a transversal for the fourth roots of unity. If $pc=0$ we can take $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb=pc=0$ we can take $pa=0$,

$baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$, $pc = 0$ we can take $pa = xbaaa$ where $0 \leq x \leq (p-1)/2$. This gives $(p+1+(p+3)\gcd(p-1,3)+\gcd(p-1,4))/2$ algebras

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb,\ ca-bab,\ cb-\omega baa,\ pa-xbaaa,\ pb,\ pc-baaa,\ \text{class } 4\rangle \ (x \text{ in a transversal for } 4^{th} \text{ roots of } 1),$$

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle \ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baab,\ babb,\ ca-bab,\ cb-\omega baa,\ pa-xbaaa,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle \ (p = 1 \bmod 4, x \text{ in a transversal for } 4^{th} \text{ root}$$

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baab,\ babb,\ ca - bab,\ cb - \omega baa,\ pa - \omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baab,\ babb,\ ca-bab,\ cb-\omega baa,\ pa-xbaaa,\ pb-baaa,\ pc,\ \text{class } 4\rangle \ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid baab,\ babb,\ ca-bab,\ cb-\omega baa,\ pa-xbaaa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3, 0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid baab,\ babb,\ ca-bab,\ cb-\omega baa,\ pa-xbaaa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle \ (p = 1 \bmod 3, 0 \leq x \leq (p-1)/2).$$

41.46.2 Case 2

Let $\lambda = 0$, $\mu = -\omega$. Then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \pm(\omega\beta a + \alpha b) + \varepsilon c, \\ c' &= (\alpha^2 - \omega\beta^2)c \end{aligned}$$

with $b'a'a'a' = \pm(\alpha^2 - \omega\beta^2)^2 baaa$. So we can take $pc = 0$ or $baaa$, and if $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we can take $pa = pb = 0$, or $pa = 0$, $pb = baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. So we have $2 + \gcd(p-1,3)$ algebras.

$$\langle a, b, c \mid baab,\ babb + \omega baaa,\ ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb + \omega baaa,\ ca - bab,\ cb - \omega baa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb + \omega baaa,\ ca - bab,\ cb - \omega baa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb + \omega baaa,\ ca - bab,\ cb - \omega baa,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baab,\ babb + \omega baaa,\ ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

41.46.3 Case 3

If $\mu = \omega$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \varepsilon c, \\ c' &= \alpha^2 c \end{aligned}$$

with $b'a'a'a' = \alpha^4 baaa$ or

$$\begin{aligned} a' &= \beta b + \gamma c, \\ b' &= \omega\beta a + \varepsilon c, \\ c' &= -\omega\beta^2 c \end{aligned}$$

with $b'a'a'a' = -\omega^2\beta^4 baaa$ or

$$\begin{aligned} a' &= -\lambda\beta a + \beta b + \gamma c, \\ b' &= -\omega\beta a + \lambda\beta b + \varepsilon c, \\ c' &= (\lambda^2 - \omega)\beta^2 c \end{aligned}$$

with $b'a'a'a' = (\lambda^2 - \omega)^2\beta^4 baaa$ or

$$\begin{aligned} a' &= \omega\alpha a - \lambda\alpha b + \gamma c, \\ b' &= \omega\lambda\alpha a - \omega\alpha b + \varepsilon c, \\ c' &= -\omega(\lambda^2 - \omega)\alpha^2 c \end{aligned}$$

with $b'a'a'a' = -\omega^2(\lambda^2 - \omega)^2\alpha^4 baaa$.

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = pb = 0$. If $pc = 0$ then we get the following number of orbits for pa, pb.

If $p = 1 \bmod 4$ the number of orbits is $1 + (p + 3)\gcd(p - 1, 3)/4$.

If $p = 3 \bmod 4$ then the number of orbits is $1 + (p + 1)\gcd(p - 1, 3)/4$ if $\lambda^2 - \omega$ is not a square, and $1 + (p + 5)\gcd(p - 1, 3)/4$ if it is a square.

So the number of orbits for the two cases when $\lambda^2 - \omega$ is a square, and for when it is not, is $2 + (p + 3)\gcd(p - 1, 3)/2$. For the total number of algebras we need to add 2.

41.46.4 Case 4

If $\mu = 0$ we have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \varepsilon c, \\ c' &= \alpha^2 c \end{aligned}$$

with $b'a'a'a' = \alpha^4 baaa$ or

$$\begin{aligned} a' &= \omega\alpha a - 2\lambda\alpha b + \gamma c, \\ b' &= 2\omega\lambda\alpha a - \omega\alpha b + \varepsilon c, \\ c' &= \omega(\omega - 4\lambda^2)\alpha^2 c \end{aligned}$$

with $b'a'a'a' = -\omega^2(\omega - 4\lambda^2)^2\alpha^4 baaa$.

If $4\lambda^2 - \omega$ is a square then we can take $pc = 0$ or $baaa$, and if $4\lambda^2 - \omega$ is not a square we can take $pc = 0$, $baaa$ or $\omega baaa$. If $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ then the number of orbits for pa, pb is $1 + (p + 3)\gcd(p - 1, 3)/2$ if $\omega^2 - 4\lambda^2\omega$ is a square, and $1 + (p + 1)\gcd(p - 1, 3)/2$ if not. Note that $4\lambda^2 - \omega$ and $\omega^2 - 4\lambda^2\omega$ are each squares for $(p - 5 + \gcd(p - 1, 4))/4$ values of λ in the range $1 \leq \lambda \leq (p - 1)/2$.

If $4\lambda^2 - \omega$ and $\omega^2 - 4\lambda^2\omega$ are both squares the number of algebras is $2 + (p+3)\gcd(p-1, 3)/2$.

If $4\lambda^2 - \omega$ is a square and $\omega^2 - 4\lambda^2\omega$ is not, the number is $2 + (p + 1)\gcd(p - 1, 3)/2$.

If $4\lambda^2 - \omega$ is a not a square and $\omega^2 - 4\lambda^2\omega$ is, the number is $3 + (p + 3)\gcd(p - 1, 3)/2$.

If neither of $4\lambda^2 - \omega$ and $\omega^2 - 4\lambda^2\omega$ are squares the number is $3 + (p+1)\gcd(p-1, 3)/2$.

So the total number of algebras for all $1 \leq \lambda \leq (p - 1)/2$ is

$$\frac{5}{4}p - \frac{1}{4} + (\frac{1}{4}p^2 + \frac{1}{4}p - \frac{3}{2})\gcd(p - 1, 3) + \frac{1}{4}(\gcd(p - 1, 3) - 1)\gcd(p - 1, 4)$$

41.46.5 Case 5

If $\lambda^2 - \mu$ is not a square, $\mu \neq \omega$, $\lambda \neq 0$ if $\mu = -\omega$ we have

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha b + \varepsilon c, \\ c' &= \alpha^2 c \end{aligned}$$

with $b'a'a'a' = \alpha^4 baaa$ or

$$\begin{aligned} a' &= (\omega + \mu)\alpha a - 2\lambda\alpha b + \gamma c, \\ b' &= 2\omega\lambda\alpha a - (\omega + \mu)\alpha b + \varepsilon c, \\ c' &= ((\omega + \mu)^2 - 4\omega\lambda^2)\alpha^2 c \end{aligned}$$

with $b'a'a'a' = -((\omega + \mu)^2 - 4\omega\lambda^2)^2\alpha^4 baaa$.

If $-((\omega + \mu)^2 - 4\omega\lambda^2)$ is a square we can take $pc = 0$, $baaa$ or $\omega baaa$ and if it is not a square we can take $pc = 0$ or $baaa$. If $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we have the number of orbits of pa, pb is $1 + (p + 3)\gcd(p - 1, 3)/2$ if $((\omega + \mu)^2 - 4\omega\lambda^2)$ is a square and $1 + (p + 1)\gcd(p - 1, 3)/2$ if not. Of the $(p - 3)/2$ orbits of (λ, μ) with $\lambda^2 - \mu$ not a square, $\mu \neq \omega$, $\lambda \neq 0$ if $\mu = -\omega$, the number with $((\omega + \mu)^2 - 4\omega\lambda^2)$ a square is $(p - 1 - \gcd(p - 1, 4))/4$. (This is proved in Appendix G.)

The total number of algebras arising in this case is

$$\frac{5}{4}p - \frac{13}{4} + (\frac{1}{4}p^2 - \frac{1}{4}p - 1)\gcd(p - 1, 3) - \frac{1}{4}(\gcd(p - 1, 3) + 1)\gcd(p - 1, 4).$$

The total number of algebras from cases 2 to 5 is

$$\frac{5}{2}p + \frac{5}{2} + \frac{1}{2}(p^2 + p)\gcd(p - 1, 3) - \frac{1}{2}\gcd(p - 1, 4).$$

(See orb6.173 for a program to generate these algebras.)

41.47 Descendant of 6.174

Algebra 6.174 has presentation

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - bab,\ pb,\ pc,\ \text{class } 3\rangle\ (0 \leq \lambda \leq (p - 1)/2),$$

and if L is a descendant of 6.174 of order p^7 then L_4 is generated by $baaa$.

We need to distinguish between the cases $\lambda = 0$ and $\lambda \neq 0$. If $\lambda = 0$ it has $(p + 1)^2/4$ descendants of order p^7 and p-class 4, and if $\lambda \neq 0$ it has $p(p + 1)/2$. The total number of descendants of order p^7 is $\frac{1}{4}p^3 + \frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{4}$.

So ørst consider the case when $\lambda = 0$. Adding suitable scalar multiples of baa to c and a suitable scalar multiple of c to a we can take $ca - bab = cb - \omega baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'b' = c'b' - \omega b'a'a' = 0$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \delta b + \varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 (with α, δ independantly equal to ± 1), which gives

$$\begin{aligned} b'a'a'a' &= \alpha\delta baaa, \\ pa' - b'a'b' &= \alpha(pa - bab) + \omega\alpha\delta\varepsilon baaa, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ with $0 \leq x \leq (p-1)/2$. If $pc \neq 0$ we can take $pb = 0$ and $pa - bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. If $pc = 0$ we can take $pa - bab = 0$ and $pb = ybaaa$ with $0 \leq y \leq (p-1)/2$. So we have $(p+1)^2/4$ algebras.

$\langle a, b, c \mid ca-bab,\ cb-\omega baa,\ pa-bab-ybaaa,\ pb,\ pc-xbaaa, \text{ class } 4\rangle$ $(0 < x \leq (p-1)/2, 0 \leq y \leq (p-1)/2)$,

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - bab,\ pb - ybaaa,\ pc, \text{ class } 4\rangle$ $(0 \leq y \leq (p-1)/2)$.

Now consider the case when $\lambda \neq 0$. Then, as above, we can take $ca-bab = cb-\omega baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'b' = c'b' - \omega b'a'a'$ then

$$\begin{aligned} a' &= \pm a + \lambda\gamma c, \\ b' &= \pm b - \omega\gamma c, \\ c' &= c \end{aligned}$$

modulo L_2, which gives

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' - \lambda b'a'a' - b'a'b' &= \pm(pa - \lambda baa - bab) + \lambda\gamma pc - (\lambda^2 - \omega)^2\gamma baaa, \\ pb' &= \pm pb - \omega\gamma pc, \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ with $0 \leq x < p$. If $pc \neq 0$ we can take $pb = 0$ and $pa-\lambda baa-bab = ybaaa$ with $0 \leq y \leq (p-1)/2$. If $pc = 0$ we can take $pa - \lambda baa - bab = 0$ and $pb = ybaaa$ with $0 \leq y \leq (p-1)/2$. So for each $\lambda \neq 0$ we have $p(p+1)/2$ algebras.

$\langle a, b, c \mid ca-bab,\ cb-\omega baa,\ pa-bab-ybaaa,\ pb,\ pc-xbaaa, \text{ class } 4\rangle$ $(0 < x < p, 0 \leq y \leq (p-1)/2)$,

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - bab,\ pb - ybaaa,\ pc, \text{ class } 4\rangle$ $(0 \leq y \leq (p-1)/2)$.

This gives $\frac{1}{4}p^3 + \frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{4}$ algebras in all.

41.48 Descendants of 6.175

Algebra 6.175 has presentation

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \omega bab,\ pb,\ pc, \text{ class } 3\rangle$ $(0 \leq \lambda \leq (p-1)/2)$,

and as in the case of algebra 6.174, if $\lambda = 0$ it has $(p+1)^2/4$ descendants of order p^7 and p-class 4, and if $\lambda \neq 0$ it has $p(p+1)/2$. The total number of descendants of order p^7 is $\frac{1}{4}p^3 + \frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{4}$.

If L is a descendant of 6.174 of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiples of baa to c and a suitable scalar multiple of c to a we can take $ca - bab = cb - \omega baa = 0$.

As in the descendants of 6.174 we have $(p+1)^2/4$ algebras with $\lambda = 0$

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \omega bab - ybaaa,\ pb,\ pc - xbaaa,\ \text{class } 4\rangle$ $(0 < x \le (p-1)/2, 0 \le y \le (p-1)/2)$,

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \omega bab,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle$ $(0 \le y \le (p-1)/2)$.

and $p(p+1)/2$ algebras for each value of λ with $0 < \lambda \le (p-1)/2$.

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \omega bab - ybaaa,\ pb,\ pc - xbaaa,\ \text{class } 4\rangle$ $(0 < x < p, 0 \le y \le (p-1)/2)$,

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \omega bab,\ pb - ybaaa,\ pc,\ \text{class } 4\rangle$ $(0 \le y \le (p-1)/2)$.

This gives $\frac{1}{4}p^3 + \frac{1}{4}p^2 + \frac{1}{4}p + \frac{1}{4}$ algebras in all.

41.49 Descendants of 6.176

Algebras 6.176 and 6.177 between them have $p(p-1) + \frac{p}{2}\gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Algebra 6.176 has presentation

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.176 of order p^7 then L_4 is generated by $babb$. Adding suitable scalar multiples of baa to c and c to b we can take $ca - bab = cb - \omega baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'b' = c'b' - \omega b'a'a'$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b, \\ c' &= \alpha^2 c \end{aligned}$$

with $\alpha^4 = 1$ and this gives

$$\begin{aligned} b'a'b'b' &= \alpha^2 babb, \\ pa' - b'a'a' &= \alpha(pa - baa) + \gamma pc - \gamma babb, \\ pb' &= \alpha^{-1} pb, \\ pc' &= \alpha^2 pc. \end{aligned}$$

We can take $pc = xbabb$ where $0 \le x < p$, and we can take $pb = 0$ or $pb = ybabb$ where y lies in a transversal for the fourth roots of unity. If $x \ne 1$ we can take $pa - baa = 0$. If $x = 1$ and $pb = 0$ then we can take $pa - baa = 0$ or $ybabb$ where y lies in a transversal for the fourth roots of unity. And if $x = 1$ and $pb \ne 0$ then we can take $pa - baa = zbabb$ for $0 \le z < p$.

So we have $p + 2p(p-1)/\gcd(p-1, 4)$ algebras.

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa,\ pb,\ pc - xbabb,\ \text{class } 4\rangle$ $(0 \le x < p)$,

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa,\ pb - ybabb,\ pc - xbabb,\ \text{class } 4\rangle$ $(0 \le x < p$, y in a tranversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa - ybabb,\ pb,\ pc - babb,\ \text{class } 4\rangle$ (y in a tranversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - baa - zbabb,\ pb - ybabb,\ pc - babb,\ \text{class } 4\rangle$ $(0 < z < p$, y in a tranversal for 4^{th} roots of

41.50 Descendants of 6.177

Algebra 6.177 only occurs when $p = 1 \bmod 4$, and has presentation

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \omega baa,\ pb,\ pc, \text{ class } 3 \rangle,$$

and if L is a descendant of 6.177 of order p^7 then L_4 is generated by $babb$. As in the descendants of 6.176, adding suitable scalar multiples of baa to c and c to b we can take $ca - bab = cb - \omega baa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'b' = c'b' - \omega b'a'a'$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^{-1} b, \\ c' &= \alpha^2 c \end{aligned}$$

with $\alpha^4 = 1$ and this gives

$$\begin{aligned} b'a'b'b' &= \alpha^2 babb, \\ pa' - b'a'a' &= \alpha(pa - baa) + \gamma pc - \omega\gamma babb, \\ pb' &= \alpha^{-1} pb, \\ pc' &= \alpha^2 pc. \end{aligned}$$

We have $p + 2p(p-1)/\gcd(p-1, 4)$ algebras.

$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \omega baa,\ pb,\ pc - xbabb, \text{ class } 4 \rangle$ $(0 \leq x < p)$,

$\langle a, b, c \mid ca{-}bab,\ cb{-}\omega baa,\ pa{-}\omega baa,\ pb{-}ybabb,\ pc{-}xbabb, \text{ class } 4 \rangle$ $(0 \leq x < p$, y in a tranversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca{-}bab,\ cb{-}\omega baa,\ pa{-}\omega baa{-}ybabb,\ pb,\ pc{-}\omega babb, \text{ class } 4 \rangle$ (y in a tranversal for 4^{th} roots of 1),

$\langle a, b, c \mid ca{-}bab,\ cb{-}\omega baa,\ pa{-}\omega baa{-}zbabb,\ pb{-}ybabb,\ pc{-}\omega babb, \text{ class } 4 \rangle$ $(0 < z < p$, y in a tranversal for 4^{th} roots

41.51 Descendants of 6.178

Algebra 6.178 has presentation

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa - \mu bab,\ pb - \nu baa - \xi bab,\ pc, \text{ class } 3 \rangle,$$

where we write $A = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$, and A ranges over a set of representatives for the orbits of non-singular 2×2 matrices under the action

$$A \to \frac{1}{\det P} PAP^{-1}$$

as P ranges over non-singular matrices

$$P = \begin{pmatrix} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{pmatrix}.$$

These algebras are terminal unless $\xi = -\lambda$. The number of orbits of non-singular matrices with $\xi = -\lambda$ is $(3p-1)/2$. The rank 2 matrices split up into one orbit of size $p-1$ (matrices $\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}$), $p-1$ orbits of size $(p^2-1)/2$ (including two orbits of elements

$\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}$), and $(p-1)/2$ orbits of size p^2-1. (This is proved in Appendix D.) In all, 6.178 has $(3p^2-1)/2$ descendants of order p^7 and p-class 4.

If L is a descendant of order p^7 of any of these algebras, then by adding suitable scalar multiples of baa and bab to c we can take $ca - bab = cb - \omega baa = 0$. (This relies on the fact that the matrix A is non-singular.) In each case L_4 is generated by $baaa$ or (when $\mu = 0$) $baab$, and we give the most general form (modulo L_2) for generators a', b', c' for L which satisfy the same relations as a, b, c modulo L_4, and for which $c'a' - b'a'b' = c'b' - \omega b'a'a' = 0$. The $p-1$ orbits of size $(p^2-1)/2$ have stabilizers of order 4, consisting of the $+$ matrix with $\alpha = \pm 1$ and $\beta = 0$ and two $-$ matrices. We can take $pc = xbaaa$ with $0 \leq x \leq (p-1)/2$ in these cases. The $(p-1)/2$ orbits of size p^2-1 have stabilizers of order 2, consisting of the $+$ matrix with $\alpha = \pm 1$ and $\beta = 0$. In these cases we can take $pc = xbaaa$ with $0 \leq x < p$.

41.51.1 Case 1

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - bab,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle$$

First consider $A = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}$. This is stabilized by plus and minus matrices P with $\alpha^2 - \omega\beta^2 = 1$. We have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \pm(\omega\beta a + \alpha b) + \varepsilon c, \\ c' &= (\alpha^2 - \omega\beta^2)c \end{aligned}$$

modulo L_2, with $\alpha^2 - \omega\beta^2 = 1$, which gives

$$\begin{aligned} b'a'a'a' &= \pm baaa, \\ pa' - b'a'b' &= \alpha(pa - bab) + \beta(pb - \omega baa) + \gamma pc \pm 3\omega\varepsilon baaa, \\ pb' - \omega b'a'a' &= \pm\omega\beta(pa - bab) \pm \alpha(pb - \omega baa) + \varepsilon pc \pm 3\omega^2\gamma baaa, \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ where $0 \leq x \leq (p-1)/2$.

First consider the case when $pc = 0$. Then we can take $pa - bab = pb - \omega baa = 0$.

Next consider the case when $pc = xbaaa$ with $0 < x \leq (p-1)/2$. Then we can take $pa - bab = 0$. Let $pb - \omega baa = ybaaa$. We are restricted to the $+$ transformation with $\beta y + \gamma x + 3\omega\varepsilon = 0$, and this gives

$$pb' - \omega b'a'a' = (\alpha y - \frac{\beta xy + \gamma x^2}{3\omega} + 3\omega^2\gamma)b'a'a'a',$$

and so we can take $\alpha = 1$, $\beta = 0$, and choose γ so that $pb' - \omega b'a'a' = 0$. (Checked again 25/7/03)

So we have $(p+1)/2$ algebras with $pa - bab = pb - \omega baa = 0$ and $pc = xbaaa$ with $0 \leq x \leq (p-1)/2$.

41.51.2 Case 2

$$\langle a, b, c \,|\, ca - bab,\ cb - \omega baa,\ pa - baa,\ pb + bab,\ pc,\ \text{class } 3\rangle$$

Next consider $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This is stabilized by the plus matrix with $\beta = 0$ and $\alpha = \pm 1$, and by two minus matrix with $\alpha^2 = -1$, $\beta = 0$ in the case when $p = 1 \bmod 4$. Again we can take $pa - baa = pb + bab = 0$, and we can take $pc = xbaab$ where $0 \leq x < p$ if $p = 3 \bmod 4$, and $0 \leq x \leq (p-1)/2$ if $p = 1 \bmod 4$. (Checked again 25/7/03)

41.51.3 Case 3

Now consider $A = \begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix}$. Here we are stabilized by both plus and minus matrices with $\alpha = \pm 1$ and $\beta = 0$ and we can take $pa - ybab = pb + \omega ybaa = 0$, and we can take $pc = xbaaa$ where $0 \leq x \leq (p-1)/2$. (Checked again 25/7/03)

41.51.4 Case 4

Now consider $A = \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}$, with $y \neq 0$. Here we are stabilized by plus and matrix with $\alpha = \pm 1$ and $\beta = 0$, and by two minus matrices. We can take $pa - baa - ybab = pb + \omega ybaa + bab = 0$ and $pc = xbaaa$ where $0 \leq x \leq (p-1)/2$. (Checked again 25/7/03)

41.51.5 Case 5

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - ybab,\ pb - zbaa,\ pc,\ \text{class } 3 \rangle$$

And now consider $A = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$, $z \neq \pm \omega y$. This has stabilizer of order 4, and is stabilized by both plus and minus matrices with $\beta = 0$ and $\alpha = \pm 1$. We have

$$\begin{aligned} pa' - yb'a'b' &= \pm(pa - ybab) + (\omega y + 2z)\varepsilon baaa + \gamma pc, \\ pb' - zb'a'a' &= \pm(pb - zbaa) + (2\omega z + z^2 y^{-1})\gamma baaa + \varepsilon pc, \\ pc' &= pc, \\ b'a'a'a' &= baaa, \end{aligned}$$

or

$$\begin{aligned} pa' - yb'a'b' &= \pm(pa - ybab) - (\omega y + 2z)\varepsilon baaa + \gamma pc, \\ pb' - zb'a'a' &= \mp(pb - zbaa) - (2\omega z + z^2 y^{-1})\gamma baaa + \varepsilon pc, \\ pc' &= pc, \\ b'a'a'a' &= -baaa. \end{aligned}$$

So we can take $pc = \lambda baaa$ with $0 \leq \lambda \leq (p-1)/2$. If $(\omega y + 2z)(2\omega z + y^{-1}z^2)$ is not a square then we can take tails on pa and pb to be zero. If $(\omega y + 2z)(2\omega z + y^{-1}z^2) = \lambda^2$ then $pc = \pm\lambda baaa$ gives problems, though for all other values of pc we can take tails on pa and pb to be zero.

If $\omega y + 2z = 0$ or $2\omega z + y^{-1}z^2 = 0$ then $pc = 0$ gives problems. If $\omega y + 2z = 0$ and $pc = 0$ then we can take $pb = 0$ and take the tail on pa to be $\mu baaa$ where $0 \leq \mu \leq (p-1)/2$. And if $2\omega z + y^{-1}z^2 = 0$ we can take $pa = 0$ and take the tail on pb to be $\mu baaa$ where $0 \leq \mu \leq (p-1)/2$.

If $(\omega y + 2z)(2\omega z + y^{-1}z^2) = \lambda^2$ where $\lambda^2 \neq 0$, then only one of $\pm\lambda baaa$ occurs as the value of pc, and for that value we can take tail on pa to be zero, and tail on pb equal to $\mu baaa$ with $0 \leq \mu \leq (p-1)/2$.

So we have $(p+1)/2$ orbits, with an extra $(p-1)/2$ orbits if $(\omega y + 2z)(2\omega z + y^{-1}z^2)$ is a square.

41.51.6 Case 6

Finally, consider the case when $A = \begin{pmatrix} 1 & y \\ z & -1 \end{pmatrix}$ with $z \neq -\omega y$, and with A not in an orbit with $(1,1)$-entry equal to 0. (There are $(p-1)/2$ orbits of this kind.) They have stabilizers of order 2 consisting of plus matrices with $\alpha = \pm 1$ and $\beta = 0$. We have L_4 generated by $baab$ and

$$\begin{aligned} pa' - b'a'a' - yb'a'b' &= \pm(pa - baa - ybab) + ((-\omega y - z)\gamma + (-\omega y^2 - 2yz - 1)\varepsilon)baab + \gamma pc, \\ pb' - zb'a'a' + b'a'b' &= \pm(pb - zbaa + bab) + ((-2\omega yz - \omega - z^2)\gamma + (\omega y + z)\varepsilon)baab + \varepsilon pc, \\ pc' &= pc, \\ b'a'a'b' &= baab, \end{aligned}$$

$$\begin{pmatrix} \lambda + (-\omega y - z) & (-\omega y^2 - 2yz - 1) \\ (-2\omega yz - \omega - z^2) & \lambda + (\omega y + z) \end{pmatrix}$$

Determinant: $\lambda^2 - (1+yz)\left(2(\omega y + z)^2 + \omega(1+yz)\right)$

We can take $pc = \lambda baab$ with $0 \leq \lambda < p$. If $(1+yz)\left(2(\omega y + z)^2 + \omega(1+yz)\right)$ is not a square we can take the tails on pa and pb to be zero. Note that $(x^2+yz)\left(2(\omega y + z)^2 + \omega(x^2+yz)\right)$ cannot be zero in this situation. But if $(1+yz)\left(2(\omega y + z)^2 + \omega(1+yz)\right) = \lambda^2$ then we an extra $(p-1)/2$ algebras for $pc = \pm\lambda baab$, with the extra algebras either having the tail on pa zero and the tail on pb equal to $\mu baaa$ (or $\mu baab$ when $y = 0$) with $0 \leq \mu \leq (p-1)/2$ or having the tail on pb zero and the tail on pa equal to $\mu baaa$ (or $\mu baab$ when $y = 0$) with $0 \leq \mu \leq (p-1)/2$. So we get an extra $p-1$ algebras when $(1+yz)\left(2(\omega y + z)^2 + \omega(1+yz)\right)$ is a square.

41.51.7 Grand totals

In general we are concerned with non-singular matrices $A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$. Since A is non-singular we want $x^2 + yz \neq 0$.

The number of descendants of this algebra depends on two parameters:

$$u = (x^2+yz)\left(2(\omega y + z)^2 + \omega(x^2+yz)\right) \text{ and } v = (z - \omega y)^2 - 4\omega x^2.$$

- If u and v are both squares in $\mathbb{Z}_p$ then we have p descendants.
- If u is not a square and v is a square we have $(p+1)/2$ descendants.
- If neither of u and v are squares we have p descendants.
- If u is a square and v is not a square then we have $2p-1$ descendants.
- There is one exception to the above ¯ if $z = -\omega y$ then u is not a square, but there are $(p+1)/2$ descendants whether or not v is a square.

It seems that for any øxed x, y, z the question of whether u or v are squares is pretty random (not PORC?), but the overall number of descendants over all x, y, z depends on the number, k say, of possibilities for x, y, z with $x^2 + yz \neq 0$ and u a square.

I conjecure that $k = (p+1)(p^2-1)/2$, and have checked this for all $p < 100$, $p > 3$. This conjecture is proved correct in Appendix E. Hence, the number of descendants of these algebras over all x, y, z with $x^2 + yz \neq 0$ is $\frac{3}{2}p^2 - \frac{1}{2}$.

Algebra 6.179 has presentation

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa - \lambda baa,\ pb - \mu baa,\ pc - bab,\ \text{class } 3\rangle,$$

where λ and $-\lambda$ give isomorphic algebras for any given μ, so that we get distinct algebras if we let $0 \leq \lambda \leq (p-1)/2$, $0 \leq \mu < p$, but this algebra is terminal unless $\lambda = \mu = 0$. In this case we have $(p^4 + p^2)/2$ descendants of order p^7 and p-class 4.

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa,\ pa,\ pb,\ pc - bab,\ \text{class } 3\rangle.$$

If L is a descendant of this algebra of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of baa to c we can take $ca - bab = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' - b'a'b' = 0$ then

$$\begin{aligned} a' &= a, \\ b' &= \pm b, \\ c' &= c \end{aligned}$$

modulo L_2, with

$$\begin{aligned} b'a'a'a' &= \pm baaa, \\ c'b' - \omega b'a'a' &= \pm(cb - \omega baa), \\ pa' &= pa, \\ pb' &= \pm pb, \\ pc' - b'a'b' &= pc - bab. \end{aligned}$$

We can take $cb - \omega baa = xbaaa$, $pa = ybaaa$, $pb = zbaaa$, $pc - bab = tbaaa$ with $0 \leq x, z, t < p$, $0 \leq y \leq (p-1)/2$, and $t \leq (p-1)/2$ if $y = 0$. So we have $p^2(p^2+1)/2$ algebras

$$\langle a, b, c \mid ca - bab,\ cb - \omega baa - xbaaa,\ pa - ybaaa,\ pb - zbaaa,\ pc - bab - tbaaa,\ \text{class } 4\rangle$$

with x, y, z, t as speciøed above.

41.53 Summary

Algebra 4.3 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 2\rangle,$$

and it has $3p^2 + 13p + 37 + \gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^6 and p-class 3, with presentations labelled 6.118 ˘ 6.179. The corresponding group is number 24 in Theorem 2 of our paper on groups of order p^6. We list the capable algebras among 6.118 ˘ 6.179, and for each such algebra we give the number of descendants of order p^7 and p-class 4.

Algebra 6.118 has $11 + 4\gcd(p-1, 3)$ descendants of order p^7.

Algebra 6.119 has $(p-1)/2 + 3 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7.

Algebra 6.120 has $(p-1)/2 + 3 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7.

Algebra 6.121 has $2p + 4 + 2\gcd(p-1, 3)$ descendants of order p^7.

Algebra 6.122 is a one parameter family of algebras, with $(p+1)/2$ algebras in the family, but only one algebra in the family is capable. It has $p+2$ descendants of order p^7.

Algebra 6.125 has $p+1$ descendants of order p^7.

Algebra 6.127 has $3p+4+6\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.131 has $15+(p+10)\gcd(p-1,3)+\gcd(p-1,4)+\gcd(p-1,7)$ descendants of order p^7.

Algebra 6.132 has $(p+1+3(p+1)\gcd(p-1,3)+\gcd(p-1,4))/2$ descendants of order p^7.

Algebra 6.133 has $(p+1+3(p+1)\gcd(p-1,3)+\gcd(p-1,4))/2$ descendants of order p^7.

Algebra 6.134 has $3p-1+\gcd(p-1,3)$ descendants of order p^7.

Algebra 6.135 is a one parameter family of algebras, with p algebras in the family, but only two algebras in the family are capable. One has $p^2+2p-1+\gcd(p-1,3)+\gcd(p-1,4)+\gcd(p-1,5)$ descendants of order p^7, and the other has 4 descendants of order p^7.

Algebra 6.138 is a one parameter family of algebras, with p algebras in the family, but only one of these algebras is capable. It has $p(p+1)/2$ descendants of order p^7.

Algebra 6.139 has $(p+3)/2+\gcd(p-1,3)$ descendants of order p^7.

Algebras 6.140 and 6.141 between then have $(3p+1)/2$ descendants of order p^7.

Algebra 6.142 is a one parameter family of algebras, with p algebras in the family, but only one of these algebras is capable. It has $p(p+1)/2$ descendants of order p^7.

Algebra 6.143 has $(p+3)/2+\gcd(p-1,3)$ descendants of order p^7.

Algebras 6.144 and 6.145 between then have $(3p+1)/2$ descendants of order p^7.

Algebra 6.146 is a one parameter family of algebras, with p algebras in the family, but only one of these algebras is capable. It has $2p^2+3p$ descendants of order p^7.

Algebra 6.148 is a one parameter family of algebras, with $1+\gcd(p-1,4)$ algebras in the family, but only one of these algebras is capable. It has $p^3+p^2+p+(p+2)\gcd(p-1,3)+\gcd(p-1,5)$ descendants of order p^7.

Algebra 6.150 has $4p+14+(\frac{1}{2}p^2+2p+\frac{13}{2})\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.151 has $p^2+p+2+(p+1)\gcd(p-1,3)$ descendants of order p^7.

Algebra 6.152 has $\frac{1}{2}p^2+p+\frac{3}{2}+\gcd(p-1,3)+\frac{p+1}{2}\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.153 has $p(p+1)/2$ descendants of order p^7.

Algebra 6.154 has $\frac{1}{2}p^2+p+\frac{3}{2}+\gcd(p-1,3)+\frac{p+1}{2}\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.155 has $p(p+1)/2$ descendants of order p^7.

Algebra 6.156 has p descendants of order p^7.

Algebra 6.157 has $2p-1$ descendants of order p^7.

Algebra 6.158 has p descendants of order p^7.

Algebra 6.159 is a one parameter family of algebras, with p algebras in the family, but only one of these algebras is capable. It has $(p^3+p^2)/2$ descendants of order p^7.

Algebra 6.160 is a one parameter family of algebras, with p algebras in the family, but only one of these algebras is capable. It has $(p^3+p^2)/2$ descendants of order p^7.

Algebra 6.160A is a one parameter family of algebras, with $(p+1)/2$ algebras in the family, but only one of these algebras is capable. It has $(p+3)/2$ descendants of order p^7.

Algebra 6.161 is a one parameter family of algebras, with $p-1$ algebras in the family, but only one of these algebras is capable. It has $2p-1$ descendants of order p^7.

Algebra 6.162 is a one parameter family of algebras, with $p-1$ algebras in the family, but only one of these algebras is capable. It has $2p-1$ descendants of order p^7.

Algebras 6.163 - 6.167 form a related family of p^2-p algebras, $\frac{5}{2}p-\frac{9}{2}+\frac{1}{2}\gcd(p-1,4)$ of which are capable. These algebras have a total of $\frac{1}{2}p^3+2p^2-5p+\frac{1}{2}+\frac{p}{2}\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.168 has $p^3+2p^2+2p+2+\gcd(p-1,3)$ descendants of order p^7.

Algebra 6.172 is a two parameter family of $p(p+1)/2$ algebras, but only one of these algebras is capable. It has $(p^4+p^2)/2$ descendants of order p^7.

Algebra 6.173 has $3p+3+\frac{1}{2}(p^2+2p+3)\gcd(p-1,3)$ descendants of order p^7.

Algebra 6.174 is a one parameter family of $(p+1)/2$ algebras, all but one having $p(p+1)/2$ descendants of order p^7, and one having $(p+1)^2/4$ descendants of order p^7 (This gives $(p^3+p^2+p+1)/4$ descendants in all.)

Algebra 6.175 is a one parameter family of $(p+1)/2$ algebras, all but one having $p(p+1)/2$ descendants of order p^7, and one having $(p+1)^2/4$ descendants of order p^7 (This gives $(p^3+p^2+p+1)/4$ descendants in all.)

Algebra 6.176 has $p+2p(p-1)/\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.177 only occurs when $p=1 \bmod 4$ and has $p+2p(p-1)/\gcd(p-1,4)$ descendants of order p^7. Thus between them 6.176 and 6.177 have $p(p-1)+\frac{p}{2}\gcd(p-1,4)$ descendants of order p^7.

Algebra 6.178 is a four parameter family of $p^2+(p+1)/2-\gcd(p-1,4)/2$ algebras, but only $(3p-1)/2$ of these algebras are capable. These $(3p-1)/2$ capable algebras have a total of $\frac{3}{2}p^2-\frac{1}{2}$ descendants of order p^7.

Algebra 6.179 is a two parameter family of $p(p+1)/2$ algebras, only one of which is capable. This capable algebra has $(p^4+p^2)/2$ descendants of order p^7.

So algebra 4.3 has a total of $5p+37+\gcd(p-1,4)$ capable descendants of order p^6 and p-class 3, and these have a total of

$$p^4+4p^3+17p^2+39p+72+(p^2+9p+47)\gcd(p-1,3)+(2p+8)\gcd(p-1,4)+2\gcd(p-1,5)+\gcd(p-1,7)$$

descendants of order p^7 and p-class 4.

42 Grandchildren of algbebra 25 (5.8)

Algebra 5.8 has 4 descendants of order p^6 (6.180 ˇ 6.183), but only 6.182 and 6.183 are capable. The corresponding group is number 25 in the list of capable groups of order p^5 from our paper. Algebra 6.182 has 4 descendants of order p^7 and p-class 4, and algebra 6.183 has 2.

6.182	4
6.183	2

So altogether algebra 5.8 has 6 grandchildren of order p^7 and p-class 4.

42.1 Descendants of 6.182

Algebra 6.182 has presentation

$$\langle a,b,c \mid ba,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.182 of order p^7 then L_4 is generated by p^3a. Adding a suitable scalar multiple of pa to b we can take $p^2b=0$, and adding a suitable scalar multiple of p^2a to c we can take $pc=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if in addition $p^2b'=pc'=0$ then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned}
p^3a' &= \alpha p^3a, \\
b'a' &= \alpha\delta ba + \alpha\varepsilon ca + (\beta\varepsilon - \gamma\delta)cb, \\
c'a' &= \alpha\xi ca + \beta\xi cb, \\
c'b' &= \delta\xi cb.
\end{aligned}$$

We can take $cb = 0$ or p^3a and if $cb = p^3a$ we can take $ba = ca = 0$. If $cb = 0$ we can take $ca = 0$ or p^3a and if $ca = p^3a$ we can take $ba = 0$. And ønally, if $ca = cb = 0$ we can take $ba = 0$ or p^3a. So we have four algebras.

$$\langle a, b, c \mid ba,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ba,\ ca,\ cb - p^3a,\ p^2b,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid ba,\ ca - p^3a,\ cb,\ p^2b,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid ba - p^3a,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 4\rangle.$$

42.2 Descendants of 6.183

Algebra 6.183 has presentation

$$\langle a, b, c \mid ba - p^2a,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.183 of order p^7 then L_4 is generated by p^3a. Adding a suitable scalar multiple of pa to b we can take $p^2b = 0$, and adding a suitable scalar multiple of p^2a to c we can take $pc = 0$. And adding suitable scalar multiples of pb to b and c we can take $ba - p^2a = ca = 0$. Scaling c we can take $cb = 0$ or p^3a. So we have two algebras

$$\langle a, b, c \mid ba - p^2a,\ ca,\ cb,\ p^2b,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ba - p^2a,\ ca,\ cb - p^3a,\ p^2b,\ pc,\ \text{class } 4\rangle.$$

43 Grandchildren of algebra 26 (5.9)

Algebra 5.9 has 23 descendants, but only 9 of them (6.184, 6.187 ˘ 6.192, 6.197, 6.198) are capable. The corresponding group is number 26 from the list of capable groups of order p^5 from our paper. Algebra 6.184 has 9 descendants of order p^7 and p-class 4, 6.187 has $11 + 4\gcd(p-1, 3) + 2\gcd(p-1, 4)$, 6.188 has $3 + 2\gcd(p-1, 3)$, 6.189 has $\frac{5}{2}p + \frac{7}{2}$, 6.190 has 2, 6.191 has $\frac{5}{2}p + \frac{7}{2}$, 6.192 has 2, 6.197 has $11 + 4\gcd(p-1, 3) + 2\gcd(p-1, 4)$, and 6.198 has $4 + \gcd(p-1, 3)$.

6.184	9
6.187	$11 + 4\gcd(p-1, 3) + 2\gcd(p-1, 4)$
6.188	$3 + 2\gcd(p-1, 3)$
6.189	$(5p + 7)/2$
6.190	2
6.191	$(5p + 7)/2$
6.192	2
6.197	$11 + 4\gcd(p-1, 3) + 2\gcd(p-1, 4)$
6.198	$4 + \gcd(p-1, 3)$

So, in all, algebra 5.9 has $5p + 49 + 11\gcd(p-1, 3) + 4\gcd(p-1, 4)$ grandchildren of order p^7 and p-class 4.

Algebra 6.184 has presentation

$$\langle a,b,c \,|\, baa,\ bab,\ ca,\ cb,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.184 of order p^7 then L_4 is generated by p^3a. Adding suitable scalar multiples of p^2a to b and c we can take $pb = pc = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $p^2b' = pc' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c \text{ modulo } L_2, \\ c' &= \xi c + \rho ba \text{ modulo } L_3 + pL, \end{aligned}$$

and

$$\begin{aligned} p^3a' &= \alpha p^3 a, \\ b'a'a' &= \alpha^2\delta baa + \alpha\beta\delta bab, \\ b'a'b' &= \alpha\delta^2 bab, \\ c'a' &= \alpha\rho baa + \beta\rho bab + \alpha\xi ca + \beta\xi cb, \\ c'b' &= \delta\rho bab + \delta\xi cb. \end{aligned}$$

We can take $bab = 0$, p^3a or ωp^3a.

If $bab = 0$ we can take $baa = 0$ or p^3a, and if $bab \neq 0$ we can take $baa = 0$.

If $baa = bab = 0$ then we can take $cb = 0$ or p^3a. If $baa = bab = cb = 0$ we can take $ca = 0$ or p^3a. And if $baa = bab = 0$, $cb = p^3a$ we can take $ca = 0$. (3 algebras.)

If $baa = p^3a$, $bab = 0$ then we need $\delta = \alpha^{-1}$. We can take $cb = 0$ or p^3a and $ca = 0$. (2 algebras.)

If $bab = p^3a$ or ωp^3a, $baa = 0$ then we need $\beta = 0$ and $\delta = \pm 1$. We can take $cb = 0$ and $ca = 0$ or p^3a. (4 algebras.)

$$\langle a,b,c \,|\, baa,\ bab,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab,\ ca - p^3a,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab,\ ca,\ cb - p^3a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa - p^3a,\ bab,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa - p^3a,\ bab,\ ca,\ cb - p^3a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab - p^3a,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab - p^3a,\ ca - p^3a,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab - \omega p^3a,\ ca,\ cb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, baa,\ bab - \omega p^3a,\ ca - p^3a,\ cb,\ pb,\ pc,\ \text{class } 4\rangle.$$

Algebra 6.187 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.187 of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of baa to c we can take $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $p^2b' = pc' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c \text{ modulo } L_2, \\ c' &= \xi c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \alpha^3 \delta baaa, \\ b'a'b' &= \alpha\delta^2 bab, \\ p^2a' &= \alpha p^2 a, \\ c'b' &= \delta\xi cb. \end{aligned}$$

If $p^2a \neq 0$ we can add suitable scalar multiples of pa to b and c so that $pb = pc = 0$. And if $p^2a = 0$ then we have

$$\begin{aligned} pb' &= \delta pb + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $p^2a = 0$ or $baaa$.

Consider the case when $p^2a = baaa$. Then we need $\delta = \alpha^{-2}$, and (as mentioned above) we can take $pb = pc = 0$. We then have

$$\begin{aligned} b'a'a'a' &= \alpha baaa, \\ b'a'b' &= \alpha^{-3} bab, \\ c'b' &= \alpha^{-2}\xi cb. \end{aligned}$$

So we can take $cb = 0$ or $baaa$ and we can take $bab = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$. ($2 + 2\gcd(p-1, 4)$ algebras.)

Now consider the case when $p^2a = 0$. We can take bab and cb independently equal to 0 or $baaa$.

If $bab = cb = 0$ then we can take $pc = 0$ or $baaa$. If $pc = baaa$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. ($2 + \gcd(p-1, 3)$ algebras.)

If $bab = 0$, $cb = baaa$ then we need $\xi = \alpha^3$. We can take $pc = 0$ or $baaa$. If $pc = baaa$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. ($2 + \gcd(p-1, 3)$ algebras.)

If $bab = baaa$, $cb = 0$ then we need $\delta = \alpha^2$, but again we can take $pc = 0$ or $baaa$. If $pc = baaa$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. ($2 + \gcd(p-1, 3)$ algebras.)

Finally, if $bab = cb = baaa$ then we need $\delta = \alpha^2$, $\xi = \alpha^3$ so that

$$\begin{aligned} b'a'a'a' &= \alpha^5 baaa, \\ pb' &= \alpha^2 pb + \varepsilon pc, \\ pc' &= \alpha^3 pc. \end{aligned}$$

We can then take $pc = 0$, $baaa$ or $\omega baaa$. If $pc \neq 0$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. ($3 + \gcd(p-1,3)$ algebras.)

So we have $11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - \omega baaa,\ ca,\ cb,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - \omega^2 baaa,\ ca,\ cb,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab - \omega^3 baaa,\ ca,\ cb,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - \omega baaa,\ ca,\ cb - baaa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - \omega^2 baaa,\ ca,\ cb - baaa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab - \omega^3 baaa,\ ca,\ cb - baaa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ p^2a,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ p^2a,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ p^2a,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ p^2a,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle.$$

43.3 Descendants of 6.188

Algebra 6.188 has presentation

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.188 of order p^7 then L_4 is generated by $baaa$. Adding suitable scalar multiples of c to a and b we can take $cb - baa = bab = 0$, and addding a suitable scalar multiple of baa to c we can take $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $b'a'b' = c'b - b'a'a' = c'a' = 0$ then

$$\begin{aligned}
a' &= \alpha a + \beta b \text{ modulo } L_2,\\
b' &= \delta b \text{ modulo } L_2,\\
c' &= \alpha^2 c - \alpha\beta ba \text{ modulo } L_3 + pL,
\end{aligned}$$

and

$$\begin{aligned}
b'a'a'a' &= \alpha^3\delta baaa,\\
p^2a' &= \alpha p^2 a.
\end{aligned}$$

So we can take $p^2a = 0$ or $baaa$. If $p^2a = baaa$ then adding suitable scalar multiples of pa to b and c we can take $pb = pc = 0$. And if $p^2a = 0$ we have

$$\begin{aligned}
pb' &= \delta pb,\\
pc' &= \alpha^2 pc
\end{aligned}$$

so we can take $pc = 0$ or $baaa$ and we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

We have $3 + 2\gcd(p-1, 3)$ algebras

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - baaa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega^2 baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3).$$

43.4 Descendants of 6.189

Algebra 6.189 has presentation

$$\langle a, b, c \,|\, bab,\ ca,\ cb,\ p^2a,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.189 of order p^7 then L_4 is generated by $baaa$. Adding suitable scalar multiple of pa and baa to c we can take $ca = cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \pm a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c \text{ modulo } L_2, \\ c' &= \xi c \text{ modulo } L_4, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \pm\delta baaa, \\ b'a'b' &= \pm\delta^2 bab, \\ p^2a' &= \pm p^2 a, \\ pc' &= \xi pc. \end{aligned}$$

If $p^2a \neq 0$ then we can add a suitable scalar multiple of pa to b so that $pb - baa = 0$, and if $p^2a = 0$ we have

$$pb' - b'a'a' = \delta(pb - baa) + \varepsilon pc \mp \beta\delta bab.$$

So if any of bab, p^2a or pc are non-zero then we can take $pb - baa = 0$, but if $bab = p^2a = pc = 0$ then we can take $pb - baa = xbaaa$ with $0 \leq x \leq (p-1)/2$.

We can take $pc = 0$ or baa and we can take $bab = 0$ or $baaa$. If $bab = 0$ we can take $p^2a = 0$ or $baaa$ and if $bab = baaa$ then we can take $p^2a = xbaaa$ with $0 \leq x < p$.

We have $\frac{5}{2}p + \frac{7}{2}$ algebras.

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - baa - xbaaa,\ pc,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a - baaa,\ pb - baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ p^2a - xbaaa,\ pb - baa,\ pc,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - baa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a - baaa,\ pb - baa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ p^2a - xbaaa,\ pb - baa,\ pc - baaa,\ \text{class } 4\rangle\ (0 \leq x < p).$$

43.5 Descendants of 6.190

Algebra 6.190 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.190 of order p^7 then L_4 is generated by $baaa$. Adding suitable scalar multiples of c to b, pa and baa to c and b and c to a we can take

$$bab = ca = cb - baa = pb - baa = pc = 0.$$

If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition

$$b'a'b' = c'a' = c'b' - b'a'a' = pb' - b'a'a' = pc' = 0$$

then

$$\begin{aligned} a' &= \pm a \text{ modulo } L_2, \\ b' &= \delta b \text{ modulo } L_2, \\ c' &= c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \pm\delta baaa, \\ p^2a' &= \pm pa. \end{aligned}$$

So we can take $p^2a = 0$ or $baaa$.

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a - baaa,\ pb - baa,\ pc,\ \text{class } 4\rangle.$$

43.6 Descendants of 6.191

Algebra 6.191 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descentants of 6.189, and we have $\frac{5}{2}p + \frac{7}{2}$ algebras.

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega baa - xbaaa,\ pc,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a - baaa,\ pb - \omega baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ p^2a - xbaaa,\ pb - \omega baa,\ pc,\ \text{class } 4\rangle\ (0 \le x < p),$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a,\ pb - \omega baa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ p^2a - baaa,\ pb - \omega baa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ p^2a - xbaaa,\ pb - \omega baa,\ pc - baaa,\ \text{class } 4\rangle\ (0 \le x < p).$$

43.7 Descendants of 6.192

Algebra 6.192 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.190, and we have

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a,\ pb - \omega baa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ p^2a - baaa,\ pb - \omega baa,\ pc,\ \text{class } 4\rangle.$$

43.8 Descendants of 6.197

Algebra 6.197 has presentation

$$\langle a, b, c \mid baa,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.197 of order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of bab to c we may take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c \text{ modulo } L_2, \\ c' &= \xi c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'b'b' &= \alpha\delta^3 babb, \\ b'a'a' &= \alpha^2\delta baa, \\ p^2a' &= \alpha p^2 a, \\ c'a' &= \alpha\xi ca. \end{aligned}$$

If $p^2a \neq 0$ we can add suitable scalar multiples of pa to b and c so that $pb = pc = 0$. And if $p^2a = 0$ we have

$$\begin{aligned} pb' &= \delta pb + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

So we can take $baa = 0$ or $babb$, $p^2a = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and $c'a' = 0$ or $babb$.

As mentioned above, if $p^2a \neq 0$ we can take $pb = pc = 0$.

If $baa = p^2a = ca = 0$ then we can take $pc = 0$ or $babb$. If $pc \neq 0$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$ or $babb$.

If $baa = p^2a = 0$, $ca = babb$ then again we can take $pc = 0$ or $babb$. If $pc \neq 0$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$ or $babb$.

If $baa = babb$ and $p^2a = ca = 0$ then we need $\alpha = \delta^2$. We can take $pc = 0$ or $babb$ and if $pc = babb$ we can take $pb = 0$. But if $pc = 0$ then we can take $pb = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 4$) $\omega^2 babb$ or $\omega^3 babb$.

Finally, if $p^2a = 0$ and $baa = ca = babb$ then we need $\alpha = \delta^2$, $\xi = \delta^3$ so that we have

$$\begin{aligned} b'a'b'b' &= \delta^5 babb, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \delta^3 pc. \end{aligned}$$

This implies that we can take $pc = 0$, $babb$ or $\omega babb$. If $pc \neq 0$ we can take $pb = 0$, and if $pc = 0$ we can take $pb = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 4$) $\omega^2 babb$ or $\omega^3 babb$.

We have $11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid baa,\ ca,\ cb,\ p^2a - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baa,\ ca,\ cb,\ p^2a - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa,\ ca,\ cb,\ p^2a - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa,\ ca - babb,\ cb,\ p^2a - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baa,\ ca - babb,\ cb,\ p^2a - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa,\ ca - babb,\ cb,\ p^2a - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa - babb,\ ca,\ cb,\ p^2a - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baa - babb,\ ca,\ cb,\ p^2a - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa - babb,\ ca,\ cb,\ p^2a - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa - babb,\ ca - babb,\ cb,\ p^2a - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baa - babb,\ ca - babb,\ cb,\ p^2a - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid baa - babb,\ ca - babb,\ cb,\ p^2a - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa,\ ca,\ cb,\ p^2a,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa,\ ca-babb,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa,\ ca-babb,\ cb,\ p^2a,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa,\ ca-babb,\ cb,\ p^2a,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb-\omega babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb-\omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$
$$\langle a,b,c \,|\, baa-babb,\ ca,\ cb,\ p^2a,\ pb-\omega^3 babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb,\ pc-\omega babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb-\omega babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb-\omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$
$$\langle a,b,c \,|\, baa-babb,\ ca-babb,\ cb,\ p^2a,\ pb-\omega^3 babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4).$$

43.9 Descendants of 6.198

Algebra 6.198 has presentation

$$\langle a,b,c \,|\, baa,\ ca-bab,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.198 of order p^7 then L_4 is generated by $babb$. Adding suitable scalar multiples of c to a and b we can take $baa = ca - bab = 0$, and adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $b'a'a' = c'a' - b'a'b' = c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a \text{ modulo } L_2, \\ b' &= \delta b \text{ modulo } L_2, \\ c' &= \delta^2 c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'b'b' &= \alpha\delta^3 babb, \\ p^2a' &= \alpha p^2 a. \end{aligned}$$

If $p^2a \neq 0$ we can add suitable scalar multiples of pa to b and c so that $pb = pc = 0$. And if $p^2a = 0$ we have

$$\begin{aligned} pb' &= \delta pb, \\ pc' &= \delta^2 pc. \end{aligned}$$

We can take $p^2a = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. As mentioned above, if $p^2a \neq 0$ we can take $pb = pc = 0$. And if $p^2a = 0$ we can take pb, pc (independantly) to be 0 or $babb$.

We have $4 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baa,\ ca - bab,\ cb,\ p^2a,\ pb - babb,\ pc - babb,\ \text{class } 4\rangle.$$

44 Grandchildren of algebra 27 (5.10)

Algebra 5.10 has 5 descendants of order p^6 (6.207 ˇ 6.211), but only 6.207 is capable. The corresponding group is number 27 from the list of capable groups of order p^5 in our paper. Algebra 6.207 has 5 descendants of order p^7 and p-class 4.

6.207 5

44.1 Descendants of 6.207

Algebra 6.207 has presentation

$$\langle a, b, c \mid baa,\ ca,\ cb,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.207 of order p^7 then L_4 is generated by p^3a. Adding suitable scalar multiples of p^2a to b and c we can take $pb - ba = pc = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $pb' - b'a' = pc' = 0 = 0$ then

$$\begin{aligned}
a' &= a + \beta b + \gamma c \text{ modulo } L_2,\\
b' &= \delta b + \varepsilon c \text{ modulo } L_2,\\
c' &= \xi c + \rho ba \text{ modulo } L_3 + pL,
\end{aligned}$$

and

$$\begin{aligned}
p^3a' &= p^3a,\\
b'a'a' &= \delta baa,\\
c'a' &= \xi ca + \beta\xi cb + \rho baa\\
c'b' &= \delta\xi cb.
\end{aligned}$$

We can take $baa = 0$ or p^3a and we can independantly take $cb = 0$ or p^3a. If either of baa or cb are non-zero then we can take $ca = 0$, and if $baa = cb = 0$ we can take $ca = 0$ or p^3a.

So we have 5 algebras.

$$\langle a, b, c \mid baa,\ ca,\ cb,\ pb - ba,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid baa,\ ca,\ cb-p^3a,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baa-p^3a,\ ca,\ cb,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baa-p^3a,\ ca,\ cb-p^3a,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid baa,\ ca-p^3a,\ cb,\ pb-ba,\ pc,\ \text{class } 4\rangle.$$

45 Grandchildren of algebra 28 (5.11)

Algebra 5.11 has four descendants (6.212 ˇ 6.215) but only 6.212 and 6.215 are capable. The corresponding group is number 28 in the list of capable groups of order p^5 from our paper. Algebra 6.212 has 4 descendants of order p^7 and p-class 4, and algebra 6.215 has 3.

6.212	4
6.215	3

So, in all, algebra 5.11 has 7 grandchildren of order p^7 and p-class 4.

45.1 Descendants of 6.212

Algebra 6.212 has presentation

$$\langle a,b,c \mid\ ca,\ cb,\ pb,\ pc-ba,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.212 of order p^7 then L_4 is generated by p^3a. Adding suitable scalar multiples of p^2a to b and c we can take $pb=pc-ba=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if in addition $pb'=pc'-b'a'=0$ then

$$\begin{aligned} a' &= a+\beta b+\gamma c \text{ modulo } L_2, \\ b' &= \delta b \text{ modulo } L_2, \\ c' &= \alpha\delta c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} p^3a' &= \alpha p^3 a, \\ c'a' &= \alpha^2\delta ca+\alpha\beta\delta cb \\ c'b' &= \alpha\delta^2 cb. \end{aligned}$$

We can take $cb=0$, p^3a or ωp^3a, and if $cb\neq 0$ we can take $ca=0$. If $cb=0$ we can take $ca=0$ or p^3a. So we have four algebras

$$\langle a,b,c \mid\ ca,\ cb,\ pb,\ pc-ba,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid\ ca-p^3a,\ cb,\ pb,\ pc-ba,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid\ ca,\ cb-p^3a,\ pb,\ pc-ba,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid\ ca,\ cb-\omega p^3a,\ pb,\ pc-ba,\ \text{class } 4\rangle.$$

45.2 Descendants of 6.215

Algebra 6.215 has presentation

$$\langle a, b, c \mid ca - p^2a,\ cb,\ pb,\ pc - ba,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.215 of order p^7 then L_4 is generated by p^3a. Adding suitable scalar multiples of p^2a to b and c we can take $pb = pc - ba = 0$. Also, adding a suitable scalar multiple of ba to c we may take $ca - p^2a = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition

$$pb' = pc' - b'a' = ca - p^2a = 0$$

then

$$\begin{aligned} a' &= a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \alpha^{-1} b \text{ modulo } L_2, \\ c' &= c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} p^3a' &= \alpha p^3 a, \\ c'b' &= \alpha^{-1} cb. \end{aligned}$$

So we can take $cb = 0$, p^3a or ωp^3a giving

$$\langle a, b, c \mid ca - p^2a,\ cb,\ pb,\ pc - ba,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - p^2a,\ cb - p^3a,\ pb,\ pc - ba,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid ca - p^2a,\ cb - \omega p^3a,\ pb,\ pc - ba,\ \text{class } 4\rangle.$$

46 Grandchildren of algebra 29 (5.12)

Algebra 5.12 has 12 descendants of order p^6(6.216 ˘ 6.227), but only 6.216, 6.218 and 6.222 are capable. The corresponding group is number 29 in the list of capable groups of order p^5. Algebra 6.216 has 4 descendants of order p^7 and p-class 4. 6.218 has $11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$, and 6.222 has $2p + 5 + 3\gcd(p-1,3) + \gcd(p-1,4)$.

6.216	4
6.218	$11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$
6.222	$2p + 5 + 3\gcd(p-1,3) + \gcd(p-1,4)$

So, in all, algebra 5.12 has $2p + 20 + 7\gcd(p-1,3) + 3\gcd(p-1,4)$ grandchildren of order p^7 and p-class 4.

46.1 Descendants of 6.216

Algebra 6.216 has presentation

$$\langle a, b, c \mid baa,\ bab,\ ca,\ cb,\ pa,\ pb,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.216 of order p^7 then L_4 is generated by p^3c. Adding suitable scalar multiples of p^2c to a and b we can take $pa = pb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $pa' = pb' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \gamma a + \delta b \text{ modulo } L_2, \\ c' &= \xi c + \rho ba \text{ modulo } L_3 + pL, \end{aligned}$$

and

$$\begin{aligned} p^3c' &= \xi pc, \\ b'a'a' &= (\alpha\delta - \beta\gamma)(\alpha baa + \beta bab), \\ b'a'b' &= (\alpha\delta - \beta\gamma)(\gamma baa + \delta bab), \\ c'a' &= \alpha\xi ca + \beta\xi cb + \alpha\rho baa + \beta\rho bab, \\ c'b' &= \gamma\xi ca + \delta\xi cb + \gamma\rho baa + \delta\rho bab. \end{aligned}$$

One possibility is that $baa = bab = 0$, in which case we can take $ca = 0$ and $cb = 0$ or p^3c.

If baa, bab are not both zero then we can take $baa = 0$, $bab = p^3c$, though we then need to take $\beta = 0$ and $\xi = \alpha\delta^2$. This gives

$$\begin{aligned} p^3c' &= \alpha\delta^2 pc, \\ c'a' &= \alpha^2\delta^2 ca, \\ c'b' &= \gamma\alpha\delta^2 ca + \alpha\delta^3 cb + \delta\rho p^3 c \end{aligned}$$

so we can take $cb = 0$ and $ca = 0$ or p^3c.

So we have four algebras:

$$\langle a, b, c \mid baa,\ bab,\ ca,\ cb,\ pa,\ pb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid baa,\ bab,\ ca,\ cb - p^3c,\ pa,\ pb,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid baa,\ bab - p^3c,\ ca,\ cb,\ pa,\ pb,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid baa,\ bab - p^3c,\ ca - p^3c,\ cb,\ pa,\ pb,\ \text{class } 4\rangle.$$

46.2 Descendants of 6.218

Algebra 6.218 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa,\ pb,\ p^2c,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.218 of order p^7 then L_4 is generated by $baaa$. Adding suitable scalar multiples of baa to c we can take $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $c'a' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \delta b \text{ modulo } L_2, \\ c' &= \xi c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \alpha^3\delta baaa, \\ b'a'b' &= \alpha\delta^2 bab, \\ p^2c' &= \xi p^2c, \\ c'b' &= \delta\xi cb. \end{aligned}$$

If $p^2c \neq 0$ we can add suitable scalar multiples of pc to a and b so that $pa = pb = 0$, and if $p^2c = 0$ we have

$$\begin{aligned} pa' &= \alpha pa + \beta pb, \\ pb' &= \delta pb. \end{aligned}$$

We can take $bab = 0$ or $baaa$ and independantly take $p^2c = 0$ or $baaa$. If $pc = 0$ or $bab = 0$ we can take $cb = 0$ or $baaa$. If $bab = p^2c = baaa$ we need $\delta = \alpha^2$, $\xi = \alpha^5$ which gives

$$\begin{aligned} b'a'a'a' &= \alpha^5 baaa, \\ c'b' &= \alpha^7 cb, \end{aligned}$$

so we can take $cb = 0$, $baaa$ or $\omega baaa$.

If $p^2c = 0$ then we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb \neq 0$ we can take $pa = 0$. If $bab = pb = 0$ we can take $pa = 0$ or $baaa$, and if $bab = baaa$, $pb = 0$ then we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

We have $11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$ algebras:

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - \omega baaa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid bab,\ ca,\ cb - baaa,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa - \omega baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa - \omega baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa - \omega^2 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa - \omega^2 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa - \omega^3 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid bab - baaa,\ ca,\ cb - baaa,\ pa - \omega^3 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4).$$

46.3 Descendants of 6.222

Algebra 6.222 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.222 of order p^7 then L_4 is generated by $baaa$. Adding suitable scalar multiples of baa to c we can take $ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $ca = 0$, then

$$\begin{aligned} a' &= \alpha a + \beta b \text{ modulo } L_2, \\ b' &= \delta b \text{ modulo } L_2, \\ c' &= \alpha^2 c - \alpha\beta ba \text{ modulo } L_3 + pL, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \alpha^3\delta baaa, \\ b'a'b' &= \alpha\delta^2 bab, \\ p^2c' &= \alpha^2 p^2 c, \\ c'b' - b'a'a' &= \alpha^2\delta(cb - baa) - 2\alpha\beta\delta bab. \end{aligned}$$

If $p^2c \neq 0$ we can add suitable scalar multiples of pc to a and b so that $pa = pb = 0$, and if $p^2c = 0$ we have

$$\begin{aligned} pa' &= \alpha pa + \beta pb, \\ pb' &= \delta pb. \end{aligned}$$

We can take $bab = 0$ or $baaa$.

If $bab = 0$ then we can take $p^2c = 0$ or $baaa$ and independantly take $cb - baa = 0$ or $baaa$. If $bab = baaa$ then we can take $cb - baa = 0$ (though we then need $\beta = 0$), and we can take $p^2c = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

If $bab = p^2c = cb - baa = 0$ then we can take $pb = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb \neq 0$ we can take $pa = 0$, and if $pb = 0$ then we can take $pa = 0$ or $baaa$.

If $bab = p^2c = 0$, $cb - baa = baaa$ then we need $\alpha = 1$ and we can take $pb = xbaaa$ with $0 \leq x < p$. If $pb \neq 0$ then we can take $pa = 0$ and if $pb = 0$ we can take $pa = 0$ or $baaa$.

If $bab = baaa$, $p^2c = cb - baa = 0$ then we need $\delta = \alpha^2$ and $\beta = 0$ so we have

$$\begin{aligned} b'a'a'a' &= \alpha^5 baaa, \\ pa' &= \alpha pa, \\ pb' &= \alpha^2 pb. \end{aligned}$$

So we can take $pb = 0$ or $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb = 0$ we can take $pa = 0$, $baaa$ or $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$, and if $pb \neq 0$ we can take $pa = 0$ or $pa = xbaaa$ where x lies in a transversal for the cube roots of unity.

We have $2p + 5 + 3\gcd(p-1,3) + \gcd(p-1,4)$ algebras

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa - baaa,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa - baaa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c - baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb,\ p^2c - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa - baaa,\ pa,\ pb - xbaaa,\ p^2c,\ \text{class } 4\rangle\ (0 < x < p),$$

$$\langle a,b,c \mid bab,\ ca,\ cb - baa - baaa,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa - baaa,\ pa - baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - \omega baaa,\ pb,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - \omega^2 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - \omega^3 baaa,\ pb,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - xbaaa,\ pb - baaa,\ p^2c,\ \text{class } 4\rangle\ (x \text{ in a transversal for } 3^{rd} \text{ roots of } 1),$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - xbaaa,\ pb - \omega baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for } 3^{rd} \text{ roots of } 1$$

$$\langle a,b,c \mid bab - baaa,\ ca,\ cb - baa,\ pa - xbaaa,\ pb - \omega^2 baaa,\ p^2c,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for } 3^{rd} \text{ roots of}$$

47 Grandchildren of algebra 30 (5.13)

Algebra 5.13 has $p+1$ descendants of order p^6 (6.228 ˇ 6.230), but only 6.228 is capable. The corresponding group is number 30 in the list of capable groups of order p^5 from our paper. Algebra 6.228 has $p+1$ descendants of order p^7 and p-class 4.

6.228 $\quad p+1$

47.1 Descendants of 6.228

Algebra 6.228 has presentation

$$\langle a, b, c \mid ca,\ cb,\ pa-ba,\ pb,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.228 then L_4 is generated by p^3c. Adding suitable scalar multiples of p^2c to a and b we may take $pa-ba=pb=0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if in addition $pa-ba=pb=0$, then

$$\begin{aligned} a' &= \alpha a+\beta b \text{ modulo } L_2, \\ b' &= b \text{ modulo } L_2, \\ c' &= \xi c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} p^3c' &= \xi p^3c, \\ c'a' &= \alpha\xi ca+\beta\xi cb, \\ c'b' &= \xi cb. \end{aligned}$$

We can take $cb=xp^3c$ with $0\leq x<p$. If $cb\neq 0$ we can take $ca=0$, and if $cb=0$ we can take $ca=0$ or p^3c. So we have $p+1$ algebras

$$\langle a, b, c \mid ca,\ cb-xp^3c,\ pa-ba,\ pb,\ \text{class } 4\rangle\ (0\leq x<p),$$

$$\langle a, b, c \mid ca-p^3c,\ cb,\ pa-ba,\ pb,\ \text{class } 4\rangle.$$

48 Grandchildren of algebra 31 (5.14)

Algebra 5.14 has 35 descendants of order p^6 (6.231 ˇ 6.265), but only 6.231, 6.256 and 6.261 are capable. The corresponding group is number 31 from the list of capable groups of order p^5 in our paper. Algebra 6.231 has $p^2+5p+14+(p+17)\gcd(p-1,3)+2\gcd(p-1,4)+\gcd(p-1,7)+\gcd(p-1,8)$ descendants of order p^7 and p-class 4, 6.256 has $20+(p+11)\gcd(p-1,3)+4\gcd(p-1,4)$, and 6.261 has $4p+2+(p^2+3p+1)\gcd(p-1,3)+(p+1)\gcd(p-1,4)$.

6.231	$p^2+5p+14+(p+17)\gcd(p-1,3)+2\gcd(p-1,4)+\gcd(p-1,7)+\gcd(p-1,8)$
6.256	$20+(p+11)\gcd(p-1,3)+4\gcd(p-1,4)$
6.261	$4p+2+(p^2+3p+1)\gcd(p-1,3)+(p+1)\gcd(p-1,4)$

So, in all, algebra 5.14 has

$$p^2+9p+36+(p^2+5p+29)\gcd(p-1,3)+(p+7)\gcd(p-1,4)+\gcd(p-1,7)+\gcd(p-1,8)$$

grandchildren of order p^7 and p-class 4.

48.1 Descendants of 6.231

Algebra 6.231 has $p^2+5p+14+(p+17)\gcd(p-1,3)+2\gcd(p-1,4)+\gcd(p-1,7)+\gcd(p-1,8)$ descendants of order p^7 and p-class 4.

Algebra 6.231 has presentation

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.231 of order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of bab to c we can take $cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if in addition $pa-ba=pb=0$, then

$$\begin{aligned} a' &= \alpha a+\gamma c \text{ modulo } L_2,\\ b' &= \delta b+\varepsilon c \text{ modulo } L_2,\\ c' &= \xi c \text{ modulo } L_2, \end{aligned}$$

and

$$\begin{aligned} b'a'b'b' &= \alpha\delta^3 babb,\\ b'a'a' &= \alpha^2\delta baa+\alpha\gamma\delta bac+\alpha^2\varepsilon caa+\alpha\gamma\varepsilon cac,\\ b'a'c' &= \alpha\delta\xi bac+\alpha\varepsilon\xi cac,\\ c'a'a' &= \alpha^2\xi caa+\alpha\gamma\xi cac,\\ c'a'c' &= \alpha\xi^2 cac,\\ pa' &= \alpha pa+\gamma pc,\\ pb' &= \delta pb+\varepsilon pc,\\ pc' &= \xi pc. \end{aligned}$$

We can take $cac=0$ or $babb$.

If $cac=0$ we can take bac and caa independantly equal to 0 or $babb$. If either of bac or caa are non-zero then we can take $baa=0$, and if $cac=bac=caa=0$ then we can take $baa=0$ or $babb$.

If $cac=babb$ then we can take $bac=caa=0$, and we can take $baa=0$ or $babb$.

48.1.1 Case 1

Let $baa=bac=caa=cac=0$. Then we can take $pc=0$ or $babb$, and if $pc=babb$ we can take $pa=pb=0$. If $pc=0$ we can take $pb=0$ or $babb$ and we can take $pa=0$, $babb$ or (if $p=1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. ($3+2\gcd(p-1,3)$ algebras.)

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-\omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-\omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-babb,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-\omega babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa-\omega^2 babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc-babb,\ \text{class } 4\rangle.$$

48.1.2 Case 2

Let $baa = bac = cac = 0$, $caa = babb$. Then we need $\xi = \alpha^{-1}\delta^3$ and $\varepsilon = 0$, and so we have

$$\begin{aligned} b'a'b'b' &= \alpha\delta^3 babb, \\ pa' &= \alpha pa + \gamma pc, \\ pb' &= \delta pb, \\ pc' &= \alpha^{-1}\delta^3 pc. \end{aligned}$$

We can take $pc = 0$, $babb$ or $\omega babb$. If $pc = 0$ we can take $pb = 0$ or $babb$ and we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pc = babb$ or $\omega babb$ then we can take $pa = 0$ and we can take $pb = 0$, $babb$ or (if $p = 1 \bmod 4$) $\omega babb$. $(4 + 2\gcd(p-1,3) + \gcd(p-1,4)$ algebras.)

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - babb,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - \omega babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa - \omega^2 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb - babb,\ pc - babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb - \omega babb,\ pc - babb,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb,\ pc - \omega babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb - babb,\ pc - \omega babb,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ baa,\ bac,\ caa - babb,\ cac,\ pa,\ pb - \omega babb,\ pc - \omega babb,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

48.1.3 Case 3

Let $baa = cac = caa = 0$, $bac = babb$. Then we need $\xi = \delta^2$ and $\gamma = 0$, so we have

$$\begin{aligned} b'a'b'b' &= \alpha\delta^3 babb, \\ pa' &= \alpha pa, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \delta^2 pc. \end{aligned}$$

We can take $pc = 0$ or $babb$. If $pc = 0$ we can take $pb = 0$ or $babb$ and we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pc = babb$ we can take $pb = 0$ and we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. $(3 + 3\gcd(p-1,3)$ algebras.)

$$\langle a,b,c \mid cb,\ baa,\ bac - babb,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac - babb,\ caa,\ cac,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-babb,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega^2 babb,\ pb-babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-babb,\ pb,\ pc-babb,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega babb,\ pb,\ pc-babb,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa,\ cac,\ pa-\omega^2 babb,\ pb,\ pc-babb,\ \text{class } 4\rangle\ (p=1 \bmod 3).$$

48.1.4 Case 4

Let $baa=cac=0$ and $bac=caa=babb$. Then we need $\delta=\alpha$, $\xi=\alpha^2$ and $\alpha\gamma\delta+\alpha^2\varepsilon=0$, and so we have

$$\begin{aligned} b'a'b'b' &= \alpha^4 babb, \\ pa' &= \alpha pa+\gamma pc, \\ pb' &= \alpha pb-\gamma pc, \\ pc' &= \alpha^2 pc. \end{aligned}$$

We can take $pc=0$, $babb$ or $\omega babb$. If $pc=0$ we can take $pa=0$, $babb$ or (if $p=1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pa=pc=0$ we can take $pb=0$, $babb$ or (if $p=1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and if $pa=babb$, $\omega babb$ or $\omega^2 babb$ and $pc=0$ then we can take $pb=xbabb$ with $0\le x<p$. If $pc=babb$ or $\omega babb$ then we can take $pa=0$ and $pb=xbabb$ with $0\le x\le (p-1)/2$. ($p+2+(p+1)\gcd(p-1,3)$ algebras.)

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa-babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa-\omega babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0\le x<p),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa-\omega^2 babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0\le x<p),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb-babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb-\omega babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb-\omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb-xbabb,\ pc-babb,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c \mid cb,\ baa,\ bac-babb,\ caa-babb,\ cac,\ pa,\ pb-xbabb,\ pc-\omega babb,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2).$$

48.1.5 Case 5

Let $baa = babb$, $bac = caa = cac = 0$. The we need $\alpha = \delta^2$ and we have

$$\begin{aligned} b'a'b'b' &= \delta^5 babb, \\ pa' &= \delta^2 pa + \gamma pc, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $pc = 0$ or $babb$ and if $pc = babb$ we can take $pa = pb = 0$. If $pc = 0$ we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $pa = pc = 0$ we can take $pb = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 4$) $\omega^2 babb$ or $\omega^3 babb$, and if $pc = 0$, $pa = babb$, $\omega babb$ or $\omega^2 babb$ then we can take $pb = 0$ or $xbabb$ where x lies in a transversal for the cube roots of unity. ($p + 1 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.)

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb - \omega babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^3 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa-babb,\ bac,\ caa,\ cac,\ pa-babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (x \text{ in a transversal for the cube roots of } 1),$$

$$\langle a, b, c \mid cb,\ baa-babb,\ bac,\ caa,\ cac,\ pa-\omega babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for the cube r}$$

$$\langle a, b, c \mid cb,\ baa-babb,\ bac,\ caa,\ cac,\ pa-\omega^2 babb,\ pb-xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in a transversal for the cube}$$

$$\langle a, b, c \mid cb,\ baa - babb,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle.$$

48.1.6 Case 6

Let $baa = bac = caa = 0$, $cac = babb$. Then we need $\xi^2 = \delta^3$, so that we can write $\delta = k^2$, $\xi = \pm k^3$. We also need $\gamma = \varepsilon = 0$ so we have

$$\begin{aligned} b'a'b'b' &= \alpha k^6 babb, \\ pa' &= \alpha pa, \\ pb' &= k^2 pb, \\ pc' &= \pm k^3 pc. \end{aligned}$$

We can take $pa = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 3$) $\omega^2 babb$, $\omega^3 babb$, $\omega^4 babb$ or $\omega^5 babb$ and (independantly) take $pb = 0$ or $babb$.

If $pa = 0$ or $pb = 0$ we can take $pc = 0$ or $babb$.

If $pa \neq 0$ and $pb = babb$ then we can take $pc = 0$ or $xbabb$ where x lies in a transversal for the sixth roots of unity.

So we have $p + 3 + 6\gcd(p-1, 3)$ algebras in this case.

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa,\ pb - babb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega babb,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^2 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^3 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^4 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^5 babb,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - babb,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega babb,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^2 babb,\ pb,\ pc - babb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^3 babb,\ pb,\ pc - babb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^4 babb,\ pb,\ pc - babb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^5 babb,\ pb,\ pc - babb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - babb,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega babb,\ pb - babb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^2 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^3 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^4 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^5 babb,\ pb - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^2 babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^3 babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac,\ pa - \omega^4 babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ baa,\ bac,\ caa,\ cac - babb,\ pa - \omega^5 babb,\ pb - babb,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

where x lies in a transversal for the sixth roots of unity.

48.1.7 Case 7

Finally, let $baa = cac = babb$, $bac = caa = 0$. Then (as in Case 6) we can write $\delta = k^2$, $\xi = \pm k^3$ and we need $\alpha = k^4$, $\gamma = \varepsilon = 0$ so we have

$$\begin{aligned} b'a'b'b' &= k^{10}babb, \\ pa' &= k^4 pa, \\ pb' &= k^2 pb, \\ pc' &= \pm k^3 pc. \end{aligned}$$

We can take $pa = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 3$) $\omega^2 babb$, $\omega^3 babb$, $\omega^4 babb$ or $\omega^5 babb$.

If $pa = 0$ we can take $pb = 0$ or $\omega^i babb$ where $i = 0, 1$ if $p = 3 \bmod 4$, $i = 0, 1, 2, 3$ if $p = 5 \bmod 8$, and $i = 0, 1, 2, 3, 4, 5, 6, 7$ if $p = 1 \bmod 8$.

If $pa = pb = 0$ then we can take $pc = 0$ or $babb$ or (if $p = 1 \bmod 7$) $\omega^j babb$ where $j = 1, 2, 3, 4, 5, 6$.

If $pa = 0$, $pb = \omega^i babb$ then we can take $pc = 0$ or $pc = xbabb$ where x lies in transversal for the 8^{th} roots of unity.

If $pa = \omega^i babb$ (with $0 \leq i \leq 5$) then we can take $pb = 0$ or $xbabb$ where x lies in a transversal for the cube roots of unity.

If $pa = \omega^i babb$ and $pb = 0$ then we can take $pc = 0$ or $xbabb$ where x lies in a transversal for the sixth roots of unity.

And if $pa = \omega^i babb$ and $pb = xbabb$ with $x \neq 0$ then we can take $pc = ybabb$ where $0 \leq y \leq (p-1)/2$.

So we have a total of $p^2 + 2p - 2 + 2\gcd(p-1, 3) + \gcd(p-1, 8) + \gcd(p-1, 7)$ algebras in this case.

$$\langle a, b, c \,|\, cb,\ baa - babb,\ bac,\ caa,\ cac - babb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ baa - babb,\ bac,\ caa,\ cac - babb,\ pa,\ pb,\ pc - babb,\ \text{class } 4\rangle,$$

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa,\ pb,\ pc-\omega^j babb,\ \text{class } 4\rangle$ $(p = 1 \bmod 7,\ j = 1, 2, \ldots, 6)$,

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa,\ pb-\omega^i babb,\ pc,\ \text{class } 4\rangle$ $(0 \leq i < \gcd(p-1, 8))$,

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa,\ pb-\omega^i babb,\ pc-xbabb,\ \text{class } 4\rangle$ $(0 \leq i < \gcd(p-1, 8))$,

x in a transversal for the 8^{th} roots of unity,

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa-\omega^i babb,\ pb,\ pc,\ \text{class } 4\rangle$ $(0 \leq i < 2\gcd(p-1, 3))$,

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa-\omega^i babb,\ pb,\ pc-xbabb,\ \text{class } 4\rangle$ $(0 \leq i < 2\gcd(p-1, 3))$,

x in a transversal for the sixth roots of unity,

$\langle a, b, c \,|\, cb,\ baa-babb,\ bac,\ caa,\ cac-babb,\ pa-\omega^i babb,\ pb-xbabb,\ pc-ybabb,\ \text{class } 4\rangle$ $(0 \leq i < 2\gcd(p-1, 3))$,

x in a transversal for the cube roots of unity, $0 \leq y \leq (p-1)/2$.

48.2 Descendants of 6.256

Algebra 6.256 has $20 + (p+11)\gcd(p-1, 3) + 4\gcd(p-1, 4)$ descendants of order p^7 and p-class 4.

Algebra 6.256 has presentation

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.256 of order p^7 then L_4 is generated by $baaa$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \delta b + \varepsilon c + \lambda ba + \mu ca \text{ modulo } L_3, \\ c' &= \xi c + \rho ba + \sigma ca \text{ modulo } L_3, \end{aligned}$$

and

$$\begin{aligned} b'a'a'a' &= \alpha^3\delta baaa, \\ b'a'b' &= \alpha\delta^2 bab + 2\alpha\delta\varepsilon bac + \alpha\varepsilon^2 cac, \\ b'a'c' &= \alpha\delta\xi bac + \alpha\varepsilon\xi cac, \\ c'a'a' &= \alpha^2\rho baaa + \alpha\beta\xi bac + \alpha^2\xi caa + \alpha\gamma\xi cac, \\ c'a'c' &= \alpha\xi^2 cac, \\ c'b' &= \delta\rho bab + (\delta\sigma + \varepsilon\rho - \lambda\xi)bac + (\varepsilon\sigma - \mu\xi)cac + \delta\xi cb, \\ pa' &= \alpha pb + \beta pb + \gamma pc, \\ pb' &= \delta pb + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $cac = 0$ or $baaa$.

If $cac = 0$ we can take $bac = 0$ or $baaa$.

If $bac = cac = 0$ we can take $bab = 0$ or $baaa$.

If $bab = bac = cac = 0$ then we can take $caa = 0$ and $cb = 0$ or $baaa$.

If $bac = cac = 0$ and $bab = baaa$ then we can take $caa = 0$ and $cb = 0$ or $baaa$.

If $cac = 0$ and $bac = baaa$ then we can take $bab = caa = cb = 0$.

If $cac = baaa$ then we can take $bac = caa = cb = 0$ and $bab = 0$, $baaa$ or $\omega baaa$.

For each of these 8 choices of commutator relations we give the most general possible description (modulo L_2) for generators a', b', c' of L which satisfy the relations, give the value of $b'a'a'a'$, and then compute the possibilities for pa, pb, pc.

48.2.1 Case 1

Let $bab = bac = caa = cac = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a' &= \alpha^3\delta baaa. \end{aligned}$$

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$ or $baaa$.

So we have $3 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

48.2.2 Case 2

Let $bab = bac = caa = cac = 0$, $cb = baaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \alpha^3 c, \\ b'a'a'a' &= \alpha^3 \delta baaa. \end{aligned}$$

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$ or $baaa$.

So we have $3 + \gcd(p-1, 3)$ algebras.

$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cb - baaa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

48.2.3 Case 3

Let $bab = baaa$, $bac = caa = cac = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

So we have $2 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.

$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \,|\, cb,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

48.2.4 Case 4

Let $bab = cb = baaa$, $bac = caa = cac = 0$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \alpha^3 c, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

We can take $pc = 0$ $baaa$ or $\omega baaa$, and if $pc = baaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

So we have $3 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa - \omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle.$$

48.2.5 Case 5

Let $bab = caa = cac = cb = 0$, $bac = baaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \alpha^2 c, \\ b'a'a'a' &= \alpha^3 \delta baaa. \end{aligned}$$

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = 0$.

We can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$ or $baaa$.

So we have $3 + 2\gcd(p-1, 3)$ algebras.

$$\langle a, b, c \mid cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - baaa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac - baaa,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle.$$

48.2.6 Case 6

Let $bab = bac = caa = cb = 0$, $cac = baaa$.

$$\begin{aligned}
a' &= \alpha a + \beta b + \gamma c,\\
b' &= \alpha^{-2}\xi^2 b,\\
c' &= \xi c,\\
b'a'a'a' &= \alpha\xi^2 baaa.
\end{aligned}$$

We can take $pc = 0$ or $baaa$, and if $pc = baaa$ we can take $pa = 0$.

We can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$ baa or $\omega baaa$.

So we have $4 + 2\gcd(p-1, 3)$ algebras.

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - baaa,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - \omega baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ bab,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - \omega^2 baaa,\ pc - baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

48.2.7 Case 7

Let $bac = caa = cb = 0$, $bab = cac = baaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^2 b, \\ c' &= \pm\alpha^2 c, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

We can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = 0$ we can take $pc = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pc \neq 0$ we can take $pa = 0$.

If $pb \neq 0$ we can take $pc = xbaaa$ where $0 \leq x \leq (p-1)/2$.

If $pb = pc = 0$ then we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

So we have $1 + \frac{p+3}{2}\gcd(p-1,3) + \gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-\omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x \leq (p-1)/2),$$

$$\langle a,b,c \mid cb,\ bab-baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega^2 baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x \leq (p-1)/2).$$

48.2.8 Case 8

Let $bab = \omega baaa$, $bac = caa = cb = 0$, $cac = baaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^2 b, \\ c' &= \pm\alpha^2 c, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

This case is identical to Case 7, so we have $1 + \frac{p+3}{2}\gcd(p-1,3) + \gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-baaa,\ \text{class } 4\rangle,$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb,\ pc-\omega^2 baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (0 \le x \le (p-1)/2),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0 \le x \le (p-1)/2),$$
$$\langle a,b,c \mid cb,\ bab-\omega baaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega^2 baaa,\ pc-xbaaa,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0 \le x \le (p-1)/2).$$

48.3 Descendants of 6.261

Algebra 6.261 has $4p+2+(p^2+3p+1)\gcd(p-1,3)+(p+1)\gcd(p-1,4)$ descendants of order p^7 and p-class 4.

Algebra 6.261 has presentation

$$\langle a,b,c \mid cb-baa,\ bab,\ bac,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is descendant of 6.261 or order p^7 then L_4 is generated by $baaa$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, then

$$\begin{aligned}
a' &= \alpha a+\beta b+\gamma c \text{ modulo } L_2,\\
b' &= \delta b+\varepsilon c+\lambda ba+\mu ca \text{ modulo } L_3,\\
c' &= \alpha^2 c+\rho ba+\sigma ca \text{ modulo } L_3,
\end{aligned}$$

and

$$\begin{aligned}
b'a'a'a' &= \alpha^3\delta baaa,\\
b'a'b' &= \alpha\delta\varepsilon baaa+\alpha\delta^2 bab+2\alpha\delta\varepsilon bac+\alpha\varepsilon^2 cac,\\
b'a'c' &= \alpha^3\delta bac+\alpha^3\varepsilon cac,\\
c'a'a' &= (2\alpha^3\beta+\alpha^2\rho)baaa+\alpha^3\beta bac+\alpha^4 caa+\alpha^3\gamma cac,\\
c'a'c' &= \alpha^5 cac,\\
c'b'-b'a'a' &= \text{a great mess!}\\
pa' &= \alpha pb+\beta pb+\gamma pc,\\
pb' &= \delta pb+\varepsilon pc,\\
pc' &= \alpha^2 pc.
\end{aligned}$$

We can take $cac=0$ or $baaa$.

If $cac=0$ we can take $bac=xbaaa$ with $0 \le x < p$, and if $cac=baaa$ then we can take $bac=0$.

If $cac=0$ and $bac=xbaaa$ with $x \ne -\frac{1}{2}$ then we can take $bab=caa=0$, and if $cac=0$ and $bac=-\frac{1}{2}baaa$ then we can take $caa=0$ and $bab=0$ or $baaa$.

If $cac=baaa$ we can take $bac=caa=0$ and $bab=xbaaa$ with $0 \le x < p$.

It turns out that in every case we can take $cb-baa=0$.

For each of these choices of commutator relations we give the most general possible description (modulo L_2) for generators a',b',c' of L which satisfy the relations, give the value of $b'a'a'a'$, and then compute the possibilities for pa,pb,pc.

48.3.1 Case 1

Let $cb-baa=bab=caa=cac=0$, $bac=xbaaa$.

If $x\neq -1,-\frac{1}{2}$ we have

$$\begin{aligned}
a' &= \alpha a+\beta b+\gamma c \text{ modulo } L_2,\\
b' &= \delta b \text{ modulo } L_2,\\
c' &= \alpha^2 c \text{ modulo } L_2,\\
b'a'a'a' &= \alpha^3\delta baaa.
\end{aligned}$$

We can take $pc=0$ or $baaa$, and we can take $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

If either of pb,pc are non-zero we can take $pa=0$, and if $pb=pc=0$ we can take $pa=0$ or $baaa$.

We have $(p-2)(3+2\gcd(p-1,3))$ algebras.

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle\ (x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb,\ pc-baaa,\ \text{class } 4\rangle\ (x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-baaa,\ pc,\ \text{class } 4\rangle\ (x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-baaa,\ pc-baaa,\ \text{class } 4\rangle\ (x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-\omega baaa,\ pc-baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3, x\neq -1,-\frac{1}{2}),$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac-xbaaa,\ caa,\ cac,\ pa,\ pb-\omega^2 baaa,\ pc-baaa,\ \text{class } 4\rangle\ (p=1 \bmod 3, x\neq -1,-\frac{1}{2}),$$

If $x=-1$ we have

$$\begin{aligned}
a' &= \alpha a+\beta b \text{ modulo } L_2,\\
b' &= \delta b \text{ modulo } L_2,\\
c' &= \alpha^2 c \text{ modulo } L_2,\\
b'a'a'a' &= \alpha^3\delta baaa.
\end{aligned}$$

We can take $pc=0$ or $baaa$, and we can take $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

If $pb\neq 0$ we can take $pa=0$, and if $pb=0$ we can take $pa=0$ or $baaa$.

So we have $4+2\gcd(p-1,3)$ algebras.

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c\,|\,cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa-baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb,\ pc-baaa,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa-baaa,\ pb,\ pc-baaa,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-baaa,\ pc,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-\omega baaa,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-baaa,\ pc-baaa,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-\omega baaa,\ pc-baaa,\ \text{class }4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+baaa,\ caa,\ cac,\ pa,\ pb-\omega^2 baaa,\ pc-baaa,\ \text{class }4\rangle\ (p=1 \bmod 3).$$

And if $x=-\frac{1}{2}$ we have

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c \text{ modulo } L_2, \\ b' &= \delta b+\varepsilon c \text{ modulo } L_2, \\ c' &= \alpha^2 c \text{ modulo } L_2, \\ b'a'a'a' &= \alpha^3\delta baaa. \end{aligned}$$

We can take $pc=0$ or $baaa$, and if $pc=baaa$ we can take $pa=pb=0$.

If $pc=0$ we can take $pb=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb\neq 0$ we can take $pa=0$.

If $pb=pc=0$ we can take $pa=0$ or $baaa$.

So we have $3+\gcd(p-1,3)$ algebras.

$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa-baaa,\ pb,\ pc,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb-baaa,\ pc,\ \text{class }4\rangle,$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb-\omega baaa,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$
$$\langle a,b,c \,|\, cb-baa,\ bab,\ bac+\frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc-baaa,\ \text{class }4\rangle.$$

48.3.2 Case 2

Let $cb-baa=caa=cac=0$, $bac=-\frac{1}{2}baaa$, $bab=baaa$. We have

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c \text{ modulo } L_2, \\ b' &= \alpha^2 b+\varepsilon c \text{ modulo } L_2, \\ c' &= \alpha^2 c \text{ modulo } L_2, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

We can take $pc=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pc\neq 0$ we can take $pa=pb=0$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

So we have $1 + 2\gcd(p-1,3) + \gcd(p-1,4)$ algebras.

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa - \omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa - \omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb - \omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb - \omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc - baaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc - \omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb - baa,\ bab - baaa,\ bac + \frac{1}{2}baaa,\ caa,\ cac,\ pa,\ pb,\ pc - \omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3).$$

48.3.3 Case 3

Let $bab = xbaaa$, $cac = baaa$, $cb - baa = bac = caa = 0$. We have

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c \text{ modulo } L_2, \\ b' &= \alpha^2 b \text{ modulo } L_2, \\ c' &= \alpha^2 c \text{ modulo } L_2, \\ b'a'a'a' &= \alpha^5 baaa. \end{aligned}$$

We can take $pc = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$.

If $pc = 0$ we can take $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pc \neq 0$ we can take $pb = ybaaa$ with $0 \leq y < p$.

If either of pb or pc are non-zero then we can take $pa = 0$, and if $pb = pc = 0$ then we can take $pa = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

So we have $p(1 + (p+1)\gcd(p-1,3) + \gcd(p-1,4))$ algebras (all with $0 \leq x < p$).

$$\langle a, b, c \,|\, cb - baa,\ bab - xbaaa,\ bac,\ caa,\ cac - baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - xbaaa,\ bac,\ caa,\ cac - baaa,\ pa - baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - xbaaa,\ bac,\ caa,\ cac - baaa,\ pa - \omega baaa,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - baa,\ bab - xbaaa,\ bac,\ caa,\ cac - baaa,\ pa - \omega^2 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa-\omega^3 baaa,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4)$,

$\langle a, b, c \mid cb - baa,\ bab - xbaaa,\ bac,\ caa,\ cac - baaa,\ pa,\ pb - baaa,\ pc,\ \text{class } 4\rangle$,

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3)$,

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3)$,

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-ybaaa,\ pc-baaa,\ \text{class } 4\rangle\ (0 \le y < p)$,

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-ybaaa,\ pc-\omega baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \le y < p)$,

$\langle a, b, c \mid cb-baa,\ bab-xbaaa,\ bac,\ caa,\ cac-baaa,\ pa,\ pb-ybaaa,\ pc-\omega^2 baaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \le y < p)$.

49 Grandchildren of algebra 32 (5.15)

Algebra 5.15 has $2p + 13$ descendants of order p^6 (6.266 ˘ 6.280), but 6.266, 6.268, 6.270 and 6.272 are terminal. So algebra 5.15 has $2p + 9$ capable descendants of order p^6. The corresponding group is number 32 in the list of capable groups of order p^5 in our paper. The table below gives a list of the capable descendants of order p^6 of algebra 5.15, and for each one of these it gives the number of its descendants of order p^7 and p-class 4.

6.267	$p + 3 + 3\gcd(p - 1, 3) + \gcd(p - 1, 4) + \gcd(p - 1, 5)$
6.269	$(p + 5)/2$
6.271	$(p + 5)/2$
6.273	$1 + \gcd(p - 1, 3) + \gcd(p - 1, 4)$
6.274	$p + 2 + (2p + 3)\gcd(p - 1, 3) + \gcd(p - 1, 5)$
6.275	1
6.276	1
6.277	p
6.278*	$(5p + 3)/2$
6.279	p
6.280*	$(5p + 3)/2$

Note that 6.278 and 6.280 are both one parameter families, with p algebras in each family. In each family $p - 4$ of the algebras have one descendant of order p^7, 2 of the algebras have 2 descendants, one has $(p + 1)/2$ descendants, and one has $p + 1$ descendants.

So, altogether, algebra 5.15 has

$$10p + 16 + (2p + 7)\gcd(p - 1, 3) + 2\gcd(p - 1, 4) + 2\gcd(p - 1, 5)$$

grandchildren of order p^7 and p-class 4.

49.1 Descendants of 6.267

Algebra 6.267 has presentation

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.267 of order p^7 then L_4 is generated by $cacc$. Adding a suitable scalar multiple of cac to b we can take $cb = 0$. If a', b', c' generate L_4 and satisfy the same relations as a, b, c modulo L_4, and if $c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b, \\ b' &= b, \\ c' &= \lambda b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c'c' &= \alpha\xi^3 cacc, \\ c'a'a' &= \alpha^2\xi caa, \\ pa' - b'a' &= \alpha(pa - ba) + \beta pb, \\ pb' &= pb, \\ pc' &= \lambda pb + \xi pc. \end{aligned}$$

We can take $caa = 0$ or $cacc$. If $caa = 0$ we can take $pb = 0$ or $cacc$, and if $caa = cacc$ we can take $pb = 0$, $cacc$ or (if $p = 1 \bmod 5$) $\omega^i cacc$ with $i = 1, 2, 3, 4$.

If $pb \neq 0$ then we can take $pa - ba = pc = 0$.

If $pb = 0$ we can take $pa - ba = 0$, $cacc$ or (if $p = 1 \bmod 3$) $\omega cacc$ or $\omega^2 cacc$.

If $caa = pb = 0$ we can take $pc = 0$ or $cacc$.

If $caa = cacc$ and $pa - ba = pb = 0$ we can take $pc = 0$, $cacc$, $\omega cacc$ or (if $p = 1 \bmod 4$) $\omega^2 cacc$ or $\omega^3 cacc$.

And if $caa = cacc$, $pa - ba \neq 0$, $pb = 0$ then we can take $pc = 0$ or $xcacc$ where x is in a transversal for the cube roots of unity.

So we have $p + 3 + 3\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - cacc,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega cacc,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega^2 cacc,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb,\ pc - cacc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - cacc,\ pb,\ pc - cacc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega cacc,\ pb,\ pc - cacc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba - \omega^2 cacc,\ pb,\ pc - cacc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa,\ pa - ba,\ pb - cacc,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb - cacc,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb - \omega^i cacc,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb,\ pc - cacc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb,\ pc - \omega cacc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb,\ pc - \omega^2 cacc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba,\ pb,\ pc - \omega^3 cacc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - cacc,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - \omega cacc,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - \omega^2 cacc,\ pb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - cacc,\ pb,\ pc - xcacc,\ \text{class } 4\rangle\ (x \text{ in transversal for cube roots of } 1),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - \omega cacc,\ pb,\ pc - xcacc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for cube roots of } 1),$$

$$\langle a, b, c \mid cb,\ caa - cacc,\ pa - ba - \omega^2 cacc,\ pb,\ pc - xcacc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for cube roots of } 1).$$

49.2 Descendants of 6.269

Algebra 6.269 has presentation

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - cac,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.269 of order p^7 then L_4 is generated by $cacc$. adding a suitable scalar multiple of b to a we can take $caa = 0$, adding a suitable scalar multiple of cac to b we can take $cb = 0$, and adding a suitable scalar multiple of b to c we can take $pa - ba - cac = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'a'a' = c'b' = pa' - b'a' - c'a'c' = 0$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= b, \\ c' &= \pm c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'c'c' &= \pm\alpha cacc, \\ pb' &= pb, \\ pc' &= \pm pc. \end{aligned}$$

We can take $pc = 0$ or $cacc$. If $pc = 0$ we can take $pb = 0$ or $cacc$, and if $pc = cacc$ we can take $pb = xcacc$ with $0 \leq x \leq (p-1)/2$.

We have $(p+5)/2$ algebras

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - cac,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - cac,\ pb - cacc,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - cac,\ pb - xcacc,\ pc - cacc,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

49.3 Descendants of 6.271

Algebra 6.271 has presentation

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - \omega cac,\ pb,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost indentical to the desendants of 6.269, and we have $(p+5)/2$ algebras

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - \omega cac,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - \omega cac,\ pb - cacc,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ caa,\ pa - ba - \omega cac,\ pb - xcacc,\ pc - cacc,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

49.4 Descendants of 6.273

Algebra 6.273 has presentation

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.273 of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of ba to c we can take $pc = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, and adding a suitable scalar multiple of ca to c we can take $pb-caa=0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $pa'-b'a'-pb'-c'a'a'=pc'=0$ then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= b, \\ c' &= \alpha^{-2}c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a'a' &= \alpha caaa, \\ c'a'c' &= \alpha^{-3}cac, \\ c'b' &= -\alpha^{-3}\beta cac+\alpha^{-2}cb. \end{aligned}$$

We can take $cac = 0$, $caaa$, $\omega caaa$ or (if $p = 1 \bmod 4$) $\omega^2 caaa$ or $\omega^3 caaa$. If $cac \neq 0$ we can take $cb = 0$, and if $cac = 0$ we can take $cb = 0$, $caaa$ or (if $p = 1 \bmod 3$) $\omega caaa$ or $\omega^2 caaa$. This gives $1 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.

$$\langle a, b, c \mid cb,\ cac,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ cac-caaa,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ cac-\omega caaa,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ cac-\omega^2 caaa,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ cac-\omega^3 caaa,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb-caaa,\ cac,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb-\omega caaa,\ cac,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid cb-\omega^2 caaa,\ cac,\ pa-ba,\ pb-caa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3).$$

49.5 Descendants of 6.274

Algebra 6.274 has presentation

$$\langle a, b, c \mid cb,\ cac,\ pa-ba,\ pb,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.274 of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of caa to b we can take $pa-ba=0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $pa'-b'a'=0$ then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= b, \\ c' &= \lambda b+\xi c \end{aligned}$$

modulo L_2, and

$$\begin{aligned} c'a'a'a' &= \alpha^3\xi caaa, \\ c'a'c' &= \alpha\xi^2 cac, \\ c'b' &= \xi cb \\ pb' &= pb, \\ pc' &= \lambda pb+\xi pc. \end{aligned}$$

We can take $cac = 0$ or $caaa$ and we can (independantly) take $cb = 0$, $caaa$ or (if $p = 1 \bmod 3$) $\omega caaa$ or $\omega^2 caaa$.

If $cac = 0$ we can take $pb = 0$ or $caaa$.

If $cac = caaa$ and $cb = 0$ then we can take $pb = 0$, $caaa$ or (if $p = 1 \bmod 5$) $\omega^i caaa$ with $i = 1, 2, 3, 4$.

And if $cac = caaa$ and $cb \neq 0$ then we can take $pb = 0$ or $xcaaa$ where x lies in a transversal for the cube roots of unity.

If $pb \neq 0$ we can take $pc = 0$.

Let $pb = 0$. If $cb = 0$ we can take $pc = 0$, $caaa$ or (if $p = 1 \bmod 3$) $\omega caaa$ or $\omega^2 caaa$, and if $cb \neq 0$ then we can take $pc = xcaaa$ where $0 \leq x < p$.

We have $p + 2 + (2p + 3)\gcd(p - 1, 3) + \gcd(p - 1, 5)$ algebras.

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - caaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - \omega caaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - \omega^2 caaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb,\ pc - caaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb,\ pc - \omega caaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb,\ pc - \omega^2 caaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb - caaa,\ cac,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, cb - \omega caaa,\ cac,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, cb - \omega^2 caaa,\ cac,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, cb - caaa,\ cac - caaa,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, cb - \omega caaa,\ cac - caaa,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, cb - \omega^2 caaa,\ cac - caaa,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb - caaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - caaa,\ cac,\ pa - ba,\ pb - caaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - \omega caaa,\ cac,\ pa - ba,\ pb - caaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb - \omega^2 caaa,\ cac,\ pa - ba,\ pb - caaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb - caaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb,\ cac - caaa,\ pa - ba,\ pb - \omega^i caaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \,|\, cb - caaa,\ cac - caaa,\ pa - ba,\ pb - xcaaa,\ pc,\ \text{class } 4\rangle\ (x \text{ in transversal for the cube roots of } 1),$$

$$\langle a, b, c \,|\, cb - \omega caaa,\ cac - caaa,\ pa - ba,\ pb - xcaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for the cube roots of } 1),$$

$$\langle a, b, c \,|\, cb - \omega^2 caaa,\ cac - caaa,\ pa - ba,\ pb - xcaaa,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for the cube roots of } 1).$$

49.6 Descendants of 6.275

Algebra 6.275 has presentation

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - caa,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.275 of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of b to c we can take $cac = 0$, adding a suitable scalar multiple of ca to c we can take $cb = 0$, adding a suitable scalar multiple of ca to b we can take $pb = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, and ønally adding a suitable scalar multiple of b to a we can take $pc - caa = 0$. So we have only one algebra

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - caa,\ \text{class } 4\rangle.$$

49.7 Descendants of 6.276

Algebra 6.276 has presentation

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.275, and we have one descendant of order p^7:

$$\langle a, b, c \,|\, cb,\ cac,\ pa - ba,\ pb,\ pc - \omega caa,\ \text{class } 4\rangle.$$

49.8 Descendants of 6.277

Algebra 6.277 has presentation

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.277 of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of ba to c we can take $pc = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, and adding a suitable scalar multiple of ca to c we can take $pb - caa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $pa' - b'a' = pb' - ca'a' = pc' = 0$ then

$$\begin{aligned} a' &= \pm a + \beta b + \gamma c, \\ b' &= b, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \pm caaa, \\ c'a'c' &= \pm cac, \\ c'b' - c'a'a' &= cb - caa \mp 2\beta caaa \mp (2\gamma + \beta)cac. \end{aligned}$$

So we can take $cac = xcaaa$ with $0 \leq x < p$ and $cb - caa = 0$.

This gives p algebras

$$\langle a, b, c \,|\, cb - caa,\ cac - xcaaa,\ pa - ba,\ pb - caa,\ pc,\ \text{class } 4\rangle\ (0 \leq x < p).$$

49.9 Descendants of 6.278

Algebra 6.278 has presentation

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa, \text{ class } 3\rangle\ (0 \leq \mu < p),$$

and this case is almost identical to 6.280 below. So we have $\frac{5}{2}p + \frac{3}{2}$ algebras.

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa, \text{ class } 4\rangle\ (\mu \neq 0, -1, 2, \frac{1}{2}).$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - xcaaa, \text{ class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb - caaa,\ pc - xcaaa, \text{ class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc + caa, \text{ class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb - caaa,\ pc + caa, \text{ class } 4\rangle.$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - 2caa - xcaaa, \text{ class } 4\rangle\ (0 \leq x \leq (p-1)/2.$$

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa - ba,\ pb,\ pc - \frac{1}{2}caa, \text{ class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - caa,\ cac - caaa,\ pa - ba,\ pb,\ pc - \frac{1}{2}caa, \text{ class } 4\rangle.$$

49.10 Descendants of 6.279

Algebra 6.279 has presentation

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb - caa,\ pc, \text{ class } 3\rangle,$$

so this case is almost identical to the descendants of 6.277 and we have p descendants of order p^7.

$$\langle a, b, c \,|\, cb - \omega caa,\ cac - xcaaa,\ pa - ba,\ pb - caa,\ pc, \text{ class } 4\rangle\ (0 \leq x < p).$$

49.11 Descendants of 6.280

Algebra 6.280 has $\frac{5}{2}p + \frac{3}{2}$ descendants of order p^7 and p-class 4.

Algebra 6.280 has presentation

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa, \text{ class } 3\rangle\ (0 \leq \mu < p).$$

and if L is a descendant of 6.280 of order p^7 then L_4 is generated by $caaa$.

49.11.1 Case 1

We ørst consider the case when $\mu \neq 0,\ -\omega,\ 2\omega$ or $\frac{\omega}{2}$. Adding a suitable scalar multiple of b to c we can take $cac = 0$, adding a suitable scalar multiple of ca to c we can take $cb - \omega caa = 0$, adding a suitable scalar multiple of ca to b we can take $pb = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, and adding a suitable scalar multiple of b to a we can take $pc - \mu caa = 0$. So we have one algebra for each of these values of μ:

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - \mu caa, \text{ class } 4\rangle\ (\mu \neq 0, -\omega, 2\omega, \frac{\omega}{2}).$$

49.11.2 Case 2

Next consider the case when $\mu = 0$. Adding a suitable scalar multiple of b to c we can take $cac = 0$, adding a suitable scalar multiple of b to a we can take $cb - \omega caa = 0$, and adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'a'c' = c'b' - \omega c'a'a' = pa' - b'a' = 0$ then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \pm \xi caaa, \\ pb' &= pb \\ pc' &= \xi pc. \end{aligned}$$

So we can take $pb = 0$ or $caaa$ and $pc = xcaaa$ with $0 \leq x \leq (p-1)/2$. This gives $p+1$ algebras.

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - xcaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb - caaa,\ pc - xcaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2).$$

49.11.3 Case 3

Now consider the case when $\mu = -\omega$. Adding a suitable scalar multiple of b to c we can take $cac = 0$, adding a suitable scalar multiple of ca to c we can take $cb - \omega caa = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, and adding a suitable scalar multiple of b to a we can take $pc + \omega caa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'a'c' = c'b' - \omega c'a'a' = pa' - b'a' = pc' + \omega c'a'a' = 0$ then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \pm \xi caaa, \\ pb' &= pb. \end{aligned}$$

So we can take $pb = 0$ or $caaa$ and we have two algebras

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc + \omega caa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb - caaa,\ pc + \omega caa,\ \text{class } 4\rangle.$$

49.11.4 Case 4

Now consider the case when $\mu = 2\omega$. Adding a suitable scalar multiple of b to c we can take $cac = 0$, adding a suitable scalar multiple of ba to b we can take $cb - \omega caa = 0$, adding a suitable scalar multiple of ca to b we can take $pb = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'a'c' = c'b' - \omega c'a'a' = pb' = pa' - b'a' = 0$ then

$$\begin{aligned} a' &= \pm a + \beta b, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \pm \xi caaa, \\ pc' - 2\omega c'a'a' &= \xi(pc - 2\omega caa). \end{aligned}$$

So we can take $pc = xcaaa$ with $0 \leq x \leq (p-1)/2$, giving $(p+1)/2$ algebras

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - 2\omega caa - xcaaa,\ \text{class } 4\rangle\ (0 \leq x \leq (p-1)/2.$$

49.11.5 Case 5

Finally, consider the case when $\mu = \frac{\omega}{2}$. Adding a suitable scalar multiple of ca to b we can take $pb = 0$, adding a suitable scalar multiple of caa to b we can take $pa - ba = 0$, adding a suitable scalar multiple of ca to c we can take $cb - \omega caa$, and adding a suitable scalar multiple of b to a we can take $pc - \frac{1}{2}caa = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'b' - \omega c'a'a' = pb' = pa' - b'a' = pc' - \frac{1}{2}c'a'a' = 0$ then

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \lambda b + \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \pm \xi caaa, \\ c'a'c' &= \pm \xi^2 cac. \end{aligned}$$

So we have two algebras

$$\langle a, b, c \,|\, cb - \omega caa,\ cac,\ pa - ba,\ pb,\ pc - \frac{\omega}{2}caa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, cb - \omega caa,\ cac - caaa,\ pa - ba,\ pb,\ pc - \frac{\omega}{2}caa,\ \text{class } 4\rangle.$$

50 Grandchildren of algebra 33 (5.16)

Algebra 5.16 has $4p + 8$ descendants of order p^6 (6.281 ˇ 6.292), but only 6.281, 6.282, 6.289 and 6.290 are capable. For each of these algebras we give the number of their descendants of order p^7 and p-class 4 in the following table.

6.281	$2p^2 + 4p + 4 + 2\gcd(p-1, 3)$
6.282	$p^3 + p^2 + 2p + 2 + \gcd(p-1, 3)$
6.289	$6p$
6.290	$2p^2 + p$

The corresponding group is number 33 in the list of capable groups of order p^5 from our paper. So altogether algebra 5.16 has

$$p^3+5p^2+13p+6+3\gcd(p-1,3)$$

grandchildren of order p^7 and p-class 4.

50.1 Descendants of 6.281

Algebra 6.281 has presentation

$$\langle a,b,c \mid cb,\ caa,\ pa,\ pb-ba,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.281 of order p^7 then L_4 is generated by $cacc$. Adding a suitable scalar multiple of cac to b we can take $cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if $c'b'=0$ then

$$\begin{aligned} a' &= a, \\ b' &= \delta b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c'c' &= \xi^3 cacc, \\ c'a'a' &= \xi caa, \\ pa' &= pa, \\ pb'-b'a' &= \delta(pb-ba), \\ pc' &= \xi pc. \end{aligned}$$

We can take $caa=0$, $cacc$ or $\omega cacc$ and we can independantly take $pb-ba=0$ or $cacc$.

If $caa=0$ we can take $pc=0$, $cacc$ or $\omega cacc$. If $caa=pc=0$ then we can take $pa=0$, $cacc$ or (if $p=1 \bmod 3$) $\omega cacc$ or $\omega^2 cacc$. And if $caa=0$ and $pc=cacc$ or $\omega cacc$ then we can take $pa=xcacc$ with $0\leq x\leq (p-1)/2$.

If $caa=cacc$ or $\omega cacc$ then we can take $pa=xcacc$ with $0\leq x\leq (p-1)/2$ and $pc=ycacc$ for $0\leq y<p$.

So we have $2p^2+4p+4+2\gcd(p-1,3)$ algebras.

$$\langle a,b,c \mid cb,\ caa,\ pa,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-cacc,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega cacc,\ pb-ba,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega^2 cacc,\ pb-ba,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba,\ pc-cacc,\ \text{class } 4\rangle\ (0\leq x\leq (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba,\ pc-\omega cacc,\ \text{class } 4\rangle\ (0\leq x\leq (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa-cacc,\ pa-xcacc,\ pb-ba,\ pc-ycacc,\ \text{class } 4\rangle\ (0\leq x\leq (p-1)/2,\ 0\leq y<p),$$

$$\langle a,b,c \mid cb,\ caa-\omega cacc,\ pa-xcacc,\ pb-ba,\ pc-ycacc,\ \text{class } 4\rangle\ (0\leq x\leq (p-1)/2,\ 0\leq y<p),$$

$$\langle a,b,c \mid cb,\ caa,\ pa,\ pb-ba-cacc,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-cacc,\ pb-ba-cacc,\ pc,\ \text{class }4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega cacc,\ pb-ba-cacc,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega^2 cacc,\ pb-ba-cacc,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba-cacc,\ pc-cacc,\ \text{class }4\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba-cacc,\ pc-\omega cacc,\ \text{class }4\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa-cacc,\ pa-xcacc,\ pb-ba-cacc,\ pc-ycacc,\ \text{class }4\rangle\ (0\le x\le (p-1)/2,\ 0\le y<p),$$

$$\langle a,b,c \mid cb,\ caa-\omega cacc,\ pa-xcacc,\ pb-ba-cacc,\ pc-ycacc,\ \text{class }4\rangle\ (0\le x\le (p-1)/2,\ 0\le y<p).$$

50.2 Descendants of 6.282

Algebra 6.282 has presentation

$$\langle a,b,c \mid cb,\ caa,\ pa,\ pb-ba-cac,\ pc,\ \text{class }3\rangle,$$

and if L is a descendant of 6.282 of order p^7 then L_4 is generated by $cacc$. Adding a suitable scalar multiple of cac to b we can take $cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if $c'b'=0$ then

$$\begin{aligned} a' &= a,\\ b' &= \xi^2 b,\\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'c'c' &= \xi^3 cacc,\\ c'a'a' &= \xi caa,\\ pa' &= pa,\\ pb'-b'a'-c'a'c' &= \xi^2(pb-ba-cac),\\ pc' &= \xi pc. \end{aligned}$$

We can take $pb-ba-cac=0$ or $cacc$.

First consider the case when $pb-ba-cac=0$.

Then (as in the descendants of 6.281) we can take $caa=0$, $cacc$ or $\omega cacc$.

If $caa=0$ we can take $pc=0$, $cacc$ or $\omega cacc$. If $caa=pc=0$ then we can take $pa=0$, $cacc$ or (if $p=1 \bmod 3$) $\omega cacc$ or $\omega^2 cacc$. And if $caa=0$ and $pc=cacc$ or $\omega cacc$ then we can take $pa=xcacc$ with $0\le x\le (p-1)/2$.

If $caa=cacc$ or $\omega cacc$ then we can take $pa=xcacc$ with $0\le x\le (p-1)/2$ and $pc=ycacc$ for $0\le y<p$.

On the other hand, if $pb-ba-cac=cacc$ then we need $\xi=1$, and so we can take $caa=xcacc$, $pa=ycacc$, $pc=zcacc$ with $0\le x,y,x<p$.

So we have $p^3+p^2+2p+2+\gcd(p-1,3)$ algebras.

$$\langle a,b,c \mid cb,\ caa,\ pa,\ pb-ba-cac,\ pc,\ \text{class }4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-cacc,\ pb-ba-cac,\ pc,\ \text{class }4\rangle,$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega cacc,\ pb-ba-cac,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-\omega^2 cacc,\ pb-ba-cac,\ pc,\ \text{class }4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba-cac,\ pc-cacc,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa,\ pa-xcacc,\ pb-ba-cac,\ pc-\omega cacc,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2),$$

$$\langle a,b,c \mid cb,\ caa-cacc,\ pa-xcacc,\ pb-ba-cac,\ pc-ycacc,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2,\ 0\le y<p),$$

$$\langle a,b,c \mid cb,\ caa-\omega cacc,\ pa-xcacc,\ pb-ba-cac,\ pc-ycacc,\ \text{class } 4\rangle\ (0\le x\le (p-1)/2,\ 0\le y<p),$$

$$\langle a,b,c \mid cb,\ caa-xcacc,\ pa-ycacc,\ pb-ba-cac-cacc,\ pc-zcacc,\ \text{class } 4\rangle\ (0\le x,y,z<p).$$

50.3 Descendants of 6.289

Algebra 6.289 has presentation

$$\langle a,b,c \mid cb,\ cac,\ pa,\ pb-ba,\ pc-\lambda caa,\ \text{class } 3\rangle\ (0\le \lambda<p),$$

but this algebra is terminal unless $\lambda=0$. When $\lambda=0$ we have

$$\langle a,b,c \mid cb,\ cac,\ pa,\ pb-ba,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of caa to b we can take $pb-ba=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if $pb'-b'a'=0$ then

$$\begin{aligned} a' &= a+\gamma c,\\ b' &= \delta b,\\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \xi caaa,\\ c'a'c' &= \xi^2 cac,\\ c'b' &= \delta\xi cb,\\ pa' &= pa+\gamma pc,\\ pc' &= \xi pc. \end{aligned}$$

We can take cac and cb independantly equal to 0 or $caaa$, and we can take $pc=xcaaa$ with $0\le x<p$.

If $pc\ne 0$ then we can take $pa=0$.

If $cac=pc=0$ we can take $pa=0$ or $caaa$, and if $cac=caaa$ and $pc=0$ then we can take $pa=xcaaa$ with $0\le x<p$.

So we have $6p$ algebras.

$$\langle a,b,c \mid cb,\ cac,\ pa,\ pb-ba,\ pc-xcaaa,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb,\ cac-caaa,\ pa,\ pb-ba,\ pc-xcaaa,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb-caaa,\ cac,\ pa,\ pb-ba,\ pc-xcaaa,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb-caaa,\ cac-caaa,\ pa,\ pb-ba,\ pc-xcaaa,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb,\ cac,\ pa-caaa,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb-caaa,\ cac,\ pa-caaa,\ pb-ba,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ cac-caaa,\ pa-xcaaa,\ pb-ba,\ pc,\ \text{class } 4\rangle\ (0<x<p),$$

$$\langle a,b,c \mid cb-caaa,\ cac-caaa,\ pa-xcaaa,\ pb-ba,\ pc,\ \text{class } 4\rangle\ (0<x<p).$$

50.4 Descendants of 6.290

Algebra 6.290 has presentation

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa,\ pb - ba,\ pc - \lambda caa,\ \text{class } 3\rangle\ (0 \le \lambda < p),$$

but this algebra is terminal unless $\lambda = 0$. When $\lambda = 0$ we have

$$\langle a, b, c \,|\, cb - caa,\ cac,\ pa,\ pb - ba,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of this algebra of order p^7 then L_4 is generated by $caaa$. Adding a suitable scalar multiple of caa to b we can take $pb - ba = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $pb' - b'a' = 0$ then

$$\begin{aligned} a' &= a + \gamma c, \\ b' &= b, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} c'a'a'a' &= \xi caaa, \\ c'a'c' &= \xi^2 cac, \\ c'b' - c'a'a' &= \xi(cb - caa) - 2\gamma\xi cac, \\ pa' &= pa + \gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $cac = 0$ or $caaa$.

If $cac = 0$ we can take $cb - caa = xcaaa$, $pc = ycaaa$ with $0 \le x, y < p$. If $pc \ne 0$ we can take $pa = 0$, and if $pc = 0$ we can take $pa = 0$ or $caaa$.

If $cac = caaa$ we can take $cb - caa = 0$ though we then need $\gamma = 0$, and we can take $pa = xcaaa$, $pc = ycaaa$ with $0 \le x, y < p$.

So we have $2p^2 + p$ algebras

$$\langle a, b, c \,|\, cb - caa - xcaaa,\ cac,\ pa,\ pb - ba,\ pc - xcaaa,\ \text{class } 4\rangle\ (0 \le x, y < p),$$

$$\langle a, b, c \,|\, cb - caa - xcaa,\ cac,\ pa - caaa,\ pb - ba,\ pc,\ \text{class } 4\rangle\ (0 \le x < p),$$

$$\langle a, b, c \,|\, cb - caa,\ cac - caaa,\ pa - xcaaa,\ pb - ba,\ pc - ycaaa,\ \text{class } 4\rangle\ (0 \le x, y < p).$$

51 Grandchildren of algebra 34 (5.18)

Algebra 5.18 has $2p + 13 + 3\gcd(p-1,3) + \gcd(p-1,4)$ descendants of order p^6 (6.293 ˘ 6.318), but only 6.294 ˘ 6.299, 6.303 ˘ 6.305, 6.312 and 6.313 are capable. For each of these algebras we give the number of descendants of order p^7 and p-class 4 in the following table.

6.294	$p + 4 + 5\gcd(p-1,3) + \gcd(p-1,4)$
6.295	$(p+5)/2$
6.296	$(p+5)/2$
6.297	$2p + p\gcd(p-1,3)$
6.298	p
6.299	p
6.303	$p^2 + p + (p+1)\gcd(p-1,3) + \gcd(p-1,4)$
6.304	$p(p+1)/2$
6.305	$p(p+1)/2$
6.312	$4p + 2 + 2\gcd(p-1,3) + 4\gcd(p-1,4)$
6.313	$p + \gcd(p-1,4) + \gcd(p-1,5)$

The corresponding group is group number 34 in the list of capable groups of order p^5 in our paper. So, in all, algebra 5.18 has

$$2p^2 + 13p + 11 + (2p + 8)\gcd(p - 1, 3) + 7\gcd(p - 1, 4) + \gcd(p - 1, 5)$$

grandchildren of order p^7 and p-class 4.

51.1 Descendants of 6.294

Algebra 6.294 has presentation

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.294 of order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha\xi b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \alpha^4\xi^3 babb, \\ b'a'a' &= \alpha^3\xi baa, \\ pa' &= \alpha pa + \gamma pc, \\ pb' - c'a' &= \alpha\xi(pb - ca) + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take baa equal to 0 or $babb$.

If $baa = 0$ we can take $pc = 0$, $babb$ or $\omega babb$, and if $pc \neq 0$ we can take $pa = pb - ca = 0$.

If $baa = pc = 0$ then we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$, and we can take $pb - ca = 0$ or $babb$.

If $baa = babb$ then we need $\alpha = \xi^{-2}$ and so we have

$$\begin{aligned} b'a'b'b' &= \xi^{-5} babb, \\ pa' &= \xi^{-2} pa + \gamma pc, \\ pb' - c'a' &= \xi^{-1}(pb - ca) + \varepsilon pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $pc = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 3$) $\omega^2 babb$, $\omega^3 babb$, $\omega^4 babb$ or $\omega^5 babb$. If $pc \neq 0$ we can take $pa = pb - ca = 0$. If $baa = babb$ and $pc = 0$ then we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. If $baa = babb$, $pa = pc = 0$ then we can take $pb - ca = 0$, $babb$, $\omega babb$ or (if $p = 1 \bmod 4$) $\omega^2 babb$, $\omega^3 babb$, and if $baa = babb$, $pa \neq 0$, $pc = 0$ then we can take $pb - ca = 0$ or $xbabb$ where x lies in a transversal for the cube roots of unity.

We have $p + 4 + 5\gcd(p - 1, 3) + \gcd(p - 1, 4)$ algebras.

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc - babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc - \omega babb,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa,\ pa - babb,\ pb - ca,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa,\ pa - \omega babb,\ pb - ca,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa,\ pa - \omega^2 babb,\ pb - ca,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa,\ pa,\ pb - ca - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa,\ pa - babb,\ pb - ca - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa,\ pa - \omega babb,\ pb - ca - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa,\ pa - \omega^2 babb,\ pb - ca - babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca,\ pc - babb,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca,\ pc - \omega babb,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca,\ pc - \omega^i babb,\ \text{class } 4\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca - babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca - \omega babb,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca - \omega^2 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa,\ pb - ca - \omega^3 babb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa - babb,\ pb - ca,\ pc,\ \text{class } 4\rangle,$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa - \omega babb,\ pb - ca,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa - babb,\ pa - \omega^2 babb,\ pb - ca,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3),$$
$$\langle a, b, c \mid cb,\ baa{-}babb,\ pa{-}babb,\ pb{-}ca{-}xbabb,\ pc,\ \text{class } 4\rangle\ (x \text{ in transversal for cube roots of } 1),$$
$$\langle a, b, c \mid cb,\ baa{-}babb,\ pa{-}\omega babb,\ pb{-}ca{-}xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for cube roots of } 1),$$
$$\langle a, b, c \mid cb,\ baa{-}babb,\ pa{-}\omega^2 babb,\ pb{-}ca{-}xbabb,\ pc,\ \text{class } 4\rangle\ (p = 1 \bmod 3, x \text{ in transversal for cube roots of } 1).$$

51.2 Descendants of 6.295

Algebra 6.295 has presentation

$$\langle a, b, c \mid cb,\ baa,\ pa - bab,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.295 or order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of c to b we can take $baa = 0$, adding a suitable scalar multiple of bab to c we can take $cb = 0$, and adding a suitable scalar multiple of ca to c we can take $pb - ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $b'a'a' = c'b' = pb' - c'a' = 0$ then

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \pm b, \\ c' &= \pm \alpha^{-1} c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \pm\alpha babb, \\ pa' - b'a'b' &= \alpha(pa - bab) + \gamma pc, \\ pc' &= \pm\alpha^{-1}pc. \end{aligned}$$

We can take $pc = 0$, $babb$ or $\omega babb$, and if $pc \neq 0$ we can take $pa - bab = 0$. If $pc = 0$ then we can take $pa - bab = xbabb$ with $0 \leq x \leq (p-1)/2$. We have $(p+5)/2$ algebras

$$\langle a, b, c \mid cb,\, baa,\, pa - bab - xbabb,\, pb - ca,\, pc,\, \text{class } 4\rangle \ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid cb,\, baa,\, pa - bab,\, pb - ca,\, pc - babb,\, \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\, baa,\, pa - bab,\, pb - ca,\, pc - \omega babb,\, \text{class } 4\rangle.$$

51.3 Descendants of 6.296

Algebra 6.296 has presentation

$$\langle a, b, c \mid cb,\, baa,\, pa - \omega bab,\, pb - ca,\, pc,\, \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.295, and we have $(p+5)/2$ algebras

$$\langle a, b, c \mid cb,\, baa,\, pa - \omega bab - xbabb,\, pb - ca,\, pc,\, \text{class } 4\rangle \ (0 \leq x \leq (p-1)/2),$$

$$\langle a, b, c \mid cb,\, baa,\, pa - \omega bab,\, pb - ca,\, pc - babb,\, \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\, baa,\, pa - \omega bab,\, pb - ca,\, pc - \omega babb,\, \text{class } 4\rangle.$$

51.4 Descendants of 6.297

Algebra 6.297 has presentation

$$\langle a, b, c \mid cb,\, baa,\, pa,\, pb - ca - bab,\, pc,\, \text{class } 3\rangle,$$

and if L is a descendant of 6.297 or order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of c to b we can take $baa = 0$, adding a suitable scalar multiple of bab to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $b'a'a' = c'b' = 0$ then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^{-1}b + \varepsilon c, \\ c' &= \alpha^{-2}c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= \alpha^{-2}babb, \\ pa' &= \alpha pa, \\ pb' - c'a' &= \alpha^{-1}(pb - ca) + \varepsilon pc + 2\varepsilon babb, \\ pc' &= \alpha^{-2}pc. \end{aligned}$$

We can take $pc = xbabb$ with $0 \leq x < p$, and if $x \neq -2$ we can take $pb - ca = 0$. If $x = -2$ then we can take $pb - ca = 0$ or $babb$. If $pb - ca = babb$ then we can take $pa = ybabb$ with

$0 \leq y < p$, and if $pb - ca = 0$ then we can take $pa = 0$, $babb$ or (if $p = 1 \bmod 3$) $\omega babb$ or $\omega^2 babb$. We have $2p + p\gcd(p-1,3)$ algebras.

$$\langle a,b,c \,|\, cb,\ baa,\ pa,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - babb,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - \omega babb,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3, 0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - \omega^2 babb,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (p = 1 \bmod 3,\ 0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - xbabb,\ pb - ca - bab - babb,\ pc + 2babb,\ \text{class } 4\rangle\ (0 \leq x < p).$$

51.5 Descendants of 6.298

Algebra 6.298 has presentation

$$\langle a,b,c \,|\, cb,\ baa,\ pa - bab,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.298 of order p^7 then L_4 is generated by $babb$. Adding a suitable scalar multiple of c to b we can take $baa = 0$, adding a suitable scalar multiple of bab to c we can take $cb = 0$, and we can also take $pb - ca - bab = 0$ by adding suitable scalar multilpe of ca to c. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if

$$b'a'a' = c'b' = pb' - c'a' - b'a'b' = 0$$

then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b - \gamma c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'b'b' &= babb, \\ pa' - b'a'b' &= \pm(pa - bab) + \gamma pc - 2\gamma babb \\ pc' &= pc, \end{aligned}$$

and so we have $(3p-1)/2$ algebras

$$\langle a,b,c \,|\, cb,\ baa,\ pa - bab,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - bab - xbabb,\ pb - ca - bab,\ pc - 2babb,\ \text{class } 4\rangle\ (0 < x \leq (p-1)/2).$$

51.6 Descendants of 6.299

Algebra 6.299 has presentation

$$\langle a,b,c \,|\, cb,\ baa,\ pa - \omega bab,\ pb - ca - bab,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.298, and we have $(3p-1)/2$ descendants of order p^7:

$$\langle a,b,c \,|\, cb,\ baa,\ pa - \omega bab,\ pb - ca - bab,\ pc - xbabb,\ \text{class } 4\rangle\ (0 \leq x < p),$$

$$\langle a,b,c \,|\, cb,\ baa,\ pa - \omega bab - xbabb,\ pb - ca - bab,\ pc - 2babb,\ \text{class } 4\rangle\ (0 < x \leq (p-1)/2).$$

Algebra 6.303 has presentation

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.303 of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of baa to c we can take $cb=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_4, and if $c'b'=0$ then

$$\begin{aligned} a' &= \alpha a+\gamma c, \\ b' &= \alpha b+\varepsilon c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= \alpha^4 baaa, \\ b'a'b'-b'a'a' &= \alpha^3(bab-baa), \\ pa' &= \alpha pa+\gamma pc, \\ pb'-c'a' &= \alpha(pb-ca)+\varepsilon pc, \\ pc' &= pc. \end{aligned}$$

We can take $bab-baa=0$ or $baaa$.

If $bab-baa=0$ then we can take $pc=0$, $baaa$, $\omega baaa$ or (if $p=1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$, and if $bab-baa=baaa$ then we can take $pc=xbaaa$ with $0\le x<p$.

If $pc\ne 0$ then we can take $pa=pb-ca=0$.

If $bab-baa=pc=0$ then we can take $pa=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pa=0$ we can take $pb-ca=0$, $baaa$ or (if $p=1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and if $pa\ne 0$ we can take $pb-ca=xbaaa$ with $0\le x<p$.

If $bab-baa=baa$ and $pc=0$ then we can take $pa=xbaaa$, $pb-ca=ybaaa$ with $0\le x,y<p$.

We have $p^2+p+(p+1)\gcd(p-1,3)+\gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca-baaa,\ pc,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca-\omega baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca-\omega^2 baaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa-baaa,\ pb-ca-xbaaa,\ pc,\ \text{class } 4\rangle\ (0\le x<p),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa-\omega baaa,\ pb-ca-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0\le x<p),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa-\omega^2 baaa,\ pb-ca-xbaaa,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3, 0\le x<p),$$

$$\langle a,b,c \mid cb,\ bab-baa-baaa,\ pa-xbaaa,\ pb-ca-ybaaa,\ pc,\ \text{class } 4\rangle\ (0\le x,y<p),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc-baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc-\omega baaa,\ \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc-\omega^2 baaa,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \mid cb,\ bab-baa,\ pa,\ pb-ca,\ pc-\omega^3 baaa,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \mid cb,\ bab-baa-baaa,\ pa,\ pb-ca,\ pc-xbaaa,\ \text{class } 4\rangle\ (0<x<p).$$

51.8 Descendants of 6.304

Algebra 6.304 has presentation

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa,\ pb - ca - baa,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.304 of order p^7 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of c to b we can take $bab - baa = 0$, and adding a suitable scalar multiple of baa to c we can take $cb = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if

$$c'b' = b'a'b' - ba'a' = 0$$

then

$$\begin{aligned} a' &= \pm a + \gamma c, \\ b' &= \pm b + \gamma c, \\ c' &= c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' &= \pm pa + \gamma pc, \\ pb' - c'a' - b'a'a' &= \pm(pb - ca - baa) + \gamma pc + 2\gamma baaa \\ pc' &= pc. \end{aligned}$$

We can take $pc = xbaaa$ with $0 \leq x < p$. If $pc \neq 0$ we can take $pa = 0$, though we then need $\gamma = 0$, and we can take $pb - ca - baa = ybaaa$ with $0 \leq y \leq (p-1)/2$. If $pc = 0$ then we can take $pb - ca - baa = 0$ and $pa = ybaaa$ with $0 \leq y \leq (p-1)/2$. So we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa,\ pb - ca - baa - ybaaa,\ pc - xbaaa,\ \text{class } 4\rangle\ (0 < x < p,\ 0 \leq y \leq (p-1)/2),$$

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa - ybaaa,\ pb - ca - baa,\ pc,\ \text{class } 4\rangle\ (0 \leq y \leq (p-1)/2).$$

51.9 Descendants of 6.305

Algebra 6.305 has presentation

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa,\ pb - ca - \omega baa,\ pc,\ \text{class } 3\rangle,$$

and so this case is almost identical to the descendants of 6.304, and we have $p(p+1)/2$ algebras

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa,\ pb - ca - \omega baa - ybaaa,\ pc - xbaaa,\ \text{class } 4\rangle\ (0 < x < p,\ 0 \leq y \leq (p-1)/2),$$

$$\langle a, b, c \,|\, cb,\ bab - baa,\ pa - ybaaa,\ pb - ca - \omega baa,\ pc,\ \text{class } 4\rangle\ (0 \leq y \leq (p-1)/2).$$

51.10 Descendants of 6.312

Algebra 6.312 has presentation

$$\langle a, b, c \,|\, cb,\ bab,\ pa,\ pb - ca,\ pc,\ \text{class } 3\rangle,$$

and if L is a descendant of 6.312 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of baa to c we can take $pb-ca=0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if $pb'-c'a'=0$ then

$$\begin{aligned} a' &= \alpha a+\gamma c, \\ b' &= \alpha\xi b+\varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= \alpha^4\xi baaa, \\ b'a'b' &= \alpha^3\xi^2 bab, \\ c'b' &= \alpha\xi^2 cb, \\ pa' &= \alpha pa+\gamma pc, \\ pc' &= \xi pc. \end{aligned}$$

We can take $bab=0$ or $baaa$.

If $bab=0$ we can take $cb=0$ or $baaa$, and if $bab=baaa$ then we can take $cb=0$, $baaa$ or $\omega baaa$.

If $bab=0$ or $cb=0$ then we can take $pc=0$, $baaa$, $\omega baaa$ or (if $p=1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$, and if $bab=baaa$ and $cb=baaa$ or $\omega baaa$ then we need $\alpha=\xi=\pm 1$ and we can take $pc=xbaaa$ with $0 \leq x<p$.

If $pc \neq 0$ we can take $pa=0$.

If $bab=cb=pc=0$ then we can take $pa=0$ or $baaa$.

If $bab=pc=0$, $cb=baaa$ then we can take $pa=0$, $baaa$, $\omega baaa$ or (if $p=1 \bmod 3$) $\omega^i baaa$ with $i=2,3,4,5$.

If $bab=baaa$, $cb=pc=0$ then we can take $pa=0$, $baaa$, $\omega baaa$ or (if $p=1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$.

And if $bab=baaa$, $cb=baaa$ or $\omega baaa$ and $pc=0$ then we need $\alpha=\xi=\pm 1$ and we can take $pa=xbaaa$ with $0 \leq x<p$.

We have $4p+2+2\gcd(p-1,3)+4\gcd(p-1,4)$ algebras.

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab,\ pa-baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb-baaa,\ bab,\ pa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb-baaa,\ bab,\ pa-baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb-baaa,\ bab,\ pa-\omega baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb-baaa,\ bab,\ pa-\omega^i baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 3,\ i=2,3,4,5),$$

$$\langle a, b, c \mid cb,\ bab-baaa,\ pa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab-baaa,\ pa-baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab-baaa,\ pa-\omega baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab-baaa,\ pa-\omega^2 baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ bab-baaa,\ pa-\omega^3 baaa,\ pb-ca,\ pc,\ \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a, b, c \mid cb-baaa,\ bab-baaa,\ pa-xbaaa,\ pb-ca,\ pc,\ \text{class } 4\rangle\ (0 \leq x<p),$$

$$\langle a, b, c \mid cb-\omega baaa,\ bab-baaa,\ pa-xbaaa,\ pb-ca,\ pc,\ \text{class } 4\rangle\ (0 \leq x<p),$$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - \omega baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ bab,\ pa,\ pb - ca,\ pc - \omega^3 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb - baaa,\ bab,\ pa,\ pb - ca,\ pc - baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab,\ pa,\ pb - ca,\ pc - \omega baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb - baaa,\ bab,\ pa,\ pb - ca,\ pc - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb - baaa,\ bab,\ pa,\ pb - ca,\ pc - \omega^3 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ bab - baaa,\ pa,\ pb - ca,\ pc - baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab - baaa,\ pa,\ pb - ca,\ pc - \omega baaa, \text{ class } 4\rangle,$$

$$\langle a, b, c \mid cb,\ bab - baaa,\ pa,\ pb - ca,\ pc - \omega^2 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb,\ bab - baaa,\ pa,\ pb - ca,\ pc - \omega^3 baaa, \text{ class } 4\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid cb - baaa,\ bab - baaa,\ pa,\ pb - ca,\ pc - xbaaa, \text{ class } 4\rangle\ (0 < x < p),$$

$$\langle a, b, c \mid cb - \omega baaa,\ bab - baaa,\ pa,\ pb - ca,\ pc - xbaaa, \text{ class } 4\rangle\ (0 < x < p).$$

51.11 Descendants of 6.313

Algebra 6.313 has presentation

$$\langle a, b, c \mid cb - baa,\ bab,\ pa,\ pb - ca,\ pc, \text{ class } 3\rangle,$$

and if L is a descendant of 6.313 then L_4 is generated by $baaa$. Adding a suitable scalar multiple of c to b we can take $bab = 0$, adding a suitable scalar multiple of c to a we can take $cb - baa = 0$, and adding a suitable scalar multiple of baa to c we can take $pb - ca = 0$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_4, and if

$$c'b' - b'a'a' = b'a'b' = pb' - c'a' = 0$$

then

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^3 b, \\ c' &= \alpha^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a' &= \alpha^6 baaa, \\ pa' &= \alpha pa, \\ pc' &= \alpha^2 pc. \end{aligned}$$

We can take $pa = 0$, $baaa$ or (if $p = 1 \bmod 5$) $\omega^i baaa$ with $i = 1, 2, 3, 4$. If $pa = 0$ then we can take $pc = 0$, $baaa$, $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$, and if $pa \neq 0$ then we can take $pc = 0$ or $pc = xbaaa$ where x lies in a transversal for the øfth roots of unity. We have $p + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \mid cb - baa,\ bab,\ pa,\ pb - ca,\ pc, \text{ class } 4\rangle,$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa,\, pb-ca,\, pc-baaa,\, \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa,\, pb-ca,\, pc-\omega baaa,\, \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa,\, pb-ca,\, pc-\omega^2 baaa,\, \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa,\, pb-ca,\, pc-\omega^3 baaa,\, \text{class } 4\rangle\ (p=1 \bmod 4),$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa-baaa,\, pb-ca,\, pc,\, \text{class } 4\rangle,$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa-\omega^i baaa,\, pb-ca,\, pc,\, \text{class } 4\rangle\ (p=1 \bmod 5,\ i=1,2,3,4),$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa-baaa,\, pb-ca,\, pc-xbaaa,\, \text{class } 4\rangle\ (x \text{ in transversal for } 5^{th} \text{ roots of } 1),$$

$$\langle a,b,c \mid cb-baa,\, bab,\, pa-\omega^i baaa,\, pb-ca,\, pc-xbaaa,\, \text{class } 4\rangle\ (p=1 \bmod 5,\ i=1,2,3,4, x \text{ in transversal for } 5^{th} \text{ roots}$$

52 Grandchildren of algebra 35 (5.19)

Algebra 5.19 has 3 descendants of order p^6 (6.319 ˘ 6.321), but all three are terminal.

53 Grandchildren of algebra 36 (5.24)

Algebra 5.24 has 3 descendants of order p^6 (6.322 ˘ 6.324), but 6.323 and 6.324 are terminal. The corresponding group is number 36 in the list of capable groups of order p^5 in our paper. Algebra 6.322 has 3 descendants of order p^7 and p-class 5.

6.322 3

53.1 Descendants of 6.322

Algebra 6.322 has presentation

$$\langle a,b,c \mid ba,\, ca,\, cb,\, pb,\, pc,\, \text{class } 4\rangle,$$

and if L is a descendant of 6.322 then L_5 is generated by p^4a. Adding suitable scalar multiples of p^3a to b and c we can take $pb=pc=0$. The subalgebra $C=\langle b,c\rangle+L_2$ is a characteristic subalgebra of L. Clearly a is centralized by some non-zero element in the span of b,c, and so we may suppose that $ca=0$. Scaling b we may suppose that $ba=0$ or p^4a, and scaling c we may suppose that $cb=0$ or p^4a. However, if $cb\neq 0$ then we can subtract a suitable scalar multiple of c from a so that $ba=0$. So we have 3 algebras

$$\langle a,b,c \mid ba,\, ca,\, cb,\, pb,\, pc,\, \text{class } 5\rangle,$$

$$\langle a,b,c \mid ba-p^4a,\, ca,\, cb,\, pb,\, pc,\, \text{class } 5\rangle,$$

$$\langle a,b,c \mid ba,\, ca,\, cb-p^4a,\, pb,\, pc,\, \text{class } 5\rangle.$$

54 Grandchildren of algebra 37 (5.27)

Algebra 5.27 has $11 + 4\gcd(p-1,3) + 2\gcd(p-1,4)$ descendants of order p^6 (6.325 ˘ 6.355), but only 6.325 ˘ 6.328 are capable. The corresponding group is number 37 in the list of capable groups of order p^5 in our paper. For each of 6.325 ˘ 6.328 we give the number of descendants of order p^7 and p-class 5.

6.325	$19 + 5\gcd(p-1,3) + 6\gcd(p-1,4)$
6.326	$4p + 5 + 4\gcd(p-1,3) + 2\gcd(p-1,4) + 4\gcd(p-1,5)$
6.327	$3p + 12 + 2\gcd(p-1,3) + 3\gcd(p-1,4) + \gcd(p-1,7)$
6.328	$p^2 + 3p - 2 + (p+3)\gcd(p-1,3) + 2\gcd(p-1,4) + 2\gcd(p-1,5)$

So, altogether, algebra 5.27 has

$$p^2 + 10p + 34 + (p+14)\gcd(p-1,3) + 13\gcd(p-1,4) + 6\gcd(p-1,5) + \gcd(p-1,7)$$

grandchildren of order p^7 and p-class 5.

54.1 Descendants of 6.325

Algebra 6.325 has $19 + 5\gcd(p-1,3) + 6\gcd(p-1,4)$ descendants of order p^7 and p-class 5.

Algebra 6.325 has presentation

$$\langle a, b, c \mid bab,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

and if L is a descendant of 6.325 of order p^7 then L_5 has order p and is generated by $baaaa$ and $baaab$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_5 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\delta baaaa + \alpha^3\beta\delta baaab, \\ b'a'a'a'b' &= \alpha^3\delta^2 baaab. \end{aligned}$$

So we can assume that $baaaa = 0$ and that L_5 is generated by $baaab$, or that $baaab = 0$ and that L_5 is generated by $baaaa$.

If $baaaa = 0$ then we need $\beta = 0$. Adding a suitable scalar multiple of baa to b we can take $bab = 0$, and adding a suitable scalar multiple of $baaa$ to c we can take $cb = 0$. Then scaling c we can take $ca = 0$ or $baaab$.

If $baaab = 0$ then adding a suitable scalar multiple of $baaa$ to c we can take $ca = 0$. If If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_5 and if $b'a'a'a'b' = c'a' = 0$ then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\delta baaaa, \\ b'a'b' &= \alpha\delta^2 bab, \\ c'b' &= \delta\xi cb. \end{aligned}$$

So we can take bab and cb to independantly equal 0 or $baaaa$.

So we have six possible commutator structures for L:

$$\begin{aligned} baaaa &= bab = ca = cb = 0, \\ baaaa &= bab = cb = 0,\ ca = baaab, \\ baaab &= bab = ca = cb = 0, \\ baaab &= bab = ca = 0,\ cb = baaaa, \\ baaab &= ca = cb = 0,\ bab = baaaa, \\ baaab &= ca = 0,\ bab = cb = baaaa. \end{aligned}$$

In each case we give the most general conditions on generators a', b', c' for L which satisfy the given commutator relations, and then compute the possibilities for pa, pb, pc.

54.1.1 Case 1

Let $baaaa = bab = ca = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a'b' &= \alpha^3\delta^2 baaab. \end{aligned}$$

We can take $pc = 0$ or $baaab$ and if $pc \neq 0$ we can take $pa = pb = 0$. If $pc = 0$ we can take $pb = 0$ or $baaab$. If $pb = pc = 0$ then we can take $pa = 0$, $baaab$ or $\omega baaab$. If $pb = baaab$ and $pc = 0$ then we can take $pa = 0$, $baaab$, $\omega baaab$ or (if $p = 1 \bmod 4$) $\omega^2 baaab$ or $\omega^3 baaab$. So we have $5 + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - \omega baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - \omega^2 baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa - \omega^3 baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca,\ cb,\ pa,\ pb,\ pc - baaab,\ \text{class } 5\rangle.$$

54.1.2 Case 2

Let $baaaa = bab = cb = 0$, $ca = baaab$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \alpha^2\delta^2 c, \\ b'a'a'a'b' &= \alpha^3\delta^2 baaab. \end{aligned}$$

This case is similar to Case 1 and again we have $5 + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - \omega baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - \omega^2 baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa - \omega^3 baaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, baaaa,\ bab,\ ca - baaab,\ cb,\ pa,\ pb,\ pc - baaab,\ \text{class } 5\rangle.$$

54.1.3 Case 3

Let $baaab = bab = ca = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a'a' &= \alpha^4\delta baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$, and if $pc = baaaa$ we can take $pa = pb = 0$.
If $pc = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$, and if $pb \neq 0$ we can take $pa = 0$.
If $pb = pc = 0$ then we can take $pa = 0$ or $baaaa$.
So we have $3 + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$
$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb - \omega^2 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb - \omega^3 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb,\ pa,\ pb,\ pc - baaaa,\ \text{class } 5\rangle.$$

54.1.4 Case 4

Let $baaab = bab = ca = 0$, $cb = baaaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \alpha^4 c, \\ b'a'a'a'a' &= \alpha^4 \delta baaaa. \end{aligned}$$

This case is similar to Case 3 and we have $3 + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa - baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb - baaaa,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb - \omega baaaa,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb - \omega^2 baaaa,\ pc, \text{ class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb - \omega^3 baaaa,\ pc, \text{ class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc - baaaa, \text{ class } 5\rangle.$$

54.1.5 Case 5

Let $baaab = ca = cb = 0$, $bab = baaaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^3 b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a'a' &= \alpha^7 baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$, and if $pc = baaaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 3$) $\omega^i baaaa$ for $i = 2, 3, 4, 5$.

So we have $2 + 2\gcd(p-1, 3) + \gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa - baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa - \omega baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa - \omega^i baaaa,\ pb,\ pc, \text{ class } 5\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb - baaaa,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb - \omega baaaa,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb - \omega^2 baaaa,\ pc, \text{ class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb - \omega^3 baaaa,\ pc, \text{ class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab - baaaa,\ ca,\ cb,\ pa,\ pb,\ pc - baaaa, \text{ class } 5\rangle.$$

54.1.6 Case 6

Let $baaab = ca = 0$, $bab = cb = baaaa$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^3 b + \varepsilon c, \\ c' &= \alpha^4 c, \\ b'a'a'a'a' &= \alpha^7 baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$ or (if $p = 1 \bmod 3$) $\omega baaaa$ or $\omega^2 baaaa$, and if $pc = baaaa$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$, and if $pb \neq 0$ we can take $pa = 0$.

If $pb = pc = 0$ then we can take $pa = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 3$) $\omega^i baaaa$ for $i = 2, 3, 4, 5$.

So we have $1 + 3\gcd(p-1,3) + \gcd(p-1,4)$ algebras

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa - \omega baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa - \omega^i baaaa,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb - \omega^2 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb - \omega^3 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc - \omega baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaab,\ bab - baaaa,\ ca,\ cb - baaaa,\ pa,\ pb,\ pc - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3).$$

54.2 Descendants of 6.326

Algebra 6.326 has $4p + 5 + 4\gcd(p-1,3) + 2\gcd(p-1,4) + 4\gcd(p-1,5)$ descendants of order p^7 and p-class 5.

Algebra 6.326 has presentation

$$\langle a, b, c \mid bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

and if L is a descendant of 6.326 of order p^7 then L_5 has order p and is generated by $baaaa$ and $baaab$. If a', b', c' generate L and satisfy the same relations as a, b, c modulo L_5 then

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \xi c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^6 baaaa + \alpha^5\beta baaab, \\ b'a'a'a'b' &= \alpha^7 baaab. \end{aligned}$$

So we can assume that $baaaa = 0$ and that L_5 is generated by $baaab$, or that $baaab = 0$ and that L_5 is generated by $baaaa$.

If $baaaa = 0$ then we need $\beta = 0$. Adding a suitable scalar multiple of baa to b we can take $bab = 0$, and adding a suitable scalar multiple of $baaa$ to c we can take $cb = 0$. Then scaling c we can take $ca = 0$ or $baaab$.

If $baaab = 0$ then adding a suitable scalar multiple of b to a we can take $bab - baaa = 0$, and adding a suitable scalar multiple of $baaa$ to c we can take $ca = 0$. Then scaling c we can take $cb = 0$ or $baaaa$.

So we have four possible commutator structures:

$$\begin{aligned} baaaa &= bab - baaa = ca = cb = 0, \\ baaaa &= bab - baaa = cb = 0,\ ca = baaab, \\ baaab &= bab - baaa = ca = cb = 0, \\ baaab &= bab - baaa = ca = 0,\ cb = baaaa. \end{aligned}$$

In each case we give the most general conditions on generators a', b', c' for L which satisfy the given commutator relations, and then compute the possibilities for pa, pb, pc.

54.2.1 Case 1

Let $baaaa = bab - baaa = ca = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a'b' &= \alpha^7 baaab. \end{aligned}$$

We can take $pc = 0$ or $baaab$ and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$ or $baaab$ or (if $p = 1 \bmod 5$) $\omega^i baaab$ for $i = 1, 2, 3, 4$. If $pb = 0$ we can take $pa = 0$, $baaab$, $\omega baaab$ or (if $p = 1 \bmod 3$) $\omega^i baaab$ for $i = 2, 3, 4, 5$, and if $pb \neq 0$ we can take $pa = 0$ or $pa = xbaaa$ with x in a transversal for the øfth roots of unity. So we have $p + 1 + 2\gcd(p-1, 3) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa - baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa - \omega^i baaab,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa,\ pb - \omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa - xbaaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (x \text{ in transversal for } 5^{th} \text{ roots of } 1),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa - xbaaab,\ pb - \omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4, x \text{ in transversal for}$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc - baaab,\ \text{class } 5\rangle.$$

54.2.2 Case 2

Let $baaaa = bab - baaa = cb = 0$, $ca = baaab$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \alpha^6 c, \\ b'a'a'a'b' &= \alpha^7 baaab. \end{aligned}$$

We can take $pc = 0$ or $baaab$ and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$ or $baaab$ or (if $p = 1 \bmod 5$) $\omega^i baaab$ for $i = 1, 2, 3, 4$. If $pb = 0$ we can take $pa = 0$, $baaab$, $\omega baaab$ or (if $p = 1 \bmod 3$) $\omega^i baaab$ for $i = 2, 3, 4, 5$, and if $pb \neq 0$ we can take $pa = 0$ or $pa = xbaaa$ with x in a transversal for the øfth roots of unity. So we have $p + 1 + 2\gcd(p-1, 3) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa - baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa - \omega^i baaab,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa,\ pb - \omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa - xbaaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle\ (x \text{ in transversal for } 5^{th} \text{ roots of } 1),$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa - xbaaab,\ pb - \omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4, x \text{ in transve}$$

$$\langle a, b, c \,|\, baaaa,\ bab - baaa,\ ca - baaab,\ cb,\ pa,\ pb,\ pc - baaab,\ \text{class } 5\rangle.$$

54.2.3 Case 3

Let $baaab = bab - baaa = ca = cb = 0$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \xi c, \\ b'a'a'a'a' &= \alpha^6 baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$ and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. If $pb = 0$ we can take $pa = 0$, $baaab$ or (if $p = 1 \bmod 5$) $\omega^i baaab$ for $i = 1, 2, 3, 4$, and if $pb \neq 0$ we can take $pa = 0$ or $pa = xbaaa$ with x in a transversal for the fourth roots of unity. So we have $p + 1 + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \,|\, baaab,\ bab - baaa,\ ca,\ cb,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaa,\ ca,\ cb,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaa,\ ca,\ cb,\ pa - \omega^i baaaa,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \,|\, baaab,\ bab - baaa,\ ca,\ cb,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab - baaa,\ ca,\ cb,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa,\, pb-\omega^2 baaaa,\, pc,\, \text{class } 5\rangle$ $(p=1 \bmod 4)$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa,\, pb-\omega^3 baaaa,\, pc,\, \text{class } 5\rangle$ $(p=1 \bmod 4)$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa-xbaaaa,\, pb-baaaa,\, pc,\, \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa-xbaaaa,\, pb-\omega baaaa,\, pc,\, \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa-xbaaaa,\, pb-\omega^2 baaaa,\, pc,\, \text{class } 5\rangle$ ($p=1 \bmod 4, x$ in transversal for 4^{th} roots of

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa-xbaaaa,\, pb-\omega^3 baaaa,\, pc,\, \text{class } 5\rangle$ ($p=1 \bmod 4, x$ in transversal for 4^{th} roots of

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb,\, pa,\, pb,\, pc-baaaa,\, \text{class } 5\rangle$.

54.2.4 Case 4

Let $baaab = bab - baaa = ca$, $cb = baaaa$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= \alpha^4 c, \\ b'a'a'a'a' &= \alpha^6 baaaa. \end{aligned}$$

We can take $pc = 0$ $baaaa$ or $\omega baaaa$ and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. If $pb = 0$ we can take $pa = 0$, $baaab$ or (if $p = 1 \bmod 5$) $\omega^i baaab$ for $i = 1,2,3,4$, and if $pb \neq 0$ we can take $pa = 0$ or $pa = xbaaa$ with x in a transversal for the fourth roots of unity. So we have $p + 2 + \gcd(p-1,4) + \gcd(p-1,5)$ algebras.

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb,\, pc,\, \text{class } 5\rangle$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-baaaa,\, pb,\, pc,\, \text{class } 5\rangle$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-\omega^i baaaa,\, pb,\, pc,\, \text{class } 5\rangle$ $(p=1 \bmod 5,\ i=1,2,3,4)$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb-baaaa,\, pc,\, \text{class } 5\rangle$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb-\omega baaaa,\, pc,\, \text{class } 5\rangle$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb-\omega^2 baaaa,\, pc,\, \text{class } 5\rangle$ $(p=1 \bmod 4)$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb-\omega^3 baaaa,\, pc,\, \text{class } 5\rangle$ $(p=1 \bmod 4)$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-xbaaaa,\, pb-baaaa,\, pc,\, \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-xbaaaa,\, pb-\omega baaaa,\, pc,\, \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-xbaaaa,\, pb-\omega^2 baaaa,\, pc,\, \text{class } 5\rangle$ ($p=1 \bmod 4, x$ in transversal for 4^{th}

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa-xbaaaa,\, pb-\omega^3 baaaa,\, pc,\, \text{class } 5\rangle$ ($p=1 \bmod 4, x$ in transversal for 4^{th}

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb,\, pc-baaaa,\, \text{class } 5\rangle$,

$\langle a,b,c \mid baaab,\, bab-baaa,\, ca,\, cb-baaaa,\, pa,\, pb,\, pc-\omega baaaa,\, \text{class } 5\rangle$.

54.3 Descendants of 6.327

Algebra 6.327 has $3p+12+2\gcd(p-1,3)+3\gcd(p-1,4)+\gcd(p-1,7)$ descendants of order p^7 and p-class 5.

Algebra 6.327 has presentation

$$\langle a,b,c \mid bab,\ ca,\ cb-baaa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

and if L is a descendant of 6.327 of order p^7 then L_5 has order p and is generated by $baaaa$ and $baaab$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_5 then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \alpha^3 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\delta baaaa+\alpha^3\beta\delta baaab, \\ b'a'a'a'b' &= \alpha^3\delta^2 baaab. \end{aligned}$$

We can take $baaaa=0$ or $baaab=0$. If $baaaa=0$ then adding a suitable scalar multiple of baa to b we can take $bab=0$, and adding a suitable scalar multiple of $baaa$ to c we can take $cb-baaa=0$. Then scaling we can take $ca=0$ or $baaab$. If $baaab=0$ then adding a suitable scalar multiple of c to b we can take $bab=0$, and adding a suitable scalar multiple of $baaa$ to c we can take $ca=0$. Then scaling we can take $cb-baaa=0$ or $baaaa$.

So we have four possible commutator structures:

$$\begin{aligned} baaaa &= bab=ca=cb-baaa=0, \\ baaaa &= bab=cb-baaa=0,\ ca=baaab, \\ baaab &= bab=ca=cb-baaa=0, \\ baaab &= bab=ca=0,\ cb-baaa=baaaa. \end{aligned}$$

In each case we give the most general form (modulo L_2) of generators a',b',c' for L which satisfy the given relations.

54.3.1 Case 1

Let $baaaa=bab=ca=cb-baaa=0$.

$$\begin{aligned} a' &= \alpha a+\gamma c, \\ b' &= \delta b+\varepsilon c, \\ c' &= \alpha^3 c, \\ b'a'a'a'b' &= \alpha^3\delta^2 baaab. \end{aligned}$$

We can take $pc=0$, $baaab$ or $\omega baaab$, and if $pc\neq 0$ we can take $pa=pb=0$.

If $pc=0$ we can take $pb=0$ or $baaab$. If $pb=pc=0$ we can take $pa=0$, $baaab$ or $\omega baaab$, and if $pb=baaab$, $pc=0$ then we can take $pa=0$, $baaab$ or $\omega baaab$ or (if $p=1 \bmod 4$) $\omega^2 baaab$ or $\omega^3 baaab$.

So we have $6+\gcd(p-1,4)$ algebras.

$$\langle a,b,c \mid baaaa,\ bab,\ ca,\ cb-baaa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - baaab,\ pb,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - baaab,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - baaab,\ pb - baaab,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - \omega baaab,\ pb - baaab,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - \omega^2 baaab,\ pb - baaab,\ pc,\ \text{class } 5 \rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa - \omega^3 baaab,\ pb - baaab,\ pc,\ \text{class } 5 \rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaab,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega baaab,\ \text{class } 5 \rangle.$$

54.3.2 Case 2

Let $baaaa = bab = cb - baaa = 0$, $ca = baaab$.

$$\begin{aligned} a' &= \delta^2 a + \gamma c, \\ b' &= \delta b + \varepsilon c, \\ c' &= \delta^6 c, \\ b'a'a'a'b' &= \delta^8 baaab. \end{aligned}$$

We can take $pc = 0$, $baaab$ or $\omega baaab$, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pb = 0$ or $baaab$ or (if $p = 1 \bmod 7$) $\omega^i baaab$ for $i = 1, 2, 3, 4, 5, 6$. If $pb = pc = 0$ we can take $pa = 0$, $baaab$ or $\omega baaab$ or (if $p = 1 \bmod 3$) $\omega^i baaab$ for $i = 2, 3, 4, 5$, and if $pb \neq 0$, $pc = 0$ then we can take $pa = 0$ or $xbaaab$ where x is in a transversal for the seventh roots of unity.

So we have $p + 2 + 2\gcd(p-1, 3) + \gcd(p-1, 7)$ algebras.

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa - baaab,\ pb,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca-baaab,\ cb-baaa,\ pa-\omega^i baaab,\ pb,\ pc,\ \text{class } 5 \rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa,\ pb - baaab,\ pc,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca-baaab,\ cb-baaa,\ pa,\ pb-\omega^i baaab,\ pc,\ \text{class } 5 \rangle\ (p = 1 \bmod 7,\ i = 1, 2, 3, 4, 5, 6),$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa - xbaaab,\ pb - baaab,\ pc,\ \text{class } 5 \rangle,$$

x in a transversal for the seventh roots of unity,

$$\langle a, b, c \mid baaaa,\ bab,\ ca-baaab,\ cb-baaa,\ pa-xbaaab,\ pb-\omega^i baaab,\ pc,\ \text{class } 5 \rangle\ (p = 1 \bmod 7,\ i = 1, 2, 3, 4, 5, 6),$$

x in a transversal for the seventh roots of unity,

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa,\ pb,\ pc - baaab,\ \text{class } 5 \rangle,$$

$$\langle a, b, c \mid baaaa,\ bab,\ ca - baaab,\ cb - baaa,\ pa,\ pb,\ pc - \omega baaab,\ \text{class } 5 \rangle.$$

54.3.3 Case 3

Let $baaab = bab = ca = cb - baaa = 0$.

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= \alpha^3 c, \\ b'a'a'a'a' &= \alpha^4 \delta baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$ and we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$.

If either of pb or pc are non-zero then we can take $pa = 0$, and if $pb = pc = 0$ then we can take $pa = 0$ or $baaaa$.

So we have $3 + 2\gcd(p-1, 4)$ algebras

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega^2 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega^3 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - baaaa,\ pc - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega baaaa,\ pc - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega^2 baaaa,\ pc - baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa,\ pb - \omega^3 baaaa,\ pc - baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle.$$

54.3.4 Case 4

Let $baaab = bab = ca = 0$, $cb - baaa = baaaa$.

$$\begin{aligned} a' &= a + \beta b + \gamma c, \\ b' &= \delta b, \\ c' &= c, \\ b'a'a'a'a' &= \delta baaaa. \end{aligned}$$

We can take $pc = 0$ or $baaaa$ and we can take $pb = xbaaaa$ with $0 \leq x < p$.

If either of pb or pc are non-zero then we can take $pa = 0$, and if $pb = pc = 0$ then we can take $pa = 0$ or $baaaa$.

So we have $2p + 1$ algebras

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa - baaaa,\ pa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa - baaaa,\ pa,\ pb - xbaaaa,\ pc - baaaa,\ \text{class } 5\rangle\ (0 \leq x < p),$$

$$\langle a, b, c \,|\, baaab,\ bab,\ ca,\ cb - baaa - baaaa,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle.$$

Algebra 6.328 has $p^2+3p-2+(p+3)\gcd(p-1,3)+2\gcd(p-1,4)+2\gcd(p-1,5)$ descendants of order p^7 and p-class 5.

Algebra 6.328 has presentation

$$\langle a,b,c\,|\,bab-baaa,\ ca,\ cb-baaa,\ pa,\ pb,\ pc,\ \text{class }4\rangle,$$

and if L is a descendant of 6.328 of order p^7 then L_5 has order p and is generated by $baaaa$ and $baaab$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_5 then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c,\\ b' &= \alpha^2 b+\varepsilon c,\\ c' &= \alpha^3 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^6 baaaa+\alpha^5\beta baaab,\\ b'a'a'a'b' &= \alpha^7 baaab. \end{aligned}$$

We can take $baaaa=0$ or $baaab=0$. If $baaaa=0$ then adding a suitable scalar multiple of baa to b we can take $bab-baaa=0$, and adding a suitable scalar multiple of $baaa$ to c we can take $cb-baaa=0$. Scaling, we can take $ca=0$, $baaab$ or (if $p=1\bmod 3$) $\omega baaab$ or $\omega^2 baaabb$. If $baaab=0$ then adding a suitable scalar mutiple of c to b we can take $bab-baaa=0$, adding suitable scalar multiples of b to a and c to b we can take $cb-baaa=0$, and adding a suitable scalar multiple of $baaa$ to c we can take $ca=0$.

So we have $2+\gcd(p-1,3)$ possible commutator structures:

$$\begin{aligned} baaaa &= bab-baaa=ca=cb-baaa=0,\\ baaaa &= bab-baaa=cb-baaa=0,\ ca=baaab,\\ baaaa &= bab-baaa=cb-baaa=0,\ ca=\omega baaab\,(p=1\bmod 3),\\ baaaa &= bab-baaa=cb-baaa=0,\ ca=\omega^2 baaab\,(p=1\bmod 3),\\ baaab &= bab-baaa=ca=cb-baaa=0. \end{aligned}$$

In each case we give (modulo L_2) the most general possible generators a',b',c' for L which satisfy the given commutator relations.

54.4.1 Case 1

Let $baaaa=bab-baaa=ca=cb-baaa=0$.

$$\begin{aligned} a' &= \alpha a+\gamma c,\\ b' &= \alpha^2 b+\varepsilon c,\\ c' &= \alpha^3 c,\\ b'a'a'a'b' &= \alpha^7 baaab. \end{aligned}$$

We can take $pc=0$, $baaab$, $\omega baaab$ or (if $p=1\bmod 4$) $\omega^2 baaab$ or $\omega^3 baaab$, and if $pc\neq 0$ then we can take $pa=pb=0$.

If $pc=0$ then we can take $pb=0$, $baaab$ or (if $p=1\bmod 5$) $\omega^i baaab$ for $i=1,2,3,4$. If $pb=pc=0$ then we can take $pa=0$, $baaab$, $\omega baaab$ or (if $p=1\bmod 3$) $\omega^i baaab$ for

$i = 2, 3, 4, 5$. And if $pb \neq 0$, $pc = 0$ then we can take $pa = 0$ or $xbaaab$ with x in a transversal for the øfth roots of unity.

So we have $p + 2\gcd(p-1,3) + \gcd(p-1,4) + \gcd(p-1,5)$ algebras.

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa - baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa - \omega baaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca,\ cb{-}baaa,\ pa{-}\omega^i baaab,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3,\ i = 2, 3, 4, 5),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca,\ cb{-}baaa,\ pa,\ pb{-}\omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa - xbaaab,\ pb - baaab,\ pc,\ \text{class } 5\rangle,$$

x in a transversal for the øfth roots of unity,

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca,\ cb{-}baaa,\ pa{-}xbaaab,\ pb{-}\omega^i baaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

x in a transversal for the øfth roots of unity,

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaab,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega baaab,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega^2 baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega^3 baaab,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

54.4.2 Case 2

Let $baaaa = bab - baaa = cb - baaa = 0$, $ca = kbaaab$ $(k = 1, \omega, \omega^2)$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b + \varepsilon c, \\ c' &= c, \\ b'a'a'a'b' &= \alpha baaab, \end{aligned}$$

with $\alpha^3 = 1$.

We can take $pc = 0$ or $xbaaab$ with x in a transversal for the cube roots of unity, and if $pc \neq 0$ we can take $pa = pb = 0$.

If $pc = 0$ we can take $pa = xbaaab$ with $0 \leq x < p$ and we can take $pb = 0$ or $ybaaab$ with y in a transversal for the cube roots of unity.

So we have $p^2 - 1 + p\gcd(p-1,3)$ algebras, where in all the algebras below $0 \leq x < p$ and y lies in a transversal for the cube roots of unity:

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca - baaab,\ cb - baaa,\ pa - xbaaab,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca - baaab,\ cb - baaa,\ pa - xbaaab,\ pb - ybaaab,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab - baaa,\ ca - baaab,\ cb - baaa,\ pa,\ pb,\ pc - ybaaab,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega baaab,\ cb{-}baaa,\ pa{-}xbaaab,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega baaab,\ cb{-}baaa,\ pa{-}xbaaab,\ pb{-}ybaaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega baaab,\ cb{-}baaa,\ pa,\ pb,\ pc{-}ybaaab,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega^2 baaab,\ cb{-}baaa,\ pa{-}xbaaab,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega^2 baaab,\ cb{-}baaa,\ pa{-}xbaaab,\ pb{-}ybaaab,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaaa,\ bab{-}baaa,\ ca{-}\omega^2 baaab,\ cb{-}baaa,\ pa,\ pb,\ pc{-}ybaaab,\ \text{class } 5\rangle\ (p = 1 \bmod 3).$$

54.4.3 Case 3

Let $baaab = bab - baaa = ca = cb - baaa = 0$.

$$\begin{aligned} a' &= \alpha a + \gamma c, \\ b' &= \alpha^2 b, \\ c' &= \alpha^3 c, \\ b'a'a'a'a' &= \alpha^6 baaaa. \end{aligned}$$

We can take $pc = 0$, $baaaa$ or (if $p = 1 \bmod 3$) $\omega baaaa$ or $\omega^2 baaaa$.

If $pc \neq 0$ then we can take $pa = 0$ and $pb = 0$ or $xbaaaa$ where x lies in a transversal for the cube roots of unity.

If $pc = 0$ then we can take $pa = 0$, $baaaa$ or (if $p = 1 \bmod 5$) $\omega^i baaaa$ for $i = 1, 2, 3, 4$. If $pa = pc = 0$ then we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$. And if $pa \neq 0$, $pc = 0$ then we can take $pb = 0$ or $pb = xbaaaa$ where x lies in a transversal for the øfth roots of unity.

So we have $2p - 1 + \gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - \omega^2 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - \omega^3 baaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 4),$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa - \omega^i baaaa,\ pb,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa - baaaa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle,$$

x in a transversal for the øfth roots of unity,

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa - \omega^i baaaa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle\ (p = 1 \bmod 5,\ i = 1, 2, 3, 4),$$

x in a transversal for the øfth roots of unity,

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - baaaa,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb,\ pc - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - xbaaaa,\ pc - baaaa,\ \text{class } 5\rangle,$$

x in a transversal for the cube roots of unity,

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - xbaaaa,\ pc - \omega baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

x in a transversal for the cube roots of unity,

$$\langle a, b, c \mid baaab,\ bab - baaa,\ ca,\ cb - baaa,\ pa,\ pb - xbaaaa,\ pc - \omega^2 baaaa,\ \text{class } 5\rangle\ (p = 1 \bmod 3),$$

x in a transversal for the cube roots of unity, Algebra 5.32 has $4+2\gcd(p-1,3)$ descendants of order p^6 (6.356 ˇ 6.365), but only 6.362 is capable. The corresponding group is number 38 in the list of capable groups order p^5 in our paper. Algebra 6.362 has $p^2+7p+3+2\gcd(p-1,3)+3\gcd(p-1,4)+\gcd(p-1,5)$ descendants of order p^7 and p-class 5.

6.362 $\quad p^2+7p+3+2\gcd(p-1,3)+3\gcd(p-1,4)+\gcd(p-1,5)$

55.1 Descendants of 6.362

Algebra 6.362 has $p^2+7p+3+2\gcd(p-1,3)+3\gcd(p-1,4)+\gcd(p-1,5)$ descendants of order p^7 and p-class 5.

Algebra 6.362 has presentation

$$\langle a,b,c \mid bab,\ ca,\ cb-baa,\ pa,\ pb,\ pc,\ \text{class } 4\rangle,$$

and if L is a descendant of 6.362 then L_5 is generated by $baaaa$. Adding a suitable scalar multiple of $baaa$ to c we can take $ca=0$. If a',b',c' generate L and satisfy the same relations as a,b,c modulo L_5, and if $c'a'=0$ then

$$\begin{aligned} a' &= \alpha a+\beta b,\\ b' &= \delta b,\\ c' &= \alpha^2 c \end{aligned}$$

modulo L_2 and

$$\begin{aligned} b'a'a'a'a' &= \alpha^4\delta baaaa,\\ b'a'b' &= \alpha\delta^2 bab,\\ c'b'-b'a'a' &= \alpha^2\delta(cb-baaa)-2\alpha\beta\delta bab. \end{aligned}$$

We can take $bab=0$ or $baaaa$, and if $bab=baaaa$ we can take $cb-baa=0$. If $bab=0$ then we can take $cb-baa=0$, $baaaa$ or $\omega baaaa$. So we have four possible commutator structures:

$$\begin{aligned} bab &= ca=cb-baa=0,\\ bab &= ca=0,\ cb-baa=baaaa,\\ bab &= ca=0,\ cb-baa=\omega baaaa,\\ ca &= cb-baa=0,\ bab=baaaa. \end{aligned}$$

In each case we give the most general possible form (modulo L_2) for generators a',b',c' which satisfy the given commutator relations, and we compute the possibilities for pa,pb,pc.

55.1.1 Case 1

Let $bab=ca=cb-baa=0$.

$$\begin{aligned} a' &= \alpha a+\beta b,\\ b' &= \delta b,\\ c' &= \alpha^2 c,\\ b'a'a'a'a' &= \alpha^4\delta baaaa. \end{aligned}$$

We can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^3 baaaa$, and we can take $pc = 0$ or $baaaa$. If $pb \neq 0$ we can take $pa = 0$, and if $pb = 0$ we can take $pa = 0$ or $baaaa$.

We have $4 + 2\gcd(p-1, 4)$ algebras.

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ pc, \text{ class } 5\rangle,$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - baaaa,\ pc, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baaaa,\ pc, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaaa,\ pc, \text{ class } 5\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^3 baaaa,\ pc, \text{ class } 5\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb,\ pc - baaaa, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - baaaa,\ pc - baaaa, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega baaaa,\ pc - baaaa, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaaa,\ pc - baaaa, \text{ class } 5\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa,\ pb - \omega^3 baaaa,\ pc - baaaa, \text{ class } 5\rangle \ (p = 1 \bmod 4),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa - baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa,\ pa - baaaa,\ pb,\ pc - baaaa, \text{ class } 5\rangle.$$

55.1.2 Case 2

Let $bab = ca = 0$, $cb - baa = kbaaaa$ with $k = 1$ or ω.

$$\begin{aligned} a' &= \pm a + \beta b, \\ b' &= \delta b, \\ c' &= c, \\ b'a'a'a'a' &= \delta baaaa. \end{aligned}$$

We can take $pb = xbaaaa$ with $0 \leq x < p$ and $pc = 0$ or $baaaa$. If $pb \neq 0$ we can take $pa = 0$. If $pb = pc = 0$ we can take $pa = 0$ or $baaaa$, and if $pb = 0$, $pc = baaaa$ then we can take $pa = xbaaaa$ with $0 \leq x \leq (p-1)/2$.

We have $5p + 1$ algebras.

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - baaaa,\ pa,\ pb - xbaaaa,\ pc, \text{ class } 5\rangle \ (0 \leq x < p),$$

$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - baaaa,\ pa,\ pb - xbaaaa,\ pc - baaaa, \text{ class } 5\rangle \ (0 \leq x < p),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - baaaa,\ pa - baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - baaaa,\ pa - xbaaaa,\ pb,\ pc - baaaa, \text{ class } 5\rangle \ (0 < x \leq (p-1)/2),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - \omega baaaa,\ pa,\ pb - xbaaaa,\ pc, \text{ class } 5\rangle \ (0 \leq x < p),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - \omega baaaa,\ pa,\ pb - xbaaaa,\ pc - baaaa, \text{ class } 5\rangle \ (0 \leq x < p),$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - \omega baaaa,\ pa - baaaa,\ pb,\ pc, \text{ class } 5\rangle,$$
$$\langle a, b, c \,|\, bab,\ ca,\ cb - baa - \omega baaaa,\ pa - xbaaaa,\ pb,\ pc - baaaa, \text{ class } 5\rangle \ (0 < x \leq (p-1)/2).$$

55.1.3 Case 3

Let $ca = cb - baa = 0$, $bab = baaaa$.

$$\begin{aligned} a' &= \alpha a, \\ b' &= \alpha^3 b, \\ c' &= \alpha^2 c, \\ b'a'a'a'a' &= \alpha^7 baaaa. \end{aligned}$$

We can take $pa = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 3$) $\omega^i baaaa$ with $i = 2, 3, 4, 5$.

If $pa \neq 0$ we can take $pc = 0$ or $xbaaaa$ with x in a transversal for the sixth roots of unity. If $pa \neq 0$ and $pc = 0$ then we can take $pb = 0$ or $xbaaaa$ with x in a transversal for the cube roots of unity, and if $pa \neq 0$ and $pc \neq 0$ we can take $pb = ybaaaa$ with $0 \leq y < p$.

If $pa = 0$ we can take $pb = 0$, $baaaa$, $\omega baaaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaaa$ or $\omega^2 baaaa$. If $pa = pb = 0$ then we can take $pc = 0$, $baaaa$ or (if $p = 1 \bmod 5$) $\omega^i baaaa$ with $i = 1, 2, 3, 4$, and if $pa = 0$, $pb \neq 0$ then we can take $pc = 0$ or $xbaaaa$ with x in a transversal for the fourth roots of unity.

We have $p^2 + 2p - 2 + 2\gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras.

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb,\ pc - baaaa,\ \text{class } 5\rangle,$$

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb,\ pc - \omega^i baaaa,\ \text{class } 5\rangle$ $(p = 1 \bmod 5,\ i = 1, 2, 3, 4)$,

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - baaaa,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega baaaa,\ pc,\ \text{class } 5\rangle,$$

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaaa,\ pc,\ \text{class } 5\rangle$ $(p = 1 \bmod 4)$,

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega^3 baaaa,\ pc,\ \text{class } 5\rangle$ $(p = 1 \bmod 4)$,

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - baaaa,\ pc - xbaaaa,\ \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega baaaa,\ pc - xbaaaa,\ \text{class } 5\rangle$ (x in transversal for 4^{th} roots of 1),

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega^2 baaaa,\ pc - xbaaaa,\ \text{class } 5\rangle$ ($p = 1 \bmod 4, x$ in transversal for 4^{th} roots of

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa,\ pb - \omega^3 baaaa,\ pc - xbaaaa,\ \text{class } 5\rangle$ ($p = 1 \bmod 4, x$ in transversal for 4^{th} roots of

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega baaaa,\ pb,\ pc,\ \text{class } 5\rangle,$$

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega^i baaaa,\ pb,\ pc,\ \text{class } 5\rangle$ $(p = 1 \bmod 3, i = 2, 3, 4, 5)$,

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - baaaa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle$ (x in transversal for cube roots of 1),

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega baaaa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle$ (x in transversal for cube roots of 1),

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega^i baaaa,\ pb - xbaaaa,\ pc,\ \text{class } 5\rangle$ ($p = 1 \bmod 3, i = 2, 3, 4, 5, x$ in transversal for

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - baaaa,\ pb - ybaaaa,\ pc - xbaaaa,\ \text{class } 5\rangle,$$

x in a transversal for the sixth roots of unity, $0 \leq y < p$,

$$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega baaaa,\ pb - ybaaaa,\ pc - xbaaaa,\ \text{class } 5\rangle,$$

x in a transversal for the sixth roots of unity, $0 \leq y < p$,

$\langle a, b, c \mid bab - baaaa,\ ca,\ cb - baa,\ pa - \omega^i baaaa,\ pb - ybaaaa,\ pc - xbaaaa,\ \text{class } 5\rangle$ $(p = 1 \bmod 3, i = 2, 3, 4, 5)$,

x in a transversal for the sixth roots of unity, $0 \leq y < p$.

56 Grandchildren of algebra 39 (4.1)

Algebra 39 has 24 capable immediate descendants of order p^6. These are 6.9 ˘ 6.21, 6.23, 6.24, 6.29, 6.33 ˘ 6.36, 6.48, 6.51, 6.52, 6.60.

56.1 Descendants of 6.9

There are 7 algebras here. I have checked that the recipe below gives 7 non-isomorphic groups for $p = 5, 7$.

Algebra 6.9 is abelian, and is isomorphic to $C_{p^2} \oplus C_{p^2} \oplus C_p \oplus C_p$. So if L is a descendant of 6.9 of order p^7 then we may suupose that L is generated by a, b, c, d, that L_2 is generated by pa, pc modulo L_3 and that L_3 is generated by $p^2 a$. Furthermore we may assume that $p^2 b = pc = pd = 0$. All commutators are scalar multiples of $p^2 a$. If a', b', c', d' satisfy the same power relations as a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, and then $p^2 a' = \alpha p^2 a$.

First consider the case when $cb = db = dc = 0$. If $ca = da = 0$ then we can assume that $ba = 0$ or $p^2 a$. If ca, da are not both zero then we can assume that $ba = da = 0$ and that $ca = p^2 a$.

Next consider the case when $dc = 0$ but db, cb are not both zero. Then we can assume that $db = dc = 0$ and that $cb = p^2 a$. We can then assume that $ba = ca = 0$ and that $da = 0$ or $p^2 a$.

Finally consider the case when $dc \neq 0$. Then we can assume that $dc = p^2 a$ and that $ca = da = cb = db = 0$ and that $ba = 0$ or $p^2 a$.

56.2 Descendants of 6.10

Algebra 6.10 has 8 descendants of order p^7. I have checked that the recipes below give 8 non-isomorphic algebras for $p = 5, 7$.

Algebra 6.10 has class 2 and relators

$$ca, da, cb, db, dc, pa, pb, pc - ba.$$

If L is a descendant of 6.10 of order p^7 then L is generated by a, b, c, d, L_2 is generated by ba , pd modulo L_3 and L_3 is generated by $p^2 d$. The elements

$$ca, da, cb, db, dc, pa, pb, pc - ba$$

are all scalar multiples of $p^2 d$. subtracting suitable multiples of pd from a, b, c we may suppose that $pa = pb = 0$ and that $pc = ba$. If a', b', c', d' generate L and satisfy the same relations as a, b, c, d modulo L_3, and if $pa' = pb' = 0$ and $pc' = b'a'$ then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & \alpha\delta - \beta\gamma & 0 \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

First consider the case when $ca = cb = 0$. We can suppose that $da = 0$ and that $db = \rho p^2 d$, $dc = \sigma p^2 d$ for some ρ, σ. Letting $\beta = \gamma = 0$ above we then have

$$\begin{aligned} d'b' &= \delta\xi db = \delta\rho p^2 d', \\ d'c' &= \alpha\delta\xi dc = \alpha\delta\sigma p^2 d' \end{aligned}$$

so we can assume that $\rho, \sigma = 0, 1$.

Next consider the case when ca, cb are not both zero then we can assume that $ca = p^2 d$ and that $cb = 0$. We can then choose ν so that $da = 0$. Let $db = \rho p^2 d$, $dc = \sigma p^2 d$. Letting $\beta = \gamma = 0$ above, and letting $\xi = \alpha^2\delta$ so that $c'a' = p^2 d'$ we then have

$$\begin{aligned} d'b' &= \delta\rho p^2 d', \\ d'c' &= \alpha\delta\sigma p^2 d' \end{aligned}$$

so that again we can take $\rho, \sigma = 0, 1$.

56.3 Descendants of 6.11

Algebra 6.11 has 39 descendants of order p^7. I have checked that the recipes below give 39 non-isomorphic groups for $p = 5, 7$.

Algebra 6.11 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc,\ pd.$$

If L is a descendant of 6.11 of order p^7 then L is generated by a, b, c, d, L_2 is generated by ba, pa modulo L_3 and L_3 is generated by baa, bab and p^2a. The elements ca, da, cb, db, dc, pb, pc, pd all lie in L_3. We consider three possibilities:

$$\begin{aligned} baa &= 0,\ bab = 0, \\ baa + \beta bab &= 0,\ bab \neq 0, \\ baa &\neq 0,\ bab = 0. \end{aligned}$$

56.3.1 Case 1

Let $baa = bab = 0$. Then L_3 is generated by p^2a, and by subtracting suitable multiples of pa from b, c, d we may suppose that $pb = pc = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as a, b, c, d modulo L_2 then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $cb = db = dc = 0$ then we can assume that $da = 0$ and that $ca = 0$ or p^2a.

If $dc = 0$ but cb, db are not both zero then we may assume that $cb = p^2a$, $db = dc = 0$. This restricts us to

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \varepsilon\lambda & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Let $ca = \rho p^2 a$, $da = \sigma p^2 a$. Then

$$\begin{aligned} c'a' &= \varepsilon\lambda^2 ca + \lambda\beta cb + \varepsilon\lambda\mu da, \\ d'a' &= \varepsilon\lambda\xi da. \end{aligned}$$

We can always choose β so that $c'a' = 0$, and so we can assume that $ca = 0$, $da = 0$ or $p^2 a$.

If $dc \neq 0$ then we can assume that $dc = p^2 a$, that $cb = db = 0$. This restricts us to

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \lambda\xi - \mu\nu & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, and we can take $\lambda = \xi = 1$, $\mu = \nu = 0$ and choose γ, δ so that $c'a' = d'a' = 0$. So we can assume that $dc = p^2 a$ and that $ca = da = cb = db = 0$.

56.3.2 Case 2

Let $baa + \beta bab = 0$, $bab \neq 0$. Then replacing a by $a + \beta b$ we can assume that $baa = 0$ and that L_3 is spanned by bab.

First, consider the case when $p^2 a \neq 0$. Then we can assume that $p^2 a = bab$ or ωbab and that $pb = pc = pd = 0$. This restricts us to

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \pm 1 & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

By subtracting suitable scalar multiples of ba from c and d we can assume that $cb = db = 0$. If $dc \neq 0$ then we can assume that $dc = bab$ and that $ca = da = 0$. And if $dc = 0$ then we can assume that $da = 0$ and that $ca = 0$ or bab.

Next consider the case when $p^2 a = 0$. We consider

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. As above, we can assume that $cb = db = 0$.

If $dc \neq 0$ then we can assume that $dc = bab$ and that $ca = da = 0$. We are then restricted to a', b', c', d' of the form

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, where $\lambda\xi - \mu\nu = \alpha\varepsilon^2$. So we can assume that $pd = 0$, $pc = 0$ or bab. If $pc = bab$ we can assume that $pb = 0$. But if $pc = pd = 0$ then we can assume that $pb = 0$ or bab.

If $dc = 0$ then we can assume that $da = 0$ and that $ca = 0$ or bab. Recall that we also assume that $cb = db = 0$. Consider the case when $ca = da = 0$, so that c, d are central. Then if one of pc, pd is non-zero we can assume that $pb = pd = 0$ and that $pc = bab$. And if $pc = pd = 0$ then we can assume that $pb = 0$ or bab. On the other hand, if $da = dc = 0$, $ca = bab$ then we are restricted to a', b', c', d' of the form

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \varepsilon^2 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So if $pd \neq 0$ we can assume that $pb = pc = 0$, $pd = bab$. If $pd = 0$, $pc \neq 0$ then we can assume that $pb = 0$, $pc = bab$. And if $pc = pd = 0$ then we can assume that $pb = 0$ or bab.

56.3.3 Case 3

Let $baa \neq 0$, $bab = 0$ then L_3 is spanned by baa.

First consider the case when $p^2a \neq 0$. Then we can assume that $p^2a = baa$ and that $pb = pc = pd = 0$. We are then restricted to a', b', c', d' of the form

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^{-1} & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

By subtracting suitable multiples of ba from c and d we can assume that $ca = da = 0$. If $dc \neq 0$ then we can assume that $dc = baa$ and that $cb = db = 0$. And if $dc = 0$ we can assume that $db = 0$, $cb = 0$ or baa.

Next consider the case when $p^2a = 0$. Again, subtracting suitable multiples of ba from c, d we may suppose that $ca = da = 0$. And we may assume that either $dc = baa$, $db = cb = 0$, or that $db = dc = 0$, $cb = 0$ or baa.

If $dc = baa$, $db = dc = 0$ then we are restricted to a', b', c', d' of the form

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

with $\lambda\xi - \mu\nu = \alpha^2\varepsilon$. So we can assume that $pb = 0$, baa or ωbaa, $pc = 0$ or baa, $pd = 0$.

And if $cb = db = dc = 0$ then we can consider a', b', c', d' of the form

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

so we can assume that $pd = 0$, $pc = baa$, $pb = 0$ or that $pc = pd = 0$, $pb = 0$, baa or ωbaa.

Finally if $db = dc = 0$, $cb = baa$ we are restricted to

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \alpha^2 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

so we can assume that $pd = baa$, $pb = pc = 0$, or $pd = 0$, $pc = baa$, $pb = 0$, or $pc = pd = 0$, $pb = 0$, baa or ωbaa.

Algebra 6.12 has 8 descendants of order p^7. I have checked that the recipes below give 8 non-isomorphic groups for $p = 5, 7$.

Algebra 6.12 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc,\ pb - ba,\ pc,\ pd.$$

If L is a descendant of 6.12 of order p^7 then L_3 is generated by baa, p^2a, and $ca,\ da,\ cb,\ db,\ dc,\ pb-ba,\ pc,\ pd$ all lie in L_3. We deal with the cases $baa = 0$ and $baa \neq 0$ separately.

56.4.1 $baa = 0$

If $baa = 0$ then L_3 is generated by p^2a. By subtracting suitable multiples of pa from b, c, d we may assume that $pb = ba$, $pc = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as a, b, c, d modulo L_3, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

Consider the case when $cb = db = dc = 0$. Then we can assume that $da = 0$ and that $ca = 0$ or p^2a.

Next, consider the case when $dc = 0$ but one of cb, db is non-zero. Then we can assume that $db = 0$, $cb = p^2a$. We are then restricted to

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \varepsilon^{-1} & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $da = p^2a$ and $ca = 0$ or $da = 0$ and $ca = 0$.

Finally consider the case when $dc \neq 0$. Then we can assume that $dc = p^2a$ and that $ca = da = cb = db = 0$.

56.4.2 $baa \neq 0$

If $baa \neq 0$ then we have $p^2a = \rho baa$ for some ρ. But then, if we let $a' = a - \rho b$ then we have

$$p^2a' = p^2a - \rho p^2b = 0.$$

So we can assume that $p^2a = 0$. Subtracting suitable multiples of ba from c, d we may assume that $pc = pd = 0$. Let $pb = ba + \rho baa$. If we let $a' = a + \rho pa$ then $pb = ba'$. So we can also asume that $pb = ba$. And by subtracting suitable multiples of pa from c, d we can assume that $cb = db = 0$. If a', b', c', d' generate L, and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $cb = db = dc = 0$, then we can assume that $da = 0$ and that $ca = 0$ or baa. But if $dc \neq 0$ then we can assume that $dc = baa$ and that $ca = da = cb = db = 0$.

56.5 Descendants of 6.13

Algebra 6.13 has 11 descendants of order p^7. I have checked that the recipes below give 11 non-isomorphic groups for $p = 5, 7$.

Algebra 6.13 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc,\ pb,\ pc - ba,\ pd.$$

If L is a descendant of 6.13 of order p^7 then L_3 is generated by p^2a, and ca, da, cb, db, dc, pb, $pc-ba$, pd all lie in L_3. By subtracting suitable scalar multiples of pa from b, c, d we can assume that $pb = pd = 0$, $pc = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \alpha\varepsilon & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

We can assume that $dc = 0$ or p^2a, and we can (independently) assume that $db = 0$ or p^2a.

First consider the case when $db = dc = 0$. Then we can assume that $da = 0$ or p^2a. If $da = 0$ then we can assume that $cb = 0$, p^2a or ωp^2a. If $cb \neq 0$ we can assume that $ca = 0$, but if $cb = 0$ then we can assume that $ca = 0$ or p^2a. If $da = p^2a$, then we can take $ca = 0$ (though we then require $\beta = 0$), and we can $cb = 0$, p^2a or ωp^2a.

Next consider the case when $db = p^2a$, $dc = 0$. Then we can assume that $da = cb = 0$, though we then require $\beta = \mu = 0$. We can assume that $ca = 0$ or p^2a.

Now consider the case when $db = 0$, $dc = p^2a$. Then we can assume that $da = cb = 0$, though we then require $\gamma = \eta = 0$. We can also assume that $ca = 0$.

Finally consider the case when $db = dc = p^2a$. We then require $\alpha = 1$ and $\xi = \varepsilon^{-1}$. We can assume that $da = 0$, though we then require $\beta + \gamma = 0$. We can also assume that $cb = 0$, though we then require $\eta = \mu$. And ønally we can also assume that $ca = 0$.

56.6 Descendants of 6.14

Algebra 6.14 has 39 descendants of order p^7. I have checked that the recipes below give 39 non-isomorphic groups for $p = 5, 7$.

Algebra 6.14 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc,\ pa,\ pb,\ pd.$$

If L is a descendant of 6.14 of order p^7 then L_3 is generated by baa, bab, p^2c, and ca, da, cb, db, dc, pa, pb, pd all lie in L_3. We consider three situations: $baa = bab = 0$, $bab = p^2c = 0$, $bab = 0$, $p^2c = baa$.

56.6.1 $baa = bab = 0$

If $baa = bab = 0$ then L_3 is generated by p^2c. By subtracting suitable scalar multiples of pc from a, b, d we can assume that $pa = pb = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ \varepsilon & \zeta & 0 & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $dc = 0$ or p^2c, and (independently) we can assume that $db = 0$, $da = 0$ or p^2c.

First consider the case when $da = db = dc = 0$. Then we can assume that $cb = 0$ and that $ca = 0$ or p^2c.

Next consider the case when $da = p^2c$, $db = dc = 0$. We then need $\varepsilon = 0$. We can assume that $ca = 0$ and that $cb = 0$ or p^2c.

Now consider the case when $dc = p^2c$ and $da = db = 0$. Then we can assume that $ca = cb = 0$.

Finally consider the case when $da = dc = p^2c$ and $db = 0$. We then need $\varepsilon = 0$, $\alpha = \lambda$, $\xi = 1$. We can then assume that $ca = cb = 0$.

56.6.2 $bab = p^2c = 0$

If $bab = p^2c = 0$ then L_3 is generated by baa. By subtracting suitable scalar multiples of ba from c, d we may suppose that $ca = da = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ 0 & \zeta & 0 & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can suppose that $dc = 0$ or baa and (independently) that $db = 0$ or baa.

When $db = dc = 0$ we can assume that $cb = 0$ or baa.

When $dc = 0$, $db = baa$ we can assume that $cb = 0$.

When $dc = baa$, $db = 0$ we can also assume that $cb = 0$.

And when $db = dc = baa$ we can also assume that $cb = 0$.

So consider the case when $cb = db = dc = 0$. We can assume that $pd = baa$, $pa = pb = 0$, or $pd = 0$, $pb = baa$ or ωbaa, $pa = 0$, or $pd = pb = 0$, $pa = 0$ or baa.

Next, consider the case when $cb = baa$, $db = dc = 0$. We then need $\lambda = \alpha^2$. We can assume that $pd = baa$, $pa = pb = 0$, or $pd = 0$, $pb = baa$ or ωbaa, $pa = 0$, or $pd = pb = 0$, $pa = 0$ or baa.

Now consider the case when $db = baa$, $cb = dc = 0$. we then need $\mu = 0$, $\xi = \alpha^2$. Again, we can assume that $pd = baa$, $pa = pb = 0$, or $pd = 0$, $pb = baa$ or ωbaa, $pa = 0$, or $pd = pb = 0$, $pa = 0$ or baa.

And now consider the case when $dc = baa$, $cb = db = 0$. We then need $\eta = 0$, $\lambda = \alpha^2\zeta\xi^{-1}$. We can assume that $pd = baa$, $pa = 0$ and $pb = 0$, baa or ωbaa, or that $pd = 0$, $pb = baa$ or ωbaa, $pa = 0$, or $pd = pb = 0$, $pa = 0$ or baa.

Finally consider the case when $db = dc = baa$ and $cb = 0$. We then need $\lambda = \zeta$, $\mu = \eta$, $\xi = \alpha^2$. Once again we can assume that $pd = baa$, $pa = pb = 0$, or $pd = 0$, $pb = baa$ or ωbaa, $pa = 0$, or $pd = pb = 0$, $pa = 0$ or baa.

56.6.3 $bab = 0$, $p^2c = baa$

If $bab = 0$ and $p^2c = baa$ then L_3 is generated by baa. By subtracting suitable scalar multiples of pc from a, b, d we can assume that $pa = pb = pd = 0$. and subtracting suitable scalar multiples of ba from c, d we may suppose that $ca = da = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ 0 & \zeta & 0 & \eta \\ 0 & 0 & \alpha^2\zeta & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We may assume that $dc = 0$ or baa. If $dc = 0$ then we can assume that $db = 0$ or baa. And if $dc = baa$ then we need $\xi = 1$ so that we can assume that $db = 0$, baa or ωbaa. If either of dc or db are non-zero then we can assume that $cb = 0$. But if $db = dc = 0$ then we can assume that $cb = 0$ or baa.

56.7 Descendants of 6.15

Algebra 6.15 has $2p + 6$ descendants of order p^7. I have checked that the recipes below give $2p + 6$ non-isomorphic groups for $p = 5, 7, 11$.

Algebra 6.15 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc,\ pa - ba,\ pb,\ pd.$$

If L is a descendant of 6.13 of order p^7 then L_3 is generated by p^2c, and ca, da, cb, db, dc, $pa-ba$, pb, pd all lie in L_3. By subtracting suitable scalar multiples of pc from a, b, d we can assume that $pb = pd = 0$, $pa = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ 0 & 1 & 0 & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can suppose that $dc = 0$ or p^2c. And we can independently assume that $db = 0$ or p^2c. If $db = 0$ then we can assume that $da = 0$ or p^2c, but if $db = p^2c$ then we can assume that $da = 0$.

So consider the case when $da = db = dc = 0$. We can assume that $cb = xp^2c$ for some $0 \leq x < p$, and if $x \neq 0$ we can assume that $ca = 0$. If $x = 0$ we can assume that $ca = 0$ or p^2c.

Next consider the case when $da = p^2c$, $db = dc = 0$. Then we can assume that $cb = xp^2c$ for some $0 \leq x < p$, and we can assume that $ca = 0$ (even if $x = 0$).

Now consider the case when $da = dc = 0$, $db = p^2c$. Then we need $\beta = 0$. We can assume that $cb = 0$, and we can assume that $ca = 0$ or p^2c.

And now consider the case when $da = db = 0$, $dc = p^2c$. We can assume that $ca = cb = 0$.

Next consider the case when $da = dc = p^2c$, $db = 0$. We can assume that $ca = cb = 0$.

And ønally consider the case when $db = dc = p^2c$, $da = 0$. We can assume that $ca = cb = 0$.

56.8 Descendants of 6.16

Algebra 6.16 has 6 descendants of order p^7. I have checked that the recipes below give 6 non-isomorphic groups for $p = 5, 7, 11$.

Algebra 6.16 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc - ba,\ pb,\ pc,\ pd.$$

If L is a descendant of 6.16 of order p^7 then L_3 is generated by p^2a, and ca, da, cb, db, $dc-ba$, pb, pc, pd all lie in L_3. By subtracting suitable scalar multiples of pa from b, c, d we can assume that $pb = pc = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha(\lambda\xi - \mu\nu) & 0 & 0 \\ 0 & -(\gamma\mu - \delta\lambda) & \alpha\lambda & \alpha\mu \\ 0 & -(\gamma\xi - \delta\nu) & \alpha\nu & \alpha\xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

We can assume that $db = 0$ and that $cb = 0$ or p^2a.

First consider the case when $cb = db = 0$. We can assume that $dc - ba = 0$ or p^2a, and if $dc - ba = p^2a$ then (by adjusting γ, δ) we can assume that $ca = da = 0$. If $dc - ba = 0$ then we can assume that $da = 0$ and that $ca = 0$ or p^2a.

Next, consider the case when $db = 0$ and $cb = p^2a$. We then need $\nu = 0$ and $\xi = \alpha^{-1}\lambda^{-2}$ so that

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & -(\gamma\mu - \delta\lambda) & \alpha\lambda & \alpha\mu \\ 0 & -\gamma\alpha^{-1}\lambda^{-2} & 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can then assume that $dc - ba = 0$, though we then also need $\gamma = 0$ giving

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & \delta\lambda & \alpha\lambda & \alpha\mu \\ 0 & 0 & 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $da = 0$, p^2a or ωp^2a, and (by adjusting β) we can assume that $ca = 0$.

56.9 Descendants of 6.17

Algebra 6.17 has $p + 7$ descendants of order p^7. I have checked that the recipes below give $p + 7$ non-isomorphic groups for $p = 5, 7$.

Algebra 6.17 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc - ba,\ pb - ba,\ pc,\ pd.$$

If L is a descendant of 6.17 of order p^7 then L_3 is generated by p^2a, and ca, da, cb, db, $dc - ba$, $pb - ba$, pc, pd all lie in L_3. By subtracting suitable scalar multiples of pa from b, c, d we can assume that $pb - ba = pc = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & \lambda\xi - \mu\nu & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $db = 0$ and that $cb = 0$ or p^2a.

First consider the case when $cb = db = 0$. Then we can assume that $da = 0$ and that $ca = 0$ or p^2a. And independently we can assume that $dc - ba = 0$ or p^2a.

Next, consider the case when $cb = p^2a$, $db = 0$. We then need $\nu = 0$ and $\xi = \lambda^{-2}$ giving

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \lambda^{-2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $ca = 0$ (by adjusting β) and we can assume that $dc - ba = 0$ or p^2a. If $dc - ba = 0$ we can assume that $da = 0$ or p^2a or ωp^2a, but if $dc - ba = p^2a$ then we need $\lambda = 1$, and so we have $da = xp^2a$ with $0 \leq x < p$.

56.10 Descendants of 6.18

Algebra 6.18 has $3p+5$ descendants of order p^7. I have checked that the recipes below give $3p+5$ non-isomorphic groups for $p = 5, 7, 11$.

Algebra 6.18 has class 2 and relators

$$ca,\ da,\ cb,\ db,\ dc - ba,\ pb,\ pc - ba,\ pd.$$

If L is a descendant of 6.18 of order p^7 then L_3 is generated by p^2a, and ca, da, cb, db, $dc - ba$, pb, $pc - ba$, pd all lie in L_3. By subtracting suitable scalar multiples of pa from b, c, d we can assume that $pb = pc - ba = pd = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & -\alpha\nu & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & \delta\varepsilon + \mu\nu & \alpha\varepsilon & \mu \\ 0 & \nu & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $db = 0$ or p^2a. If $db = 0$ then we can assume that $cb = 0$, p^2a or ωp^2a. and if $db = p^2a$ then we can assume that $cb = 0$.

First consider the case when $cb = db = 0$. We can assume that $dc - ba = 0$ or p^2a. If $cb = db = dc - ba = 0$ then we can assume that $da = xp^2a$ with $0 \leq x < p$. If $x \neq 0$ we can assume that $ca = 0$, and if $x = 0$ then we can assume that $ca = 0$ or p^2a. If $cb = db = 0$, $dc - ba = p^2a$ then we can assume that $ca = da = 0$.

Next, consider the case when $cb = 0$, $db = p^2a$. We then need $\varepsilon = \alpha$ and $\mu = 0$, giving

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & -\alpha\nu & \delta \\ 0 & \alpha & 0 & 0 \\ 0 & \alpha\delta & \alpha^2 & 0 \\ 0 & \nu & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $dc - ba = 0$, though we then need $\delta = 0$. We can also assume that $da = 0$ though we then need $\beta = 0$. And we can assume that $ca = 0$, p^2a or ωp^2a.

If $cb = p^2a$ or ωp^2a and $db = 0$ then we can assume that $ca = dc - ba = 0$ and that $da = xp^2a$ with $0 \leq x < p$.

So there are $3p+5$ algebras here.

56.11 Descendants of 6.19

Algebra 6.19 has 97 descendants of order p^7. I have checked that the recipes below give 97 non-isomorphic groups for $p = 5, 7$.

Algebra 6.19 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd.$$

If L is a descendant of 6.19 of order p^7 then the commutator structure of L is the same as one of 7.29 ˘ 7.41 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we may assume that one of the following sets of commutator relations holds.

$$cb, baa, caa, bac, cac, da, db, dc,$$
$$cb, baa, caa, bac, cac, da, db, dc - bab,$$
$$cb, baa, caa, bac, cac, da - bab, db, dc,$$

$$cb, baa, caa - bab, bac, cac, da, db, dc,$$
$$cb, baa, caa - bab, bac, cac, da, db, dc - bab,$$
$$cb, baa, caa, bac, cac + bab, da, db, dc,$$
$$cb, baa, caa, bac, cac + \omega bab, da, db, dc,$$
$$cb, baa, caa, bac, cac + bab, da - bab, db, dc,$$
$$cb, baa, caa, bac, cac + \omega bab, da - bab, db, dc,$$
$$cb, bab, bac, caa, cac, da, db, dc,$$
$$cb, bab, bac, caa, cac, da, db - baa, dc,$$
$$cb, bab, bac, caa, cac, da, db, dc - baa,$$
$$cb - baa, bab, bac, caa, cac, da, db, dc.$$

56.11.1 Case 1

Let L satisfy the following relators.

$$cb, baa, caa, bac, cac, da, db, dc.$$

Then L_3 is generated by bab. The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = bab$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is isomorphic to one of 6.231 ˇ 6.237 from the descendants of 5.14.

56.11.2 Case 2

Let L satisfy the relators

$$cb, baa, caa, bac, cac, da, db, dc - bab.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \alpha\varepsilon^2\lambda^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb = pc = 0$.

If $pd = 0$ and $pc \neq 0$ we can assume that $pc = bab$ and that $pb = 0$ and $pa = 0$, bab or ωbab.

If $pc = pd = 0$ then we can assume that $pa = 0$, bab or ωbab, and independently assume that $pb = 0$ or bab.

56.11.3 Case 3

Let L satisfy the relators

$$cb, baa, caa, bac, cac, da - bab, db, dc.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \varepsilon^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb = pc = 0$.

If $pd = 0$ and $pc \neq 0$ we can assume that $pc = bab$ and that $pa = pb = 0$.

If $pc = pd = 0$ then we can assume that $pa = 0$, bab or ωbab, and independently assume that $pb = 0$ or bab.

56.11.4 Case 4

Let L satisfy the relators

$$cb, baa, caa - bab, bac, cac, da, db, dc.$$

Then L_3 is generated by bab. The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = bab$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is isomorphic to one of 6.238 ˇ 6.243 from the descendants of 5.14.

56.11.5 Case 5

Let L satisfy the relators

$$cb, baa, caa - bab, bac, cac, da, db, dc - bab.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & -\alpha^{-1}\beta\varepsilon & \eta \\ 0 & 0 & \alpha^{-1}\varepsilon^2 & \mu \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb = pc = 0$.

If $pd = 0$ and $pc \neq 0$ we can assume that $pc = bab$ or ωbab and that $pa = pb = 0$.

If $pc = pd = 0$ and $pb \neq 0$ then we can assume that $pb = bab$ and that $pa = 0$.

And if $pb = pc = pd = 0$ then we can assume that $pa = 0$, bab or ωbab.

56.11.6 Case 6

Let L satisfy the relators

$$cb, baa, caa, bac, cac + bab, da, db, dc.$$

The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = bab$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is isomorphic to one of 6.244 ˇ 6.250 from the descendants of 5.14.

56.11.7 Case 7

Let L satisfy the relators

$$cb, baa, caa, bac, cac + \omega bab, da, db, dc.$$

The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = bab$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is isomorphic to one of 6.251 ˇ 6.255 from the descendants of 5.14.

56.11.8 Case 8

Let L satisfy the relators

$$cb, baa, caa, bac, cac + bab, da - bab, db, dc.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & \pm\zeta & \pm\varepsilon & \mu \\ 0 & 0 & 0 & \varepsilon^2 - \zeta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb = pc = 0$.

If $pd = 0$ and $pb \neq \pm pc$, we can assume that $pc = 0$, though we have to distinguish between the cases $pb = 0$ and $pb \neq 0$. First consider the case when $pb = pc = pd = 0$. Then we can assume that $pa = 0$ or bab. If $pb \neq 0$ then we can assume that $pb = bab$, $pc = pd = 0$. We then need $\varepsilon = \alpha^{-1}$, $\zeta = 0$, so we can assume that $pa = 0$, bab or ωbab.

If $pd = 0$ and $pb = \pm pc \neq 0$, then we can assume that $pb = pc = bab$. We then need

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} (\varepsilon - \zeta)^{-1} & 0 & 0 & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & \zeta & \varepsilon & \mu \\ 0 & 0 & 0 & \varepsilon^2 - \zeta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pa = \rho bab$ then

$$pa' = (\varepsilon - \zeta)^{-1} bab = (\varepsilon^2 - \zeta^2)^{-1} b'a'b'$$

so we can assume that $pa = 0$ or bab.

56.11.9 Case 9

Let L satisfy the relators

$$cb, baa, caa, bac, cac + \omega bab, da - bab, db, dc.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & \pm\omega\zeta & \pm\varepsilon & \mu \\ 0 & 0 & 0 & \varepsilon^2 - \omega\zeta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb = pc = 0$.

If $pd = 0$ we can assume that $pc = 0$, but we have to distinguish between the case $pb = 0$ and the case $pb \neq 0$. If $pb = pc = pd = 0$, we can assume that $pa = 0$ or bab. but if $pb \neq 0$, $pc = pd = 0$ then we need $\zeta = 0$ giving

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \alpha^{-1} & 0 & \eta \\ 0 & 0 & \pm\alpha^{-1} & \mu \\ 0 & 0 & 0 & \alpha^{-2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pb = bab$ and that $pa = 0$ or bab or $\omega babb$.

56.11.10 Case 10

Let L satisfy the relators

$$cb, bab, bac, caa, cac, da, db, dc.$$

Then L_3 is generated by baa. The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = baa$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c \rangle \oplus \langle d \rangle$, and $\langle a, b, c \rangle$ is isomorphic to one of 6.256 ˇ 6.260 from the descendants of 5.14.

56.11.11 Case 11

Let L satisfy the relators

$$cb, bab, bac, caa, cac, da, db - baa, dc.$$

Then L_3 is generated by baa. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = baa$, $pc = 0$ or baa and that $pa = pb = 0$.

If $pd = 0$, $pc \neq 0$ we can assume that $pc = baa$, $pa = pb = 0$.

If $pc = pd = 0$ and $pb \neq 0$ we can assume that $pb = baa$ or ωbaa and that $pa = 0$.

And if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or baa.

56.11.12 Case 12

Let L satisfy the relators

$$cb, bab, bac, caa, cac, da, db, dc - baa.$$

Then L_3 is generated by baa. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \alpha^2\varepsilon\lambda^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = baa$, $pb = 0$ baa or ωbaa and that $pa = pc = 0$.

If $pd = 0$, $pc \neq 0$ we can assume that $pc = baa$, $pa = 0$, and $pb = 0$, baa or ωbaa.

If $pc = pd = 0$ and $pb \neq 0$ we can assume that $pb = baa$ or ωbaa and that $pa = 0$.

And if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or baa.

56.11.13 Case 13

Let L satisfy the relators

$$cb - baa, bab, bac, caa, cac, da, db, dc.$$

Then L_3 is generated by baa. The generator d is central, and so if $pd \neq 0$ we may suppose that $pd = baa$, and that $pa = pb = pc = 0$. On the other hand if $pd = 0$ then $L = \langle a, b, c\rangle \oplus \langle d\rangle$, and $\langle a, b, c\rangle$ is isomorphic to one of 6.261 ˘ 6.265 from the descendants of 5.14.

56.12 Descendants of 6.20

Algebra 6.20 has $6p + 35$ descendants of order p^7. I have checked that the recipes below give $6p + 35$ non-isomorphic groups for $p = 5, 7$.

Algebra 6.20 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa - ba,\ pb,\ pc,\ pd.$$

If L is a descendant of 6.20 of order p^7 then L_3 is generated by caa and cac, and cb, da, db, dc, $pa - ba$, pb, pc, pd all lie in L_3. The elements caa and cac must be linearly dependent, and we distinguish two cases: $caa = 0$ and $cac = 0$.

56.12.1 Case 1

First consider the case when $caa = 0$, so that L_3 is generated by cac. Adding suitable scalar multiples of ca to b and d we may suppose that $cb = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & \delta \\ 0 & 1 & 0 & \eta \\ 0 & \lambda & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can asssume that $db = 0$ or cac. If $db = cac$ then we can assume that $da = 0$, and if $db = 0$ then we can assume that $da = 0$ or cac.

Consider the case when $da = db = dc = cb = 0$. Then $b'a' = \alpha ba$. If $pd \neq 0$ then we can assume that $pd = cac$ and that $pa - ba = pb = pc = 0$. If $pd = 0$ and $pb \neq 0$ then we can assume that $pb = cac$ and that $pa - ba = pc = 0$. If $pb = pd = 0$ then we can assume that $pa - ba = 0$, cac or ωcac and (independently) that $pc = 0$ or cac.

Next, consider the case when $cb = db = dc = 0$, $da = cac$. Then $b'a' = \alpha ba + \alpha\eta cac$, so that

$$pa' - b'a' = \alpha(pa - ba) + \beta pb + \delta pd - \alpha\eta cac.$$

We also need $\xi = \mu^2$. If $pd \neq 0$ then we can assume that $pd = cac$ and that $pa - ba = pb = pc = 0$. If $pd = 0$ and $pb \neq 0$ then we can assume that $pb = cac$ and that $pa - ba = pc = 0$. If $pb = pd = 0$ then we can assume that $pa - ba = 0$ (by adjusting η) and that $pc = 0$ or cac.

Finally, consider the case when $cb = da = dc = 0$, $db = cac$. We then need $\beta = 0$ and $\xi = \alpha\mu^2$, and we have $b'a' = \alpha ba - \delta cac$ so that

$$pa' - b'a' = \alpha(pa - ba) + \delta pd + \delta cac.$$

If $pd \neq 0$ we can assume that $pd = xcac$ for some $x \neq 0$ $(0 < x < p)$ and that $pc = pd = 0$. If $x \neq -1$ we can assume that $pa - ba = 0$. But if $pd = -cac$ then we can assume that $pb = pc = 0$ and that $pa - ba = cac$ or ωcac. If $pd = 0$ and $pb \neq 0$ then we can assume that $pb = cac$ and (by adjusting δ) that $pa - ba = pc = 0$. If $pb = pd = 0$ then we can assume that $pa - ba = 0$ (by adjusting δ) and that $pc = 0$ or cac.

56.12.2 Case 2

Now consider the case when $cac = 0$, so that L_3 is generated by caa. By adding a suitable scalar multiple of ca to d we can assume that $da = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 1 & 0 & \eta \\ 0 & \lambda & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can asssume that $db = 0$ or caa. If $db = caa$ then we can assume that $cb = dc = 0$, and if $db = 0$ then we can assume that $dc = 0$ or caa. If $db = 0$ and $dc = caa$ then we can assume that $cb = 0$, but if $db = dc = 0$ then we can assume that $cb = 0$, caa or ωcaa.

First consider the case when $db = caa$ and $cb = da = dc = 0$. Adding a suitable scalar multiple of ca to b we can assume that $pa = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 1 & 0 & \eta \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \alpha^2\mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = xcaa$ for some x with $0 \leq x < p$. And we can independently assume that $pc = 0$, caa or ωcaa. If $pd \neq 0$ then we can assume that $pb = 0$, and if $pd = 0$ then we can assume that $pb = 0$ or caa.

Next consider the case when $cb = da = db = 0$, $dc = caa$. Adding a suitable scalar multiple of ca to b we can assume that $pa = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & \mu & \nu \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa. If $pd = 0$ we can assume that $pb = 0$ or caa, and if $pd = caa$ we can assume that $pb = 0$ or caa or ωcaa. If either of pb or pd are non-zero we can assume that $pc = 0$. But if $pb = pd = 0$ then we can assume that $pc = 0$ or caa or ωcaa.

Now consider the case when $db = dc = 0$, $cb = caa$ or ωcaa. Again, adding a suitable scalar multiple of ca to b we can assume that $pa = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \pm 1 & \beta & \gamma & \delta \\ 0 & 1 & 0 & \eta \\ 0 & \lambda & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa, and if $pd = caa$ then we can assume that $pb = pc = 0$. If $pd = 0$ then we can assume that $pb = 0$ or caa. If $pb \neq 0$ we can assume that $pc = 0$, but if $pb = pd = 0$ then we can assume that $pc = xcaa$ with $0 \leq x < p$.

Finally, consider the case when $cb = da = db = dc = 0$. Again, adding a suitable scalar multiple of ca to b we can assume that $pa = ba$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & 1 & 0 & \eta \\ 0 & \lambda & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa, and if $pd = caa$ then we can assume that $pb = pc = 0$. If $pd = 0$ then we can assume that $pb = 0$, caa. If $pb \neq 0$, $pd = 0$ we can assume that $pc = 0$, but if $pb = pd = 0$ then we can assume that $pc = 0$, caa or ωcaa.

56.13 Descendants of 6.21

Algebra 6.21 has $13p + 27$ descendants of order p^7. I have checked that the recipes below give $13p + 27$ non-isomorphic groups for $p = 5, 7$.

Algebra 6.21 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa,\ pb - ba,\ pc,\ pd.$$

If L is a descendant of 6.21 of order p^7 then L_3 is generated by caa and cac, and cb, da, db, dc, pa, $pb - ba$, pc, pd all lie in L_3. We distinguish two cases: $caa = 0$ and $cac = 0$.

56.13.1 Case 1

Let $caa = 0$. Then L_3 is generated by cac. By adding suitable scalar multiples of ca to b and d we can assume that $cb = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $da = 0$ or cac and independently assume that $db = 0$ or cac.

Let $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = cac$ and that $pa = pb - ba = pc = 0$. If $pd = 0$ we can independently assume that $pc = 0$ or cac, and that $pb - ba = 0$ or cac. If $pc = 0$ we can assume that $pa = 0$, cac or ωcac. But if $pc = cac$ then we need $\mu = 1$, and so we have $pa = xcac$ for some x with $0 \leq x < p$.

Next let $cb = db = dc = 0$, $da = cac$. We let $pd = xcac$ for some x with $0 \leq x < p$. If $x \neq 0$ then we can take $pa = pc = 0$, and if $x \neq 1$ we can take $pb - ba = 0$. However if $x = 1$ then we can take $pb - ba = 0$ or cac. (the problem is that altering η also alters ba.) So suppose that $pd = 0$. As mentioned above, we can take $pb - ba = 0$. If $pc \neq 0$ then we can take $pc = cac$, $pa = xcac$ for some x with $0 \leq x < p$. And if $pc = 0$ we can take $pa = 0$, cac or ωcac.

Now let $cb = dc = da = 0$, $db = cac$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \mu & \nu \\ 0 & 0 & 0 & \varepsilon^{-1}\mu^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can take $pd = cac$ and $pa = pb - ba = pc = 0$. If $pd = 0$ then we can take $pb = ba$, and we can take $pc = 0$ or cac. If $pc = 0$ we can take $pa = 0$, cac or ωcac, but if $pc = cac$ then we need $\mu = 1$, so we have to take $pa = xcac$ for some x with $0 \leq x < p$.

Finally, let $cb = dc = 0$, $da = db = cac$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & 1 & 0 & \eta \\ 0 & 0 & \mu & \nu \\ 0 & 0 & 0 & \mu^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Let $pd = xcac$ where $0 \leq x < p$. If $x \neq 0$ then we can take $pa = pc = 0$. If $x \neq 0, 1$ then we can take $pa = pc = pb - ba = 0$. But if $x = 1$ then we can take $pa = pc = 0$, $pb - ba = 0$, cac or ωcac. So consider the case when $pd = 0$. We can take $pb = ba$, and we can take $pc = 0$ or cac. If $pc = 0$ we can take $pa = 0$, cac or ωcac, but if $pc = cac$ then we need $\mu = 1$ so we have to take $pa = xcac$ for some x with $0 \leq x < p$.

56.13.2 Case 2

Now let $cac = 0$ so that L_3 is generated by caa. adding a suitable scalar multiple of ca to d, we may assume that $da = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & \lambda & 0 & \eta \\ 0 & 0 & \mu & \nu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can suppose that $dc = 0$ or caa, and independently assume that $db = 0$ or caa. If either of db or dc is non-zero we can assume that $cb = 0$, but if $db = dc = 0$ then we can assume that $cb = 0$ or caa.

Let $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = caa$ and that $pa = pb - ba = pc = 0$. If $pd = 0$ we can still assume that $pb = ba$. And we can assume that $pc = xcaa$ for some x with $0 \leq x < p$. If $x \neq 0$ then we can assume that $pa = 0$, but if $x = 0$ we can assume that $pa = 0$ or caa.

Next let $cb = da = db = 0$, $dc = caa$. If $pd \neq 0$ we can assume that $pd = caa$ and that $pa = pb - ba = pc = 0$. If $pd = 0$ we can still assume that $pb = ba$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pc = xcaa$ for some x with $0 \leq x < p$. If $x \neq 0$ then we can assume that $pa = 0$, but if $x = 0$ we can assume that $pa = 0$ or caa.

Now let $cb = da = dc = 0$, $db = caa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & \lambda & 0 & \eta \\ 0 & 0 & \lambda\xi & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can take $pd = caa$ (though we then need $\lambda = 1$) and we can take $pa = 0$, $pb = ba$, $pc = xcaa$ for some x with $0 \leq x < p$. If $pd = 0$ we can still take $pb = ba$, and we can take $pc = xcaa$ for some x with $0 \leq x < p$. If $pc \neq 0$ we can take $pa = 0$ and if $pc = pd = 0$ we can take $pa = 0$ or caa.

And now let $cb = da = 0$, $db = dc = caa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \varepsilon & \eta \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can take $pd = caa$ and $pa = pb - ba = pc = 0$. If $pd = 0$ we can still take $pb = ba$, and we can take $pc = xcaa$ for some x with $0 \leq x < p$. If $pc \neq 0$ we can take $pa = 0$ and if $pc = pd = 0$ we can take $pa = 0$ or caa.

Finally let $cb = caa$ and let $da = db = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & \delta \\ 0 & 1 & 0 & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can take $pd = caa$ and $pa = pb - ba = pc = 0$. If $pd = 0$ we can still take $pb = ba$, and we can take $pc = xcaa$ for some x with $0 \leq x < p$. If $pc \neq 0$ we can take $pa = 0$ and if $pc = pd = 0$ we can take $pa = 0$ or caa.

56.14 Descendants of 6.23

Algebra 6.23 has $3p + 41 + 8\gcd(p-1, 3) + 2\gcd(p-1, 4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$.

Algebra 6.23 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa,\ pb - ca,\ pc,\ pd.$$

If L is a descendant of 6.23 of order p^7 then L_3 is generated by baa and bab, and cb, da, db, dc, pa, $pb - ca$, pc, pd all lie in L_3. We distinguish three cases: $baa = 0$, $bab = 0$, and $baa = bab$.

56.14.1 $baa = 0$

If $baa = 0$ then L_3 is generated by bab. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha\lambda & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Adding suitable scalar multiples of ba to c, d we may assume that $cb = db = 0$. We can assume that $dc = 0$ or bab. If $dc = bab$ we can assume that $da = 0$, but if $dc = 0$ then we can assume that $da = 0$ or bab.

First consider the case when $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ then we can assume that $pc = 0$ or

bab. If $pc = bab$ we can assume that $pa = pb - ca = 0$. If $pc = pd = 0$ then we can assume that $pb - ca = 0$ or bab, and we can independently assume that $pa = 0$, bab or ωbab.

Next consider the case when $cb = da = db = 0$, $dc = bab$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \delta \\ 0 & \alpha\lambda & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \alpha^3\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ then we can still assume that $pb = ca$, and we can assume that $pc = 0$ or bab. If $pc = pd = 0$ we can assume that $pa = 0$, bab or ωbab. If $pc = bab$, $pd = 0$ then we need $\lambda = \alpha^{-3}$ so that $b'a'b' = \alpha^{-3}bab$ and we can assume that $pa = 0$, bab or ωbab or (when $p = 1 \bmod 4$) $\omega^2 bab$ or $\omega^3 bab$.

And ønally consider the case when $cb = db = dc = 0$, $da = bab$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha\lambda & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \alpha^2\lambda^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = bab$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ then we can still assume that $pb = ca$, and we can assume that $pc = 0$ or bab. If $pc \neq 0$ we can assume that $pa = 0$, and if $pc = 0$ we can assume that $pa = 0$, bab or ωbab.

56.14.2 $\quad bab = 0$

If $bab = 0$ then L_3 is generated by baa. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha\lambda & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Adding suitable scalar multiples of ba to d we may assume that $da = 0$. We can assume that $dc = 0$ or baa. If $dc = baa$ we can assume that $db = 0$, but if $dc = 0$ then we can assume that $db = 0$ or baa. If either of db or dc are non-zero then we can assume that $cb = 0$. But if $da = db = dc = 0$ then we can assume that $cb = 0$ or baa.

First consider the case when $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ then we can still assume that $pb = ca$, and we can assume that $pc = 0$ or baa, or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pc \neq 0$ we can assume that $pa = 0$, and if $pc = 0$ we can assume that $pa = 0$ or baa.

Next, consider the case when $cb = da = dc = 0$, $db = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha\lambda & \zeta & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = 0$, and we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pd = 0$ we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$ and that $pb = ca$. If $pc \neq 0$ we can assume that $pa = 0$, and if $pc = pd = 0$ we can assume that $pa = 0$ or baa.

And now consider the case when $cb = da = db = 0$, $dc = baa$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha\lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$ and that $pb = ca$. If $pc \neq 0$ we can assume that $pa = 0$, and if $pc = pd = 0$ we can assume that $pa = 0$ or baa.

Finally, consider the case when $cb = baa$, $da = db = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha^3 & \zeta & \eta \\ 0 & 0 & \alpha^2 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$ and that $pb = ca$. If $pc \neq 0$ we can assume that $pa = 0$, and if $pc = pd = 0$ we can assume that $pa = 0$, baa, ωbaa or (when $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$.

56.14.3 $\quad baa = bab$

If $baa = bab$ then L_3 is generated by baa. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha & \zeta & \eta \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. By adding suitable scalar multiples of ba to c and d we may assume that $cb = db = 0$. And we may assume that $dc = 0$ or baa. If $dc = baa$ we can assume that $da = 0$, and if $dc = 0$ we can assume that $da = 0$ or baa.

Consider the case when $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pc \neq 0$ then we can assume that $pa = 0$ and $pb = ca$. If $pc = pd = 0$ and $pb - ca \neq 0$ we can assume that $pb - ca = baa$ or ωbaa and that $pa = xbaa$ for some x with $0 \leq x < p$. And if $pb - ca = pc = pd = 0$ then we can assume that $pa = 0$, baa or ωbaa.

Next consider the case when $cb = db = dc = 0$, $da = baa$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha & \zeta & \eta \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = pb - ca = pc = 0$. If $pd = 0$ we can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pc \neq 0$ then we can assume that $pa = pb - ca = 0$. If $pc = pd = 0$ we can assume that $pa = 0$, baa or ωbaa and that $pb - ca = 0$.

Finally, consider the case when $cb = da = db = 0$, $dc = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \delta \\ 0 & \alpha & \gamma & \eta \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Let $pd = xbaa$ where $0 \leq x < p$. If $x \neq 0, 1$ then we can assume that $pa = pb - ca = pc = 0$, and if $x = 1$ we can assume that $pa = pc = 0$, $pb - ca = 0$, baa or ωbaa. So let $pd = 0$. We can assume that $pc = 0$, baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$, and that $pb = ca$. If $pc \neq 0$ then we can assume that $pa = 0$. If $pc = pd = 0$ we can assume that $pa = 0$, baa or ωbaa.

56.15 Descendants of 6.24

Algebra 6.24 has 5 descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$.

Algebra 6.24 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa - ba,\ pb - ca,\ pc,\ pd.$$

If L is a descendant of 6.24 of order p^7 then L_3 is generated by bab with $caa = -bab$, and cb, da, db, dc, $pa - ba$, $pb - ca$, pc, pd all lie in L_3. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\zeta & \gamma & \delta \\ 0 & 1 & \zeta & \eta \\ 0 & 0 & \alpha^{-1} & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Adding suitable scalar multiples of ba to c and d we can assume that $cb = db = 0$, and adding a sutiable scalar multiple of ca to d we may assume that $da = 0$. We can assume that $dc = 0$ or bab.

First consider the case when $cb = da = db = dc = 0$. If $pd \neq 0$ we can assume that $pd = bab$ and that $pa - ba = pb - ca = pc = 0$. If $pd = 0$ we can assume that $pa - ba = pb - ca = 0$ and that $pc = 0$, bab or ωbab.

And now consider the case when $cb = da = db = 0$, $dc = bab$. We need more detailed information about possible generating sets a', b', c', d' satisfying the same relations as those already speciøed for a, b, c, d than we have needed before. We have

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\zeta & \gamma & \delta & x & y \\ 0 & 1 & \zeta & \eta & z & t \\ 0 & 0 & \alpha^{-1} & \mu & \alpha^{-1}\eta - \zeta\mu & u \\ 0 & 0 & 0 & \alpha^2 & -\alpha^2\zeta & \alpha\gamma - \alpha^2\zeta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ ba \\ ca \end{pmatrix}$$

modulo L_3. Now $pba = -bab$, so by adjusting ζ we may assume that $pd = 0$, and by adjusting μ we may assume that $pc = 0$. Then, by adjusting t, u we may assume that $pa - ba = pb - ca = 0$.

56.16 Descendants of 6.29

Algebra 6.29 has $10p+69+\gcd(p-1,3)+\gcd(p-1,4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p=5,7$. I have also checked using orb6.29 that this formula gives the right number of algebras for $p=5,7,11,13$.

Algebra 6.29 has class 2 and relators

$$cb,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd-ca.$$

If L is a descendant of 6.29 of order p^7 then L_3 is generated by baa and bab, and cb, da, db, dc, pa, pb, pc, $pd-ca$ all lie in L_3. If a',b',c',d' generate L and satisfy the same relations as those already speciøed for a,b,c,d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We distinguish two cases: $baa=0$ and $bab=0$.

56.16.1 $baa=0$

Consider the case when $baa=0$, so that L_3 is generated by bab. We then need $\beta=0$. Adding suitable scalar multiple of ba to c and d we may assume that $cb=db=0$. We can assume that $dc=0$, bab or ωbab. If $dc\neq 0$ we can assume that $da=0$, and if $dc=0$ then we can assume that $da=0$ or bab.

First consider the case when $cb=da=db=dc=0$. We can assume that $pc=0$ or bab. If $pc=bab$ we can assume that $pa=pb=0$ and that $pd-ca=0$ or bab. If $pc=0$ then we can independently assume that $pa=0$, bab or ωbab, that $pb=0$ or bab, and that $pd-ca=0$ or bab.

Next consider the case when $cb=db=dc=0$ and $da=bab$. If a',b',c',d' generate L and satisfy the same relations as those already speciøed for a,b,c,d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \alpha^{-1}\varepsilon^2 & 0 \\ 0 & 0 & 0 & \varepsilon^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd-ca=0$ or bab. If $pd-ca=0$ we can assume that $pc=0$, bab or ωbab, and if $pd-ca=bab$ we can assume that $pc=xbab$ for some x with $0\leq x<p$. If $pc\neq 0$ we can assume that $pa=pb=0$. If $pc=0$ and $pd-ca=bab$ then we need $\alpha=1$, and so we can take $pb=0$ or bab. If $pb=0$ we can assume that $pa=0$, bab or ωbab and if $pb=bab$ then we can take $pa=xbab$ for some x with $0\leq x<p$. If $pc=pd-ca=0$ then we can assume that $pa=0$, bab or ωbab and that $pb=0$ or bab.

And ønally consider the case when $cb=da=db=0$ and $dc=bab$ or ωbab. If a',b',c',d' generate L and satisfy the same relations as those already speciøed for a,b,c,d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \pm\varepsilon & 0 \\ 0 & 0 & 0 & \pm\alpha\varepsilon \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc=0$ or bab and that $pd-ca=0$ or bab. First consider the case when $pc=pd-ca=bab$. Then we need $\alpha=1$ and $\varepsilon=\pm1$, and so we can take

$pb = 0$ and $pa = xbab$ for some x with $0 \le x < p$. If $pc = bab$, $pd - ca = 0$, then we need $\alpha\varepsilon = \pm 1$ and so we can take $pb = 0$, $pa = 0$, bab or ωbab. If $pc = 0$, $pd - ca = bab$ then we need $\varepsilon = \pm 1$ and so we can take $pa = xbab$ for some x with $0 \le x < p$ and $pb = 0$ or bab. Finally, if $pc = pd - bab = 0$ then we can take $pa = 0$, bab or ωbab and $pb = 0$ or bab.

56.16.2 $bab = 0$

Now let $bab = 0$, so that L_3 is generated by baa. By adding a suitable scalar multiple of ba to d we can assume that $da = 0$, and by adding a suitable scalar multiple of ba to c we can assume that $pd = ca$. We can assume that $dc = 0$ or baa and that $cb = 0$ or baa. If $dc \ne 0$ then we can assume that $db = 0$, and if $dc = 0$ then we can assume that $db = 0$ or baa.

First consider the case when $cb = da = db = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pc \ne 0$ then we can assume that $pa = pb = 0$, $pc = baa$. If $pc = 0$, $pb \ne 0$ we can assume that $pa = 0$, $pb = baa$ or ωbaa. And if $pb = pc = 0$ then we can assume that $pa = 0$ or baa.

Next, consider the case when $cb = da = db = 0$, $dc = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \alpha^{-1}\lambda^2 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pb = 0$, baa or ωbaa and that $pc = 0$ or baa. If pb or $pc \ne 0$ we can assume that $pa = 0$, and if $pb = pc = 0$ we can assume that $pa = 0$ or baa or ωbaa.

And now consider the case when $da = db = 0$, $cb = dc = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \alpha^3 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If $pb = pc = 0$ we can assume that $pa = 0$, baa, ωbaa or (when $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. If either of pb or pc are non-zero then we can assume that $pa = 0$. We can assume that $pb = 0$, baa or ωbaa. If $pb = 0$ we can assume that $pc = 0$ or baa or (when $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pb \ne 0$ then we need $\alpha = \pm 1$ and so we can take $pc = xbaa$ with $0 \le x \le (p-1)/2$.

Now consider the case when $cb = baa$, $da = db = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$ or baa, and if $pc = baa$ we can assume that $pa = pb = 0$. If $pc = 0$, $pb \neq 0$ we can assume that $pa = 0$ and that $pb = baa$ or ωbaa. And if $pb = pc = 0$ we can assume that $pa = 0$ or baa.

Next, consider the case when $cb = da = dc = 0$, $db = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$ or baa, and if $pc = baa$ we can assume that $pa = pb = 0$. If $pc = 0$, $pb \neq 0$ we can assume that $pa = 0$ and that $pb = baa$ or ωbaa. And if $pb = pc = 0$ we can assume that $pa = 0$ or baa.

And ønally consider the case when $da = dc = 0$, $cb = db = baa$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & \beta & \gamma & 0 \\ 0 & \varepsilon & \zeta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$ or baa, and if $pc = baa$ we can assume that $pa = pb = 0$. If $pc = 0$, $pb \neq 0$ we can assume that $pa = 0$ and that $pb = xbaa$ with $0 < x < p$. And if $pb = pc = 0$ we can assume that $pa = 0$ or baa.

56.17 Descendants of 6.33

Algebra 6.33 has $39+\gcd(p-1,3)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5$.

Algebra 6.33 has class 2 and relators

$$cb,\ da,\ db - ba,\ dc,\ pa,\ pb,\ pc,\ pd.$$

It is convenient to give another presentation for 6.33. If we let $a' = a + d$, $b' = c$, $c' = b$, $d' = d$ then we see that 6.33 has a presentation with generators a, b, c, d and relators

$$ca,\ da,\ cb,\ db,\ pa,\ pb,\ pc,\ pd,$$

so that the derived algebra of 6.33 is generated by ba, dc. If L is a descendant of 6.33 of order p^7 then the commutator structure of L is the same as one of 7.44 ˘ 7.47 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we can assume that L_3 is generated by bab and that one of the following sets of commutator relations holds:

$$\begin{aligned}
&baa, ca, cb, da, db, dcc, dcd,\\
&baa, ca - bab, cb, da, db, dcc, dcd,\\
&baa, ca, cb, da, db, dcc, dcd - bab,\\
&baa, ca - bab, cb, da, db, dcc, dcd - bab.
\end{aligned}$$

56.17.1 Case 1

Let L satisfy the relators

$$baa, ca, cb, da, db, dcc, dcd.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pd = 0$ and that $pc = 0$ or bab. And we can assume that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$, and if $pa = 0$ we can assume that $pb = 0$ or bab. (8 algebras.)

56.17.2 Case 2

Let L satisfy the relators

$$baa, ca - bab, cb, da, db, dcc, dcd.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & \gamma^2 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pd = 0$ or bab. If $pd = bab$ we can assume that $pc = 0$, and if $pd = 0$ we can assume that $pc = 0$ or bab. We can also assume that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$, and if $pa = 0$ we can assume that $pb = 0$ or bab. (12 algebras.)

56.17.3 Case 3

Let L satisfy the relators

$$baa, ca, cb, da, db, dcc, dcd - bab.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & \alpha\gamma^2\xi^{-2} & 0 \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, or

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & \gamma \\ \alpha\gamma^2\xi^{-2} & 0 & 0 & 0 \\ \nu & \xi & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Considering transformations of the ørst kind we see that we can assume that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$, and if $pa = 0$ we can assume that $pb = 0$ or bab. Similarly we can assume that $pc = 0$, bab or ωbab. If $pc \neq 0$ we can assume that $pd = 0$, and if $pc = 0$ we can assume that $pd = 0$ or bab. Transformations of the second kind allow us to interchange the pair a, b with the pair c, d, and so the 16 possibilities described above are reduced to 10.

56.17.4 Case 4

Let L satisfy the relators

$$baa, ca - bab, cb, da, db, dcc, dcd - bab.$$

If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \xi^2 & 0 & 0 & 0 \\ \beta & \gamma & 0 & 0 \\ 0 & 0 & \gamma^2 & 0 \\ 0 & 0 & \nu & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, or

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\xi^2 & 0 \\ 0 & 0 & \beta & \gamma \\ -\gamma^2 & 0 & 0 & 0 \\ \nu & \xi & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. Considering transformations of the ørst kind we see that we can assume that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$, and if $pa = 0$ we can assume that $pb = 0$ or bab. Similarly we can assume that $pc = 0$, bab or ωbab. If $pc \neq 0$ we can assume that $pd = 0$. If $pc = 0$ we can assume that $pd = 0$ or bab, except in the case when $pa = 0$, $pb = bab$. If $pa = pc = 0$, $pb = bab$, $pd = \lambda bab$ then we need $\gamma = \xi^{-2}$ so that

$$pd' = \xi\lambda bab = \xi^3\lambda b'a'b',$$

and so if $p = 1 \bmod 3$ then we have two additional algebras with $\lambda = \omega, \omega^2$. Transformations of the second kind allow us to let a', b', c', d' equal $-c, d, -a, b$ which gives $b'a'b' = -bab$. If

$$(pa, pb, pc, pd) = (\rho, \sigma, \tau, \phi)bab$$

this gives

$$(pa', pb', pc', pd') = (\tau, -\phi, \rho, -\sigma)b'a'b',$$

which is equivalent to

$$(pa', pb', pc', pd') = (\tau, \phi, \rho, \sigma)b'a'b'.$$

So again we get 10 algebras, or 12 if $p = 1 \bmod 3$.

56.18 Descendants of 6.34

Algebra 6.34 has $5p + 38 + 2\gcd(p-1, 4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$.

Algebra 6.34 has class 2 and relators

$$cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb,\ pc,\ pd.$$

If L is a descendant of 6.34 of order p^7 then L_3 is generated by caa and cac and cb, da, $db - ba$, dc, $pa - ba$, pb, pc, pd lie in L_3. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \varepsilon\xi - \eta\nu & -\nu & \gamma & \varepsilon\xi - \eta\nu - \xi \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & \nu & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We distinguish two cases: $caa = 0$ and $cac = 0$.

56.18.1 $\quad caa = 0$

If $caa = 0$ then L_3 is generated by cac, and we require $\gamma = 0$. Adding suitable scalar multiples of ca to b and d we may assume that $cb = dc = 0$. We can also assume that $db = ba$ and that $da = 0$ or cac.

First consider the case when $cb = da = db - ba = dc = 0$. We can assume that $pd = 0$ and that $pb = 0$ or cac. If $pb = pd = 0$ then we can assume that $pa - ba = 0$, cac or ωcac and that $pc = 0$ or cac. If $pb = bab$, $pd = 0$ then again we can assume that $pa - ba = 0$, cac or ωcac and that $pc = 0$ or cac.

And now consider the case when $cb = db - ba = dc = 0$, $da = cac$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \varepsilon\lambda^2 & -\nu & 0 & \varepsilon\lambda^2 - \lambda^2 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \nu & 0 & \lambda^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pb = 0$, cac or ωcac or (when $p = 1 \bmod 4$) $\omega^2 cac$ or $\omega^3 cac$, and we can assume that $pc = 0$ or cac.

If $pb = pc = 0$ we can assume that $pd = 0$ or cac. If $pb = pc = pd = 0$ then we can assume that $pa - ba = 0$, cac or ωcac. If $pb = pc = 0$, $pd = cac$ we need $\varepsilon = \lambda^{-2}$. If we let $pa - ba = \rho cac$ then

$$\begin{aligned} pa' &= pa + (1 - \lambda^2)cac = ba + (\rho + 1 - \lambda^2)cac, \\ b'a' &= ba, \\ c'a'c' &= \lambda^2 cac \end{aligned}$$

and so

$$pa' - b'a' = (\frac{(\rho + 1)}{\lambda^2} - 1)c'a'c'.$$

So we can take $\rho = 0$, -1 or $\omega - 1$.

If $pb = 0$, $pc = cac$ we can again assume that $pd = 0$ or cac. If $pd = 0$ we can assume that $pa - ba = 0$, cac or ωcac. And if $pd = cac$ we need $\varepsilon = \lambda = 0$ and so we have $pa - ba = xcac$ for some $0 \le x < p$.

Now consider the cases when $pb \neq 0$. We can assume that $pd = 0$, though we then need $\nu = 0$ and $\lambda^4 = 1$. We can assume that $pc = 0$ or cac, and we can assume that $pa - ba = xcac$ with $0 \le x < p$ (or $0 \le x \le (p-1)/2$ in the case $p = 1 \bmod 4$).

56.18.2 $\quad cac = 0$

If $cac = 0$ then L_3 is generated by caa. By adding suitable scalar multiples of ca to b and d we can assume that $da = db - ba = 0$. We can assume that $dc = 0$ and that $cb = 0$ or caa.

Consider the case when $cb = da = db - ba = dc = 0$. We can assume that $pd = 0$, that $pb = 0$ or caa, and that $pc = 0$, caa or ωcaa. If $pc \neq 0$ we can assume that $pa - ba = 0$, and if $pc = 0$ we can assume that $pa - ba = 0$ or caa.

And now consider the case when $cb = caa$ and $da = db - ba = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \xi^{-1} & 0 & \gamma & \xi^{-1} - \xi \\ 0 & \xi^{-2} & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$, caa or ωcaa, and that $pd = 0$ or caa. If $pc \neq 0$ we can assume that $pa = ba$, and if $pd \neq 0$ we can assume that $pb = 0$. If $pd = 0$ we can assume that $pb = 0$ or caa. If $pc = pd = 0$ then we can assume that $pa - ba$ and pb are independently equal to 0 or caa. Consider the case when $pc = 0$ and $pd = caa$. Then we need $\lambda = \xi^3$. As above we can assume that $pb = 0$ (though we then need $\eta = 0$). Let $pa - ba = \rho caa$. Then

$$\begin{aligned} pa' &= \xi^{-1}pa + (\xi^{-1} - \xi)caa = \xi^{-1}ba + (\xi^{-1}(\rho+1) - \xi)caa, \\ b'a' &= \xi^{-1}pa, \\ c'a'a' &= \xi caa. \end{aligned}$$

So

$$pa' - b'a' = (\frac{\rho+1}{\xi^2} - 1)c'a'a',$$

and we can take $\rho = 0$, -1 or $\omega - 1$.

56.19 Descendants of 6.35

Algebra 6.35 has $10p + 26$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5$.

Algebra 6.35 has class 2 and relators

$$cb,\ da,\ db - ba,\ dc,\ pa,\ pb - ba,\ pc,\ pd.$$

If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \alpha - 1 \\ 0 & \varepsilon & 0 & \eta \\ \mu & 0 & \lambda & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. L_3 is generated by caa and cac, but we can assume that $cac = 0$ and that L_3 is generated by caa, though we then need $\mu = 0$. By adding suitable scalar multiples of ca to b and d we can assume that $da = db - ba = 0$. We can assume that $dc = 0$, caa or ωcaa. If $dc \neq 0$ then we can assume that $cb = 0$, and if $dc = 0$ we can assume that $cb = 0$ or caa.

First consider the case when $cb = da = db - ba = dc = 0$. We can assume that $pc = 0$, caa or ωcaa and that $pd = 0$ or caa. If $pc \neq 0$ we can assume that $pa = 0$ and if $pd \neq 0$ then we can assume that $pb - ba = 0$. If $pc = pd = 0$ we can assume that pa and $pb - ba$ are independently equal to 0 or caa. If $pc \neq 0$, $pd = 0$ then we can assume that $pa = 0$, $pb - ba = 0$ or caa. And if $pc = 0$, $pd = caa$ then we can assume that $pb - ba = 0$, and we need $\lambda = \alpha^{-2}$. Let $pa = \rho caa$ then

$$pa' = (\alpha\rho + \alpha - 1)caa = (\alpha\rho + \alpha - 1)c'a'a'.$$

So we can take $\rho = 0$ or -1.

Next consider the case when $cb = caa$, $da = db - ba = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already specified for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma & \alpha - 1 \\ 0 & \alpha^2 & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$, caa or ωcaa and that $pd = 0$ or caa. If $pc \neq 0$ we can assume that $pa = 0$ and if $pd \neq 0$ then we can assume that $pb - ba = 0$.

If $pc = pd = 0$ we can assume that pa and $pb - ba$ are independently equal to 0 or caa.

If $pc \neq 0$, $pd = 0$ then we can assume that $pa = 0$, $pb - ba = 0$ or caa.

And if $pc = 0$, $pd = caa$ then we can assume that $pb - ba = 0$, and we need $\lambda = \alpha^{-2}$. Let $pa = \rho caa$ then

$$pa' = (\alpha\rho + \alpha - 1)caa = (\alpha\rho + \alpha - 1)c'a'a'.$$

So we can take $\rho = 0$ or -1.

And øнally consider the cases when $cb = da = db - ba = 0$ and $dc = caa$ or ωcaa. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & \gamma & \pm 1 - 1 \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa, that $pc = xcaa$ for some x with $0 \leq x < p$, and that $pb - ba = 0$ or caa. If $pc \neq 0$ we can assume that we can assume that $pa = 0$. If $pc = pd = 0$ we can assume that $pa = 0$ or caa. If $pc = 0$ and $pd = caa$ we need $\lambda = 1$. If we let $pa = \rho caa$ then

$$pa' = (\pm(\rho + 1) - 1)caa = (\pm(\rho + 1) - 1)c'a'a',$$

and so we can assume that $-1 \leq \rho \leq (p - 1)/3$.

56.20 Descendants of 6.36

Algebra 6.36 has $p^3 + 9p^2 + 20p + 18 + \gcd(p - 1, 3) + \gcd(p - 1, 4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$, and the formulae for each batch of descendants were checked with orb6.36 for $p = 5, 7, 11, 13$. I also checked that group6.36 produces this number of presentations for $p = 5, 7, 11, 13$.

Algebra 6.36 has class 2 and relators

$$cb,\ da,\ db - ba,\ dc,\ pa - ba,\ pb - ba,\ pc,\ pd.$$

If L is a descendant of 6.36 of order p^7 then L_3 is generated by caa and cac. We distinguish 2 cases: $caa = 0$ and $cac = 0$.

56.20.1 $caa = 0$

If $caa = 0$ then L_3 is generated by cac. By adding suitable scalar multiples of ca to b and d we may assume that $cb = dc = 0$. We can assume that $da = 0$, cac or ωcac. If $da \neq 0$ we can assume that $db - ba = 0$, and if $da = 0$ we can assume that $db - ba = 0$ or cac.

First consider the case when $cb = da = db - ba = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \alpha - 1 \\ 0 & \alpha & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that pc and pd are independently equal to 0 or cac.

If $pc = pd = 0$ then we can assume that $pb - ba = 0$, cac or ωcac. If $pb - ba = 0$ then we can assume that $pa - ba = 0$, cac or ωcac, and if $pb - ba = cac$ or ωcac then we can assume that $pa - ba = xcac$ for some $0 \leq x < p$.

If $pc = 0$, $pd = cac$ then we need $\alpha = \lambda^{-2}$. We can assume that $pb - ba = 0$. If $pa - ba = \rho cac$ then

$$\begin{aligned} pa' &= \lambda^{-2}pa + (\lambda^{-2} - 1)cac = \lambda^{-2}ba + (\lambda^{-2}(\rho + 1) - 1)cac, \\ b'a' &= \lambda^{-2}ba, \\ c'a'c' &= cac \end{aligned}$$

so

$$pa' - b'a' = (\lambda^{-2}(\rho + 1) - 1)c'a'c'$$

and we can take $\rho = 0$, -1 or $\omega - 1$.

If $pc = cac$ and $pd = 0$ then we need $\alpha = \lambda^{-1}$. We can take $pb - ba = 0$, cac or ωcac. If $pb - ba = 0$ we can take $pa - ba = 0$, cac or ωcac, and if $pb - ba = cac$ then we can take $pa - ba = xcac$ for some x with $0 \leq x < p$.

If $pc = pd = cac$ then we need $\alpha = \lambda = 1$. We can take $pb - ba = 0$ and $pa - ba = xcac$ for some x with $0 \leq x < p$ So there are $5p + 9$ descendants of this kind.

Next consider the case when $cb = da = dc = 0$, $db - ba = cac$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 & 0 & \lambda^2 - 1 \\ 0 & \lambda^2 & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ then we can take $pd = cac$ or ωcac or (if $p = 1 \bmod 4$) $\omega^2 cac$ or $\omega^3 cac$. We then need $\lambda^4 = 1$. We can assume that $pb - ba = 0$. If $pd = ucac$ and $pa - ba = \rho cac$ then

$$pa' - b'a' = ((u - 1)(\lambda^2 - 1) + \lambda^2\rho)cac = ((u - 1)(\lambda^{-2} - 1) + \lambda^{-2}\rho)c'a'c'.$$

First consider the case $p = 3 \bmod 4$. Then $\lambda^2 = 1$, and so we can take $0 \leq \rho < p$, and we can take $pc = xcac$ for some x with $0 \leq x \leq (p - 1)/2$.

Now consider the case $p = 1 \bmod 4$. Then $\lambda^2 = \pm 1$, and so ρ and $-\rho + 2(1 - u)$ give isomorphic algebras. If $\rho \neq 1 - u$ then we need $\lambda^2 = 1$, so we can take $pc = xcac$ for some x with $0 \leq x \leq (p - 1)/2$. But if $\rho = 1 - u$ then we can take $pc = xcac$ where the non-zero values of x are a transversal for the four fourth roots of unity.

Now let $pd = 0$, and let $pa - ba = \rho cac$. Then

$$pa' - b'a' = (\lambda^{-2}(\rho - 1) + 1)c'a'c'$$

and so we can take $\rho = 1, 2, 1 + \omega$. And if $pb - ba = \sigma cac$ then

$$pb' - b'a' = (\lambda^{-2}(\sigma - 1) + 1)c'a'c'.$$

So if $\rho = 1$ then we can take $\sigma = 1, 2, 1 + \omega$, and if $\rho = 2$ or $1 + \omega$ then we can take $0 \leq \sigma < p$. If either ρ or σ is diœerent from 1 then we need $\lambda^2 = 1$ and so we can take $pc = ycac$ for some y with $0 \leq y \leq (p - 1)/2$. But if $\rho = \sigma = 1$ then we can take $pc = 0$, cac or (if $p = 1 \bmod 3$) ωcac or $\omega^2 cac$.

The number of descendants in this case is $2p^2 + 3p + \gcd(p - 1, 3) + \gcd(p - 1, 4)$.

And ønally consider the case when $cb = db - ba = dc = 0$ and $da = cac$ or ωcac. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & \alpha - 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2.

If $pd \neq 0$ we can assume that $pd = cac$, though we then need $\alpha = 1$. So we can assume that $pa - ba = xcac$, $pb - ba = ycac$, $pc = zcac$ where $0 \leq x, y < p$, $0 \leq z < (p-1)/2$.

If $pd = 0$ we can assume that $pa - ba = xcac$, $pb - ba = ycac$ where $0 \leq x, y < p$, and we can assume that $pc = 0$ or cac. So there are $p^2(p+5)/2$ descendants for each of the two values of da.

56.20.2 $\quad cac = 0$

If $cac = 0$ then L_3 is generated by caa. Adding a suitable scalar multiple of ca to d and b we may assume that $da = db - ba = 0$. We can assume that $dc = 0$, caa or ωcaa. If $dc \neq 0$ we can assume that $cb = 0$, and if $dc = 0$ we can assume that $cb = 0$ or caa.

First consider the case when $cb = da = db - ba = dc = 0$. We can assume that $pd = 0$ or caa and that $pc = 0$, caa or ωcaa.

If $pc = pd = 0$ then we can assume that $pa - ba = 0$ or caa. If $pa - caa = 0$ we can assume that $pb - ba = 0$ or caa, and if $pa - ba = caa$ we can assume that $pb - ba = xcaa$ with $0 \leq x < p$.

If $pc = 0$, $pd = caa$ then we can assume that $pb - ba = 0$ and that $pa - ba = 0$ or $-caa$.

If $pc = caa$ or ωcaa and $pd = 0$ then we can assume that $pa - ba = 0$ and that $pb - ba = 0$ or caa.

And if $pc = caa$ or ωcaa and $pd = caa$ then we can assume that $pa - ba = pb - ba = 0$.

So there are $p + 10$ descendants here.

Next consider the case when $cb = caa$, $da = db - ba = dc = 0$. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma & 0 \\ 0 & 1 & 0 & \eta \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa and that $pc = xcaa$ with $0 \leq x < p$.

If $pc = pd = 0$ we can assume that $pa - ba = 0$ or caa. If $pa - ba = 0$ then we can assume that $pb - ba = 0$ or caa, and if $pa - ba \neq 0$ we can assume that $pb - ba = ycaa$ with $0 \leq y < p$.

If $pc \neq 0$ and $pd = 0$ we can assume that $pa - ba = 0$ and that $pb - ba = 0$ or caa.

If $pc = 0$ and $pd = caa$ then we need $\lambda = 1$, and so we can assume that $pb - baa = 0$ and that $pa - ba = ycaa$ with $0 \leq y < p$.

And if pc and pd are both non-zero then we can assume that $pa - ba = pb - ba = 0$.

So there are $5p - 1$ algebras here.

And ønally consider the case when $cb = da = db - ba = 0$, $dc = caa$ or ωcaa. If a', b', c', d' generate L and satisfy the same relations as those already speciøed for a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & \gamma & \pm 1 - 1 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pd = 0$ or caa and that $pc = xcaa$ with $0 \leq x < p$.

If $pc = pd = 0$ we can assume that $pa - ba = 0$ or caa. If $pa - ba = 0$ then we can assume that $pb - ba = 0$ or caa, and if $pa - ba \neq 0$ we can assume that $pb - ba = ycaa$ with $0 \leq y < p$.

If $pc \neq 0$ and $pd = 0$ we can assume that $pa - ba = 0$ and that $pb - ba = 0$ or caa.

If $pc = 0$ and $pd = caa$ then we need $\lambda = 1$, and so we can assume that $pb - baa = xcaa$ with $0 \leq x \leq (p-1)/2$. If $pb - ba \neq 0$ then we can take $pa - ba = ycaa$ with $0 \leq y < p$, and if $pb - ba = 0$ we can take $pa - ba = ycaa$ with $-1 \leq y \leq (p-3)/2$.

And if pc and pd are both non-zero then we can assume that $pa - ba = 0$ and $pb - ba = ycaa$ with $0 \leq x \leq (p-1)/2$.

So the number of descendants is $p^2 + 3p$.

56.21 Descendants of 6.48

Algebra 6.48 has $27 + 3\gcd(p-1, 3) + 2\gcd(p-1, 4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$, and the formulae for each batch of descendants were checked with orb6.48 for $p = 5, 7, 11, 13$.

Algebra 6.48 has class 2 and relators

$$cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc,\ pd.$$

If a', b', c', d' generate 6.48, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \varepsilon & \zeta & \eta & \theta \\ 0 & 0 & \zeta\lambda & -\varepsilon\lambda \\ 0 & 0 & -\beta\lambda & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo the derived algebra of 6.48. If L is a descendant of 6.48 of order p^7 then L_3 is generated by baa, bab, bac, bad. If a', b', c', d' are as above then

$$\begin{aligned} b'a'c' &= (\alpha\zeta - \beta\varepsilon)\lambda(\zeta bac - \varepsilon bad), \\ b'a'd' &= (\alpha\zeta - \beta\varepsilon)\lambda(-\beta bac + \alpha bad). \end{aligned}$$

So we can assume that $bac = 0$, and we have to distinguish between the cases $bad = 0$ and $bad \neq 0$.

If $bac = bad = 0$ then

$$\begin{aligned} b'a'a' &= (\alpha\zeta - \beta\varepsilon)(\alpha baa + \beta bab), \\ b'a'b' &= (\alpha\zeta - \beta\varepsilon)(\varepsilon baa + \zeta bab) \end{aligned}$$

and so we can assume that $bab = 0$ and that L_3 is generated by baa. We then need $\varepsilon = 0$. Adding suitable scalar multiples of ba to d and c we may assume that $da = db - ca = 0$. We can assume that $dc = 0$ or baa. If $dc = baa$ we can assume that $cb = 0$, and if $dc = 0$ we can assume that $cb = 0$ or baa.

If $bad \neq 0$ and $bac = 0$ then L_3 is generated by bad, and again we need $\varepsilon = 0$. We can then assume that $bab = 0$, though we then need $\theta = 0$, and by adjusting δ we may assume that $baa = 0$. By adding suitable scalar multiples of ca to c and d, and by adding a suitable scalar multiple ba to c, we may assume that $da = db - ca = dc = 0$, and we may assume that $cb = 0$ or bad.

So we may assume that one of the following øve sets of commutator relations holds:

$$\begin{aligned}
bab &= bac = bad = cb = da = db - ca = dc = 0, \\
bab &= bac = bad = cb - baa = da = db - ca = dc = 0, \\
bab &= bac = bad = cb = da = db - ca = dc - baa = 0, \\
baa &= bab = bac = cb = da = db - ca = dc = 0, \\
baa &= bab = bac = cb - bad = da = db - ca = dc = 0.
\end{aligned}$$

56.21.1 Case 1

Let L satisfy

$$bab = bac = bad = cb = da = db - ca = dc = 0.$$

If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & \zeta\lambda & 0 \\ 0 & 0 & -\beta\lambda & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pc = 0$ or baa. If $pc = baa$ then we can assume that $pa = pb = pd = 0$. If $pc = 0$ we can assume that $pd = 0$ or baa, and if $pd = baa$ then we can assume that $pa = pb = 0$. If $pc = pd = 0$ then we can assume that $pb = 0$, baa or ωbaa. If $pb \neq 0$, $pc = pd = 0$ then we can assume that $pa = 0$. And if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or baa.

So there are 6 algebras here.

56.21.2 Case 2

Let L satisfy

$$bab = bac = bad = cb - baa = da = db - ca = dc = 0.$$

If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2\lambda^{-1} & \eta & \theta \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & -\beta\lambda & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pc = 0$ or baa. If $pc = baa$ then we can assume that $pa = pb = pd = 0$. If $pc = 0$ we can assume that $pd = 0$ or baa. If $pc - 0$, $pd = baa$ then we can assume that $pa = pb = 0$. If $pc = pd = 0$ then we can assume that $pa = 0$ or baa. If $pa = baa$ then we can assume that $pb = 0$ and if $pa = 0$ then we can assume that $pb = 0$, baa or ωbaa.

So there are 6 algebras here.

56.21.3 Case 3

Let L satisfy

$$bab = bac = bad = cb = da = db - ca = dc - baa = 0.$$

If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \lambda^2 & \beta & \gamma & \delta \\ 0 & \varepsilon & \eta & 0 \\ 0 & 0 & \varepsilon\lambda & 0 \\ 0 & 0 & -\beta\lambda & \lambda^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. So we can assume that $pc = 0$ or baa or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pc \neq 0$ we can assume that $pa = pb = pd = 0$.

So consider the case when $pc = 0$. We can assume that $pd = 0$ or baa and we can assume that $pb = 0$, baa, ωbaa or (if $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. If either of pb or pd are non-zero then we can assume that $pa = 0$. But if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or baa.

So there are $3 + \gcd(p-1,3) + 2\gcd(p-1,4)$ algebras here.

56.21.4 Case 4

Let L satisfy
$$baa = bab = bac = cb = da = db - ca = dc = 0.$$
If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then
$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \frac{1}{2}\alpha\varepsilon^{-1}\eta \\ 0 & \varepsilon & \eta & 0 \\ 0 & 0 & \varepsilon\lambda & 0 \\ 0 & 0 & -\beta\lambda & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
modulo L_2. So we can assume that $pc = 0$, bad or ωbad. If $pc \neq 0$ we can assume that $pa = pb = pd = 0$. If $pc = 0$ we can assume that pb and pd are independently equal to 0 or bad. If either of pb or pd is non-zero then we can assume that $pa = 0$, and if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or bad.

So there are 7 algebras here.

56.21.5 Case 5

Let L satisfy
$$baa = bab = bac = cb - bad = da = db - ca = dc = 0.$$
If a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then
$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2 & 2\alpha\delta & 0 \\ 0 & 0 & \alpha^2\lambda & 0 \\ 0 & 0 & -\beta\lambda & \alpha\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
modulo L_2. So we can assume that $pc = 0$, bad or ωbad. If $pc \neq 0$ we can assume that $pa = pb = pd = 0$. If $pc = 0$ we can assume that $pb = 0$ or bad, and we cand assume that $pd = 0$, bad or (if $p = 1 \bmod 3$) ωbad or $\omega^2 bad$. If either of pb or pd is non-zero then we can assume that $pa = 0$, and if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or bad.

So there are $5 + 2\gcd(p-1,3)$ algebras here.

56.22 Descendants of 6.51

Algebra 6.51 has $8p + 15 + (2p+5)\gcd(p-1,3) + (p+2)\gcd(p-1,4) + \gcd(p-1,5)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$, and the formulae for each batch of descendants were checked with orb6.51 for $p = 5, 7, 11, 13$.

Algebra 6.51 has class 2 and relators
$$cb,\ da,\ db - ca,\ dc,\ pa - ca,\ pb,\ pc,\ pd.$$

If a', b', c', d' generate 6.51, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta\mu & -\zeta\lambda & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo the Frattini subalgebra. If L is a descendant of 6.51 of order p^7 then L_3 is generated by baa and bab. We distinguish two cases: $baa = 0$ and $bab = 0$.

56.22.1 $\quad baa = 0$

If $baa = 0$ then L_3 is generated by bab, and we need to take $\lambda = 0$. Adding suitable scalar multiples of ba to c and d we may assume that $cb = db - ca = 0$. We can assume that $dc = 0$, bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. If $dc \neq 0$ then we can assume that $da = 0$, and if $dc = 0$ then we can assume that $da = 0$ or bab.

Consider the case when $cb = da = db - ca = dc = 0$. If a', b', c', d' generate L, and satisfy the same relations as those already speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta\mu & 0 & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If a', b', c', d' are as above then $c'a' = \zeta\mu ca$. We can assume that $pc = 0$ or bab and that $pd = 0$, bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. If either of pc, pd is non-zero then we can assume that $pa - ca = pb = 0$. If $pc = pd = 0$ we can assume that $pa - ca = 0$, bab or ωbab and that $pb = 0$ or bab. So there are $7 + 2\gcd(p-1, 3)$ algebras here.

Next consider the case when $cb = db - ca = dc = 0$ and $da = bab$. If a', b', c', d' generate L, and satisfy the same relations as those already speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta^3 & 0 & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. If a', b', c', d' are as above then $c'a' = \zeta^3 ca$. We can assume that $pd = 0$, bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$. If $pd \neq 0$ we can assume that $pa - ca = pb = 0$ and that $pc = 0$ or $xbab$ where $x \neq 0$ runs through a transversal for the cube roots of unity. If $pd = 0$ then we can assume that $pc = 0$, bab or (if $p = 1 \bmod 5$) ωbab, $\omega^2 bab$, $\omega^3 bab$, $\omega^4 bab$. If $pc \neq 0$ then we can assume that $pa - ca = pb = 0$. If $pc = pd = 0$ then we can assume that $pb = 0$, bab, ωbab or (if $p = 1 \bmod 4$) $\omega^2 bab$ or $\omega^3 bab$. If $pb \neq 0$ then we need $\zeta^4 = 1$ and if $pa - ca = ybab$ then

$$pa' - c'a' = \zeta^{-2} b'a'b'$$

so if $p = 1 \bmod 4$ we can take $0 \leq x \leq (p-1)/2$, and if $p = 3 \bmod 4$ we can take $0 \leq x < p$. If $pb = pc = pd = 0$ then we can assume that $pa - ca = 0$, bab or ωbab. So we have $3p + \gcd(p-1, 3) + \gcd(p-1, 4) + \gcd(p-1, 5)$ algebras here.

Finally, consider the case when $cb = da = db - ca = 0$, $dc = ubab$ (where $u = 1, \omega$ or ω^2). If a', b', c', d' generate L, and satisfy the same relations as those already speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta\mu & 0 & 0 & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, where $\zeta^3 = 1$. If a', b', c', d' are as above then $c'a' = \zeta\mu ca - u\delta bab$. We can assume that $pc = 0$ or bab and we can assume that $pd = xbab$ with $0 \leq x < p$. If either of pc, pd are non-zero then we can assume that $pb = 0$. If $pd = xbab$ and $pa - ca = ybab$ then

$$pa' - c'a' = (u\delta + x\delta + \zeta\mu)ybab = (u\delta\mu^{-1} + x\delta\mu^{-1} + \zeta)bab$$

and so if $x \neq -u$ we can take $pa - ca = 0$. If $x = -u$ we can take $pa - ca = 0$ or $ybab$ where $y \neq 0$ runs through a transversal for the cube roots of unity. If $pc = pd = 0$ then we can assume that $pb = 0$ or bab. So the number of algebras here is $4p - 1$ if $p = 2 \bmod 3$ and $2p + 1 + \frac{2}{3}(p-1)$ if $p = 1 \bmod 3$. However if $p = 1 \bmod 3$ we have three times this values so we have $8p + 1$ algebras. So (in general) the number of algebras is $2p - 2 + (2p+1)\gcd(p-1, 3)$.

56.22.2 $bab = 0$

If $bab = 0$ then L_3 is generated by baa. Adding suitable scalar multiples of ba to c and d we can assume that $da = db - ca = 0$. We can assume that $dc = 0$ or baa. If $dc = baa$ we can assume that $cb = 0$ and if $dc = 0$ then we can assume that $cb = 0$, baa or ωbaa.

First consider the case when $cb = da = db - ca = dc = 0$. If a', b', c', d' generate L, and satisfy the same relations as speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta\mu & -\zeta\lambda & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$ or baa, and if $pc = baa$ we can assume that $pa - ca = pb = pd = 0$. If $pc = 0$ then we can assume that $pd = 0$ or baa, and if $pd = baa$ we can assume that $pa - ca = pb = 0$. If $pc = pd = 0$ then we can assume that $pb = 0$, baa or ωbaa, and if $pb \neq 0$ we can assume that $pa = ca$. Finally, if $pb = pc = pd = 0$ we can assume that $pa - ca = 0$ or baa. So there are 6 algebras here.

Next consider the case when $cb = baa$ or ωbaa and $da = db - ca = dc = 0$. If a', b', c', d' generate L, and satisfy the same relations as speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \pm 1 & -\zeta\lambda & \gamma & \delta \\ 0 & \zeta & \eta & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & \pm\zeta^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$ or baa, and if $pc = baa$ we can assume that $pa - ca = pb = pd = 0$.

If $pc = 0$ and $pd = xbaa$ then $pd' = \pm\zeta^{-2}xb'a'a'$, so if $p = 1 \bmod 4$ then we can assume that $pd = 0$, baa or ωbaa, and if $p = 3 \bmod 4$ then we can assume that $pd = 0$ or baa. If $pd \neq 0$ we can assume that $pa - ca = pb = 0$.

If $pc = pd = 0$ then we can assume that $pb = ybaa$ with $0 \leq y < p$. If $cb = baa$ and $y \neq -1$ then we can adjust λ so that $pa - ca = 0$. Similarly, if $cb = \omega baa$ and $y \neq -\omega$ we can assume that $pa - ca = 0$. But if $cb = baa$ and $pb = -baa$ or $cb = \omega baa$ and $pb = -\omega baa$ we can assume that $pa - ca = 0$ or baa. So there are $p + 2 + \frac{1}{2}\gcd(p-1, 4)$ algebras here for each value of cb, and $2p + 4 + \gcd(p-1, 4)$ algebras altogether.

Finally consider the case when $cb = da = db - ca = 0$, $dc = baa$. If a', b', c', d' generate

L, and satisfy the same relations as speciøed for a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \zeta^{-2} & -\zeta\lambda & \gamma & \delta \\ 0 & \zeta & \eta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & \zeta^{-3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We can assume that $pc = 0$, baa or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$. If $pc \neq 0$ we can assume that $pa - ca = pb = pd = 0$.

So consider the case when $pc = 0$. We can assume that $pd = xbaa$ with $0 \leq x < p$, and that $pb = 0$, baa or ωbaa or (if $p = 1 \bmod 4$) $\omega^2 baa$ or $\omega^3 baa$. If $pd \neq 0$ we can assume that $pa - ca = 0$ by adjusting δ, and if $pd = 0$ we can still assume that $pa - ca = 0$ by adjusting η.

So there are $p + \gcd(p-1, 3) + p\gcd(p-1, 4)$ algebras here.

56.23 Descendants of 6.52

Algebra 6.52 has $4p^2 + 15p + 15 + (p+1)\gcd(p-1, 3) + \gcd(p-1, 4)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$, and the formulae for each batch of descendants were checked with orb6.52 for $p = 5, 7, 11, 13$.

Algebra 6.52 has class 2 and relators

$$cb,\ da,\ db - ca,\ dc,\ pa,\ pb,\ pc - ca,\ pd.$$

If a', b', c', d' generate 6.52, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ \varepsilon & \zeta & 0 & \theta \\ 0 & 0 & \zeta\mu & -\varepsilon\mu \\ 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo the Frattini subalgebra. If L is a descendant of 6.52 of order p^7 then L_3 is generated by baa and bab. We distinguish two cases: $baa = 0$ and $bab = 0$.

56.23.1 $baa = 0$

If $baa = 0$ then L_3 is generated by bab. Adding suitable scalar multiples of ba to c and d we can assume that $cb = db - ca = 0$. We can assume that $da = 0$ or bab. If $da = 0$ we cand assume that $dc = 0$ or bab, but if $da = bab$ then we need $\mu = \zeta^2$ and so we can take $dc = 0$ or bab or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$.

First consider the case when $cb = da = db - ca = dc = 0$. If a', b', c', d' are as above we have $c'a' = \zeta\mu ca$. We can assume that $pd = 0$ or bab, and if $pd = bab$ then we can assume that $pa = pb = pc - ca = 0$. If $pd = 0$ then we can assume that $pc - ca = 0$ or bab and that $pa = 0$, bab or ωbab. If $pa \neq 0$ we can assume that $pb = 0$, and if $pa = 0$ we can assume that $pb = 0$ or bab. So there are 9 algebras here.

Next consider the case when $cb = da = db - ca = 0$, $dc = bab$. We need $\zeta = \mu^2$, and we have $c'a' = \zeta^3 ca - \delta\zeta bab$. If $pd \neq 0$ we can assume that $pd = bab$ or (if $p = 1 \bmod 3$) ωbab or $\omega^2 bab$ and that $pa = pb = pc - ca = 0$. So assume that $pd = 0$. By adjusting δ we may assume that $pc - ca = 0$. And we can assume that $pa = 0$, bab or ωbab or (if $p = 1 \bmod 4$) $\omega^2 bab$ or $\omega^3 bab$, and if $pa \neq 0$ we can assume that $pb = 0$, but if $pa = pc = pd = 0$ we can assume that $pb = 0$ or bab or ωbab. So there are $3 + \gcd(p-1, 3) + \gcd(p-1, 4)$ algebras here.

Now consider the case when $cb = db - ca = dc = 0$, $da = bab$. We now need $\mu = \zeta^2$, and $c'a' = \zeta^3 ca - \varepsilon\zeta^2 bab$. We can assume that $pd = xbab$ with $0 \leq x < p$. If $pd \neq 0$ we can assume that $pa = pb = 0$ and if $pd \neq bab$ we can adjust ε so that $pc - ca = 0$. If $pd = bab$ then we can assume that $pc - ca = 0$ or bab. So let $pd = 0$. By adjusting ε we may assume that $pc - ca = 0$, though we then need $\varepsilon = 0$. We can then assume that $pb = 0$ or bab, and if $pb = bab$ we can assume that $pa = xbab$ for some x with $0 \leq x < p$. If $pb = pc - ca = pd = 0$ then we can assume that $pa = 0$, bab or ωbab. So there are $2p+3$ algebras here.

Finally consider the case when $cb = db - ca = 0$, $da = bab$, $dc = ubab$ where $u = 1$ or (if $p = 1 \bmod 3$) ω or ω^2. Then we need $\zeta = \mu^2$ and $\zeta^3 = 1$, and $c'a' = ca - (u\delta + \varepsilon\mu)bab$. We can assume that $pd = xbab$ with $0 \leq x < p$. If $x \neq 0, 1$ we can assume that $pa = pb = pc - ca = 0$. If $x = 1$ we can assume that $pa = pb = 0$ and that $pc - ca = 0$ or $ybab$ where $y \neq 0$ is a transversal for the cube roots of unity. If $pd = 0$ we can take $pc - ca = 0$, and we can take $pa = 0$ or $ybab$ where $y \neq 0$ is a transversal for the cube roots of unity. If $pa \neq 0$ we can take $pb = 0$, and if $pa = 0$ we can take $pb = 0$ or $ybab$ where $y \neq 0$ is a transversal for the cube roots of unity. So the number of algebras here is $4p - 3$ if $p = 2 \bmod 3$, and $2p - 1$ if $p = 1 \bmod 3$. However we have 3 values of u when $p = 1 \bmod 3$, so the total number of algebras is $3p - 3 + p\gcd(p-1, 3)$.

56.23.2 $bab = 0$

If $bab = 0$ then L_3 is generated by baa. Adding suitable scalar multiples of ba to c and d we may assume that $da = db - ca = 0$. We can assume that $dc = 0$, baa or ωbaa. If $dc = 0$ we can assume that $cb = 0$ or baa, and if $dc \neq 0$ we can assume that $cb = 0$.

First consider the case when $cb = da = db - ca = dc = 0$. If a', b', c', d' generate L, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & \zeta & 0 & \theta \\ 0 & 0 & \zeta\mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We have $c'a' = \zeta\mu ca$. We can assume that $pc - ca$ and pd are (independently) equal to 0 or baa. If $pd \neq 0$ we can assume that $pa = pb = 0$. If $pd = 0$ we can assume that $pa = 0$ or baa and that $pb = xbaa$ for some x with $0 \leq x < p$. So there are $4p + 2$ algebras here.

Next consider the case when $cb = baa$, $da = db - ca = dc = 0$. If a', b', c', d' generate L, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & \zeta & 0 & \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We have $c'a' = ca$. If $pd \neq 0$ we can assume that $pd = baa$ or ωbaa and that $pa = pb = 0$. We then require $\zeta = \pm 1$ and so we can assume that $pc - ca = xbaa$ with $0 \leq x < (p-1)/2$. So consider the case when $pd = 0$. If $pc - ca \neq 0$ then we can assume that $pc - ca = baa$ and that $pa = xbaa$, $pb = ybaa$ with $0 \leq x, y < p$. If $pc - ca = pd = 0$ then we can assume that $pa = 0$ or baa and that $pb = xbaa$ with $0 \leq x < p$. So there are $p^2 + 3p + 1$ algebras here.

Finally, consider the case when $cb = da = db - ca = 0$, $dc = baa$ or ωbaa. If a', b', c', d'

generate L, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \delta \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \pm\zeta & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2. We have $c'a' = \pm\zeta ca$. If $pd \neq 0$ we can assume that $pd = baa$ and that $pa = 0$. We then require $\zeta = \pm 1$ and so we can assume that $pc - ca = xbaa$ with $0 \leq x < (p-1)/2$, and that $pb = ybaa$ with $0 \leq y < p$. If $pd = 0$ we can assume that $pc - ca = xbaa$ with $0 \leq x < (p-1)/2$ and that $pa = 0$ or baa and $pb = ybaa$ with $0 \leq y < p$. So there are $3p(p+1)/2$ algebras here for each of the two possible values of dc.

56.24 Descendants of 6.60

Algebra 6.60 has $7 + \gcd(p-1, 3)$ descendants of order p^7. I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7$.

Algebra 6.60 has class 2 and relators

$$cb,\ da,\ db - ca,\ dc - \omega ba,\ pa,\ pb,\ pc,\ pd.$$

If a', b', c', d' generate 6.60, and satisfy the same relations as a, b, c, d, then

$$\begin{pmatrix} a' \\ b' \\ \pm c' \\ \pm d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \lambda & \mu & \nu & \xi \\ -\omega\xi & \omega\nu & \mu & -\lambda \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo the Frattini subalgebra. If L is a descendant of 6.60 of order p^7 then the commutator structure is the same as 7.53 or 7.54 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we may assume that L_3 is generated by baa and that one of following two sets of commutator relators holds:

$$bab, bac, bad, cb, da, db - ca, dc - \omega ba,$$
$$bab, bac, bad, cb - baa, da, db - ca, dc - \omega ba.$$

56.24.1 Case 1

Let L satisfy the relators

$$bab, bac, bad, cb, da, db - ca, dc - \omega ba.$$

If a', b', c', d' generate L and satisfy these relators then

$$\begin{pmatrix} a' \\ b' \\ \pm c' \\ \pm d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \lambda(\alpha^2 + \omega\delta^2) & 2\lambda\alpha\delta & 0 \\ 0 & 2\omega\lambda\alpha\delta & \lambda(\alpha^2 + \omega\delta^2) & 0 \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, and $b'a'a' = \lambda(\alpha^2 - \omega\delta^2)^2$. First consider the possible values for pb, pc. An easy calculation using Burnside's Lemma shows that there are three orbits of values for the pair pb, pc. Clearly $(0, 0)$ is in an orbit on its own. It is easy to see that all pairs $(0, xbaa)$ with $x \neq 0$ are in the same orbit, and that any orbit distinct from the orbits of $(0, 0)$ and $(0, baa)$ contains an element of the form $(baa, xbaa)$. Furthermore, $(baa, xbaa)$

is in a dioeerent orbit from $(0, baa)$ if and only if $x^2 - \omega$ is not a square. So we can assume that $(pb, pc) = (0, 0)$ or $(0, baa)$ or $(baa, xbaa)$ where $x^2 - \omega$ is not a square. (If $p = 1 \bmod 4$ then we can take $x = 0$, and if $p = 3 \bmod 4$ then we can usually take $x = 1$. However there are 9 primes less than a thousand for which we need to take $x > 1$: 43, 283, 331, 367, 571, 691, 739, 811, 919.)

If either of pb, pc is non-zero then we can take $pa = pd = 0$. If $pb = pc = 0$ then we can take $pa = 0$ and $pd = 0$ or baa.

So there are 4 algebras here.

56.24.2 Case 2

Let L satisfy the relators

$$bab, bac, bad, cb - baa, da, db - ca, dc - \omega ba.$$

If a', b', c', d' generate L and satisfy these relators then

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & (\alpha^2 + \omega\delta^2) & 2\alpha\delta & 0 \\ 0 & 2\omega\alpha\delta & (\alpha^2 + \omega\delta^2) & 0 \\ \omega\delta & -\omega\gamma & -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, and $b'a'a' = (\alpha^2 - \omega\delta^2)^2$, or

$$\begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & -(\alpha^2 + \omega\delta^2) & -2\alpha\delta & 0 \\ 0 & 2\omega\alpha\delta & (\alpha^2 + \omega\delta^2) & 0 \\ -\omega\delta & \omega\gamma & \beta & -\alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

modulo L_2, and $b'a'a' = -(\alpha^2 - \omega\delta^2)^2$. As above we can take $(pb, pc) = (0, 0)$ or $(0, baa)$ or $(baa, xbaa)$ where $x^2 - \omega$ is not a square, and if at least one of pb, pc is non-zero then we can take $pa = pd = 0$.

If $pb = pc = 0$ then we can take $pa = 0$ and $pd = 0$, baa or (if $p = 1 \bmod 3$) ωbaa or $\omega^2 baa$.

So there are $3 + \gcd(p - 1, 3)$ algebras here.

57 Grandchildren of algebra 40 (5.2)

Algebra 5.2 has 4 descendants of order p^6, but only 6.63 is capable.

57.1 Descendants of 6.63

Algebra 6.63 has four descendants of order p^7. I have checked that the recipe below gives 4 non-isomorphic groups for $p = 5, 7$. Let L be an immediate descendant of 6.63 of order p^7. Then L is generated by a, b, c, d, L_2 is generated by pa modulo L_3, L_3 is generated by p^2a modulo L_4 and L_4 is generated by p^3a. The commutators ba, ca, da, cb, db, dc are all scalar mutiples of p^3a, as are pb, pc, pd. Clearly we may assume that $pb = pc = pd = 0$. The subalgebra $B = \langle b, c, d\rangle + L_4$ is then a characteristic subalgebra.

If B is abelian then we have

$$\langle a, b, c, d \mid ba, ca, da, cb, db, dc, pb, pc, pd, \text{ class } 4\rangle,$$

$$\langle a, b, c, d \mid ba - p^3a, ca, da, cb, db, dc, pb, pc, pd, \text{ class } 4\rangle.$$

On the other hand, if B is not abelian then we may suppose that $cb = p^3a$, $db = dc = 0$. Subtracting suitable multiples of b and c from a we may suppose that $ba = ca = 0$. So we have

$$\langle a, b, c, d \mid ba, ca, da, cb - p^3a, db, dc, pb, pc, pd, \text{ class } 4\rangle,$$

$$\langle a, b, c, d \mid ba, ca, da - p^3a, cb - p^3a, db, dc, pb, pc, pd, \text{ class } 4\rangle.$$

58 Grandchildren of algebra 41 (5.3)

Of the descendants of 5.3 of order p^6, only 6.67 and 6.72 are capable.

58.1 Descendants of 6.67

The number of descendants of 6.67 is

$3 + \gcd(p - 1, 3)+$
$2 + \gcd(p - 1, 3) + \gcd(p - 1, 4)+$
$4 + \gcd(p - 1, 3)+$
$4 + \gcd(p - 1, 3) + \gcd(p - 1, 4)+$
$3 + 2\gcd(p - 1, 3)+$
$2 + 2\gcd(p - 1, 3) + \gcd(p - 1, 4) =$
$18 + 8\gcd(p - 1, 3) + 3\gcd(p - 1, 4)$

I have checked that the recipes below give this number of non-isomorphic groups for $p = 5, 7, 11, 13$.

Let L be an immediate descendant of 6.67 of order p^7. Recall that 6.67 has class 3 and satisøes the following relations.

$$bab = ca = cb = da = db = dc = pa = pb = pc = pd = 0.$$

So L is generated by a, b, c, d, L_2 is generated modulo L_3 by ba, L_3 is generated modulo L_4 by baa, and L_4 is generated by $baaa$. The commutators bab, ca, cb, da, db, dc are all scalar multiples of $baaa$, as are pa, pb, pc, pd. The commutator structure of L is the same as one of 7.55 ˇ 7.60 from the list of nilpotent Lie algebras over $\mathbb{Z}_p$ of dimension 7. So we can assume that one of the following sets of commutator relations holds.

$$\begin{aligned}
bab &= ca = cb = da = db = dc = 0,\\
bab &= baaa,\ ca = cb = da = db = dc = 0,\\
bab &= 0,\ cb = baaa,\ ca = da = db = dc = 0,\\
bab &= baaa,\ cb = baaa,\ ca = da = db = dc = 0,\\
bab &= 0,\ dc = baaa,\ ca = cb = da = db = 0,\\
bab &= baaa,\ dc = baaa,\ ca = cb = da = db = 0.
\end{aligned}$$

In each of these cases we let

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = Abaaa$$

for some 4×1 matrix A, and we investigate how A transforms under a map from L to L which preserves the relevant commutator relations.

58.1.1 Case 1

Let L satisfy

$$bab = ca = cb = da = db = dc = 0.$$

Then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} A.$$

Clearly we may assume that $pd = 0$, and if $pc \neq 0$ we can assume that $pc = baaa$ and that $pa = pb = 0$. If $pc = pd = 0$ then we can $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaaa$. If $pb \neq 0$ we may assume that $pa = 0$, and if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or $baaa$.

So there are 4 algebras if $p \neq 1 \bmod 3$, and 6 algebras if $p = 1 \bmod 3$. Thus the number of algebras is $3 + \gcd(p-1, 3)$.

58.1.2 Case 2

Next assume that L satisøes

$$bab = baaa,\ ca = cb = da = db = dc = 0.$$

Then A transforms to

$$\alpha^{-5}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2 & \zeta & \eta \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} A.$$

Once again, we can assume that $pd = 0$, and if $pc \neq 0$ then we can assume that $pc = baaa$ and that $pa = pb = 0$. If $pc = pd = 0$ and $pb \neq 0$ then we can assume that $pa = 0$ and that $pb = baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Finally, if $pb = pc = pd = 0$ then we can assume that $pa = 0$, $baaa$, $\omega baaa$, or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$. So there are 5 algebras, though we have to add 2 if $p = 1 \bmod 3$ and add another two if $p = 1 \bmod 4$. Thus the number of algebras is $2 + \gcd(p-1, 3) + \gcd(p-1, 4)$.

58.1.3 Case 3

Now assume that L satisøes

$$bab = 0,\ cb = baaa,\ ca = da = db = dc = 0.$$

Then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & \zeta & \eta \\ 0 & 0 & \alpha^3 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} A.$$

If $pd \neq 0$ we may assume that $pd = baaa$, $pb = pc = pd = 0$. If $pd = 0$ and $pc \neq 0$ then we can assume that $pc = baaa$, $pa = pb = 0$. If $pc = pd = 0$ and $pb \neq 0$ then we can assume that $pa = 0$ and that $pb = baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Finally, if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or $baaa$. So the number of algebras is $4 + \gcd(p-1, 3)$.

58.1.4 Case 4

If L satisøes

$$bab = baaa,\ cb = baaa,\ ca = da = db = dc = 0,$$

then A transforms to

$$\alpha^{-5}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2 & \zeta & \eta \\ 0 & 0 & \alpha^3 & \mu \\ 0 & 0 & 0 & \xi \end{pmatrix} A.$$

If $pd \neq 0$ we may assume that $pd = baaa$, $pb = pc = pd = 0$. If $pd = 0$ and $pc \neq 0$ then we can assume that $pc = baaa$ or $\omega baaa$, $pa = pb = 0$. If $pc = pd = 0$ and $pb \neq 0$ then we can assume that $pa = 0$ and that $pb = baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Finally, if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or $baaa$ or $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$. So the number of algebras is $4 + \gcd(p-1,3) + \gcd(p-1,4)$.

58.1.5 Case 5

If L satisøes

$$bab = 0,\ dc = baaa,\ ca = cb = da = db = 0,$$

then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} A$$

with $\lambda\xi - \mu\nu = \alpha^3\varepsilon$. We can assume that $pd = 0$ and that $pc = 0$ or $baaa$. And we can assume that $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If either of pb or pc are non-zero then we can assume that $pa = 0$, and if $pb = pc = pd = 0$ then we can assume that $pa = 0$ or $baaa$. So we have $3 + 2\gcd(p-1,3)$ algebras.

58.1.6 Case 6

Finally consider the case when L satisøes

$$bab = baaa,\ dc = baaa,\ ca = cb = da = db = 0.$$

Then A transforms to

$$\alpha^{-5}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & \nu & \xi \end{pmatrix} A$$

with $\lambda\xi - \mu\nu = \alpha^5$. We can assume that $pd = 0$ and that $pc = 0$ or $baaa$. And we can assume that $pb = 0$, $baaa$ or (if $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If either of pb or pc are non-zero then we can assume that $pa = 0$, and if $pb = pc = pd = 0$ then we can assume that $pa = 0$, $baaa$ or $\omega baaa$ or (if $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$ So we have $2 + 2\gcd(p-1,3) + \gcd(p-1,4)$ algebras.

The number of descendants of 6.72 of order p^7 is
$5+2\gcd(p-1,3)+$
$4+2\gcd(p-1,3)+$
$5+2\gcd(p-1,3)+$
$3+(p+1)\gcd(p-1,3)+\gcd(p-1,4)=$
$17+(p+7)\gcd(p-1,3)+\gcd(p-1,4)$.

I have checked that the recipes below give this number of non-isomorphic groups for $p=5,7,11,13$.

Algebra 6.72 is a four generator class 3 algebra satisfying the following relators.

$$ca,\ cb-baa,\ bab,\ da,\ db,\ dc,\ pa,\ pb,\ pc,\ pd.$$

If L is an immediate descendant of 6.72 of order p^7 then L is generated by a,b,c,d, L_2 is generated by ba modulo L_3, L_3 is generated by baa modulo L_4 and L_4 is generated by $baaa$. The commutator structure of L is the same as one of 7.61 ˘ 7.64 from the list of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So L satisøes one of the following four commutator structures

$$\begin{aligned}
ca &= cb-baa=bab=da=db=dc=0,\\
ca &= cb-baa=bab=da=db-baaa=dc=0,\\
ca &= cb-baa=bab=da=db=dc-baaa=0,\\
ca &= cb-baa=bab=da=db-baaa=dc-baaa=0.
\end{aligned}$$

In each of these cases we let

$$\begin{pmatrix} pa\\ pb\\ pc\\ pd\end{pmatrix}=Abaaa$$

for some 4×1 matrix A, and we investigate how A transforms under a map from L to L which preserves the relevant commutator relations.

58.2.1 Case 1

If L satisøes

$$ca=cb-baa=bab=da=db=dc=0$$

then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & 0 & \delta\\ 0 & \varepsilon & 0 & \eta\\ 0 & 0 & \alpha^2 & \mu\\ 0 & 0 & 0 & \xi\end{pmatrix}A.$$

So if $pd\neq 0$ we may assume that $pd=baaa$, and that $pa=pb=pc=0$. If $pd=0$ then we may assume that $pc=0$ or $baaa$, and (independently) that $pb=0$, $baaa$ or (in the case when $p=1\bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb\neq 0$ then we can assume that $pa=0$. And if $pb=0$ we can assume that $pa=0$ or $baaa$. So we have $5+2\gcd(p-1,3)$ algebras.

58.2.2 Case 2

If L satisøes

$$ca=cb-baa=bab=da=db-baaa=dc=0$$

then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & \eta \\ 0 & 0 & \alpha^2 & -2\alpha\gamma \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} A.$$

So if $pd \neq 0$ we may assume that $pd = baaa$, and that $pa = pb = pc = 0$. If $pd = 0$ then we can assume that $pc = 0$ or $baaa$, and (independently) that $pb = 0$, $baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb \neq 0$ or $pc \neq 0$ then we can assume that $pa = 0$. And if $pb = pc = 0$ we can assume that $pa = 0$ or $baaa$. So we have $4 + 2\gcd(p-1, 3)$ algebras.

58.2.3 Case 3

If L satisøes

$$ca = cb - baa = bab = da = db = dc - baaa = 0$$

then A transforms to

$$\alpha^{-3}\varepsilon^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \varepsilon & 0 & 2\alpha^{-1}\gamma\varepsilon \\ 0 & 0 & \alpha^2 & \mu \\ 0 & 0 & 0 & \alpha\varepsilon \end{pmatrix} A.$$

So if $pd \neq 0$ we may assume that $pd = baaa$ or $\omega baaa$ and that $pa = pb = pc = 0$. If $pd = 0$ then we can assume that $pc = 0$ or $baaa$, and (independently) that $pb = 0$, $baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. If $pb \neq 0$ or $pc \neq 0$ then we can assume that $pa = 0$. And if $pb = pc = 0$ we can assume that $pa = 0$ or $baaa$. So we have $5 + 2\gcd(p-1, 3)$ algebras.

58.2.4 Case 4

If L satisøes

$$ca = cb - baa = bab = da = db - baaa = dc - baaa = 0$$

then A transforms to

$$\alpha^{-5}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha^2 & 0 & \eta \\ 0 & 0 & \alpha^2 & -2\alpha\gamma + \eta \\ 0 & 0 & 0 & \alpha^3 \end{pmatrix} A.$$

So if $pd \neq 0$ we may assume that $pd = baaa$ or $\omega baaa$ and that $pa = pb = pc = 0$. If $pd = 0$ and $pc \neq 0$ then we can assume that $pc = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$, and that $pa = 0$ and that $pb = \lambda baaa$ for some λ with $0 \leq \lambda < p$. If $pc = pd = 0$ and $pb \neq 0$ then we can assume that $pa = 0$ and that $pb = baaa$ or (in the case when $p = 1 \bmod 3$) $\omega baaa$ or $\omega^2 baaa$. Finally, if $pb = pc = pd = 0$ then we can assume that $pa = 0$, $baaa$, $\omega baaa$, or (in the case when $p = 1 \bmod 4$) $\omega^2 baaa$ or $\omega^3 baaa$. So there are $3 + (p+1)\gcd(p-1, 3) + \gcd(p-1, 4)$ algebras.

59 Grandchildren of algebra 42 (5.1)

Algebra 5.1 has 7 descendants of of order p^6, but only 6.2 and 6.3 are capable.

59.1 Descendants of 6.2

Let L be an immediate descendant of 6.2 of order p^7. Then L is generated by a, b, c, d, e, L_2 is generated modulo L_3 by pa, and L_3 is generated by p^2a. All commutators and pb, pc, pd, pe lie in L_3. Subtracting suitable multiples of pa from b, c, d, e we may assume that $pb = pc = pd = pe = 0$. Then $B = \langle b, c, d, e\rangle + L_3$ is characteristic, being the largest subring B such that $pB = 0$. There are three possible commutator structures on B:

$$\begin{aligned} cb &= db = dc = eb = ec = ed = 0, \\ db &= dc = eb = ec = ed = 0, \\ db &= dc = eb = ec = 0,\ ed = cb. \end{aligned}$$

In the ørst case B is abelian and in the other two B^2 is spanned by cb.

First consider the case when B is abelian. Then we may suppose that $ca = da = ea = 0$, and that $ba = 0$ or p^2a, giving

$$\langle a, b, c, d, e \mid ba, ca, da, ea, cb, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 3\rangle, \tag{7.165}$$

$$\langle a, b, c, d, e \mid ba - p^2a, ca, da, ea, cb, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 3\rangle. \tag{7.166}$$

Nex consider the case when $db = dc = eb = ec = ed = 0$. Then we may suppose that $cb = p^2a$. Note that the centre of B is $D = \langle d, e\rangle + L_3$. Replacing a by $a + \alpha b + \beta c$ for suitable α, β we may suppose that $ba = ca = 0$. One possibility is that a is central, giving

$$\langle a, b, c, d, e \mid ba, ca, da, ea, cb - p^2a, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 3\rangle. \tag{7.167}$$

If a is not central then we may suppose that $da = p^2a$ and that $ea = 0$. So we have

$$\langle a, b, c, d, e \mid ba, ca, da - p^2a, ea, cb - p^2a, db, eb, dc, ec, ed, pb, pc, pd, pe, \text{ class } 3\rangle. \tag{7.167B}$$

Finally consider the case when $db = dc = eb = ec = 0$, $ed = cb$. Again, we may suppose that $cb = ed = p^2a$. And subtracting suitable multiples of b, c, d, e from a we may suppose that a is central. This gives

$$\langle a, b, c, d, e \mid ba, ca, da, ea, cb - p^2a, db, eb, dc, ec, ed - p^2a, pb, pc, pd, pe, \text{ class } 3\rangle. \tag{7.167C}$$

59.2 Descendants of 6.3

Let L be an immediate descendant of 6.3 of order p^7. Then L is generated by a, b, c, d, e, L_2 is generated modulo L_3 by ba, and L_3 is generated by baa, bab. All other commutators and pa, pb, pc, pd, pe lie in L_3. The commutator structure of L must be the same as one of 7.11 ˘ 7.14 from the list nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$. So we can assume that one of the follwing sets of commutator relations holds:

$$\begin{aligned} bab &= ca = da = ea = cb = db = eb = dc = ec = de = 0, \\ bab &= ca = da = ea = cb - baa = db = eb = dc = ec = de = 0, \\ bab &= ca = da = ea = cb = db = eb = dc - baa = ec = de = 0, \\ bab &= ca = da = ea = cb = db = eb - baa = dc - baa = ec = ed = 0. \end{aligned}$$

59.2.1 Case 1

Let

$$bab = ca = da = ea = cb = db = eb = dc = ec = de = 0.$$

Then c, d, e are central. Since pc, pd, pe span a space of dimension at most 1, we can suppose that $pd = pe = 0$. So the subalgebra generated by a, b, c must be isomorphic to one of 5.27 ˘ 5.31. So we have

$$\langle a,b,c,d,e \mid ca,cb,bab,da,db,dc,ea,eb,ec,ed,pa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.168}$$

$$\langle a,b,c,d,e \mid ca,cb,bab,da,db,dc,ea,eb,ec,ed,pa-baa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.169}$$

$$\langle a,b,c,d,e \mid ca,cb,bab,da,db,dc,ea,eb,ec,ed,pa,pb-baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.170}$$

$$\langle a,b,c,d,e \mid ca,cb,bab,da,db,dc,ea,eb,ec,ed,pa,pb-\omega baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.171}$$

$$\langle a,b,c,d,e \mid ca,cb,bab,da,db,dc,ea,eb,ec,ed,pa,pb,pc-baa,pd,pe, \text{ class } 3\rangle. \tag{7.172}$$

59.2.2 Case 2

Let

$$bab = ca = da = ea = cb - baa = db = eb = dc = ec = de = 0.$$

Here d and e are central, so that we can assume that $pe = 0$. So the subalgebra generated by a, b, c, d must be isomorphic to one of 6.72 ˘ 6.77. so we have

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.173}$$

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa-baa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.174}$$

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa,pb-baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.175}$$

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa,pb-\omega baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.176}$$

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa,pb,pc-baa,pd,pe, \text{ class } 3\rangle, \tag{7.177}$$

$$\langle a,b,c,d,e \mid ca,cb-baa,bab,da,db,dc,ea,eb,ec,ed,pa,pb,pc,pd-baa,pe, \text{ class } 3\rangle. \tag{7.178}$$

59.2.3 Case 3

Let

$$bab = ca = da = ea = cb = db = eb = dc - baa = ec = de = 0.$$

Now e is central, and so if $pe = 0$ then the subalgebra generated by a, b, c, d must be isomorphic to one of 6.78 ˘ 6.84, giving

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.179}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa-baa,pb,pc,pd,pe, \text{ class } 3\rangle, \tag{7.180}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb-baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.181}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb-\omega baa,pc,pd,pe, \text{ class } 3\rangle, \tag{7.182}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb,pc-baa,pd,pe, \text{ class } 3\rangle, \tag{7.183}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb-baa,pc-baa,pd,pe, \text{ class } 3\rangle, \tag{7.184}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb-\omega baa,pc-baa,pd,pe,\text{ class }3\rangle. \tag{7.185}$$

And if $pe \neq 0$ we may suppose that $pe = baa$ and that $pa = pb = pc = pd = 0$.

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb,ec,ed,pa,pb,pc,pd,pe-baa,\text{ class }3\rangle. \tag{7.186}$$

59.2.4 Case 4

Finally, let

$$bab = ca = da = ea = cb = db = eb - baa = dc - baa = ec = ed = 0.$$

The elements pa, pb, pc, pd, pe are all scalar multiples of baa. The subalgebras $B = \langle b,c,d,e\rangle + L^2$, $C = \langle c,d,e\rangle + L^2$ and $E = \langle e\rangle + L^2$ are all characteristic.

First, consider the case when $pE \neq \{0\}$. Scaling b and d by the same scale factor, we may assume that $pe = baa$. If we let $a' = a + \alpha e$, $b' = b - \beta c + \gamma d + \varepsilon e$, $c' = c + \delta e$, $d' = d + \gamma e$, $e' = e$ then a', b', c', d', e' satisfy the same commutator relations as a, b, c, d, e and $b'a'a' = baa$. So we can assume that $pa = pb = pc = pd = 0$, $pe = baa$. this gives

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb-baa,ec,ed,pa,pb,pc,pd,pe-baa,\text{ class }3\rangle. \tag{7.187}$$

If $pE = \{0\}$, but $pC \neq \{0\}$ then replacing a, b, c, d, e by a', b', c', d', e' of the form

$$\begin{aligned} a' &= a, \\ b' &= (\alpha\delta - \beta\gamma)b, \\ c' &= \alpha c + \beta d, \\ d' &= \gamma c + \delta d, \\ e' &= e, \end{aligned}$$

we may suppose that $pd = pe = 0$ and $pc \neq 0$. By scaling, we may assume that $pc = baa$. If we let

$$\begin{aligned} a' &= a + \alpha c \\ b' &= b - \beta c, \\ c' &= c, \\ d' &= d + \beta e - \alpha ba, \\ e' &= e \end{aligned}$$

then a', b', c', d', e' satisfy the same commutator relations as a, b, c, d, e. So we may assume that $pa = pb = 0$. This gives

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb-baa,ec,ed,pa,pb,pc-baa,pd,pe,\text{ class }3\rangle. \tag{7.188}$$

Next assume that $pC = \{0\}$, but that $pB \neq \{0\}$. Then by scaling we can assume that $pb = baa$ or ωbaa. And replacing a and e by $a + \alpha a$ and $e - \alpha ba$ for suitable α we may suppose that $pa = 0$. So we have

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb-baa,ec,ed,pa,pb-baa,pc,pd,pe,\text{ class }3\rangle, \tag{7.189}$$

$$\langle a,b,c,d,e \mid bab,ca,cb,da,db,dc-baa,ea,eb-baa,ec,ed,pa,pb-\omega baa,pc,pd,pe,\text{ class }3\rangle. \tag{7.190}$$

Finally, consider the case when $pB = \{0\}$. Scaling we may assume that $pa = 0$ or baa giving

$$\langle a, b, c, d, e \mid bab, ca, cb, da, db, dc - baa, ea, eb - baa, ec, ed, pa, pb, pc, pd, pe, \text{ class } 3\rangle, \tag{7.191}$$

$$\langle a, b, c, d, e \mid bab, ca, cb, da, db, dc - baa, ea, eb - baa, ec, ed, pa - baa, pb, pc, pd, pe, \text{ class } 3\rangle. \tag{7.192}$$

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$$\langle a, b, c, d, e, f \mid ba, ca, da, ea, fa, cb, db, eb, fb, dc, ec, fc, ed, fd, fe, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.2}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc, ec, fc, ed, fd, fe, pa, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.3}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc, ec, fc, ed, fd, fe, pa-ba, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.4}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc, ec, fc, ed, fd, fe, pa, pb, pc-ba, pd, pe, pf, \text{ class } 2\rangle \tag{7.5}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc-ba, ec, fc, ed, fd, fe, pa, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.6}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc-ba, ec, fc, ed, fd, fe, pa-ba, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.7}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc-ba, ec, fc, ed, fd, fe, pa, pb, pc, pd, pe-ba, pf, \text{ class } 2\rangle \tag{7.8}$$

$$\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc-ba, ec, fc, ed, fd, fe-ba, pa, pb, pc, pd, pe, pf, \text{ class } 2\rangle \tag{7.9}$$

$$\begin{aligned}&\langle a, b, c, d, e, f \mid ca, da, ea, fa, cb, db, eb, fb, dc - ba, ec, fc, ed, fd, fe - ba,\\ &pa - ba, pb, pc, pd, pe, pf, \text{ class } 2\rangle\end{aligned} \tag{7.10}$$

61 Appendix A

61.1 Case 5 in the descendants of 4.1, when L^2 has order p^3

$$\begin{aligned}
a' &= \alpha\lambda a + \beta\lambda b + \beta\mu c - \alpha\mu d,\\
b' &= \gamma\lambda a + \delta\lambda b + \delta\mu c - \gamma\mu d,\\
c' &= \gamma\nu a + \delta\nu b + \delta\xi c - \gamma\xi d,\\
d' &= -\alpha\nu a - \beta\nu b - \beta\xi c + \alpha\xi d
\end{aligned}$$

with (α, β) and (γ, δ) linearly independant, and with (λ, μ) and (ν, ξ) linearly independant. Furthermore

$$\begin{pmatrix} b'a' \\ c'a' \\ d'c' \end{pmatrix} = (\alpha\delta - \beta\gamma) \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ dc \end{pmatrix}.$$

It is easy to show that any element in L^2 can be expressed in the form $\rho b'a' + \tau d'c'$ for some a', b', c', d'. From this it is easy to see that a one dimensional subspace of L^2 can be taken to be spanned by ba or ca or $ba - \omega dc$.

So if pL has dimension 1 then we can assume that pL is spanned by ba or ca or $ba-\omega dc$.

Now suppose that pL has order p^2. Using transformations of the form

$$\begin{aligned}
a' &= \alpha\lambda a + \beta\lambda b + \beta\mu c - \alpha\mu d, \\
b' &= \gamma\lambda a + \delta\lambda b + \delta\mu c - \gamma\mu d, \\
c' &= \delta\xi c - \gamma\xi d, \\
d' &= -\beta\xi c + \alpha\xi d
\end{aligned}$$

it is easy to see that we can take pL to be one of the following øve subspaces:

$$\langle ba, ca\rangle,\ \langle ba + dc, ca\rangle,\ \langle ba + \omega dc, ca\rangle,\ \langle ba, dc\rangle,\ \langle ca, dc\rangle.$$

Now

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} \lambda^2 + \nu^2 & 2\lambda\mu + 2\nu\xi & \mu^2 + \xi^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & \omega \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} \lambda^2 + \omega\nu^2 & 2\lambda\mu + 2\omega\nu\xi & \mu^2 + \omega\xi^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}.$$

So taking $\lambda = 0$ we see that the ørst of these øve subspaces is equivalent to the last. Also, if we take $\lambda = \mu = \nu = 1$, $\xi = -1$ then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi + \mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

So every two dimensional subspace of L^2 is equivalent to one of the following three subspaces:

$$\langle ba + \omega dc, ca\rangle,\ \langle ba, dc\rangle,\ \langle ca, dc\rangle.$$

It is straightforward to show that these three subspaces are inequivalent, and so we may assume that pL is one of these three subspaces.

For the general case we have

$$\begin{aligned}
a' &= \alpha\lambda a + \beta\lambda b + \beta\mu c - \alpha\mu d, \\
b' &= \gamma\lambda a + \delta\lambda b + \delta\mu c - \gamma\mu d, \\
c' &= \gamma\nu a + \delta\nu b + \delta\xi c - \gamma\xi d, \\
d' &= -\alpha\nu a - \beta\nu b - \beta\xi c + \alpha\xi d
\end{aligned}$$

with (α,β) and (γ,δ) linearly independant, and with (λ,μ) and (ν,ξ) linearly independant. Furthermore

$$\begin{pmatrix} b'a' \\ c'a' \\ d'c' \end{pmatrix} = (\alpha\delta-\beta\gamma)\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi+\mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}\begin{pmatrix} ba \\ ca \\ dc \end{pmatrix}.$$

So we consider orbits of 4×3 matrices A (representing pa, pb, pc, pd) under transformations of the form

$$A \longmapsto (\alpha\delta-\beta\gamma)^{-1}\begin{pmatrix} \alpha\lambda & \beta\lambda & \beta\mu & -\alpha\mu \\ \gamma\lambda & \delta\lambda & \delta\mu & -\gamma\mu \\ \gamma\nu & \delta\nu & \delta\xi & -\gamma\xi \\ -\alpha\nu & -\beta\nu & -\beta\xi & \alpha\xi \end{pmatrix} A \begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi+\mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}^{-1}.$$

We note that if we multiply $\alpha,\beta,\gamma,\delta$ through by a factor k (in the expression above), and multiply λ,μ,ν,ξ through by a factor l, then the image of A is multiplied by a factor $k^{-1}l^{-1}$. So we can ignore the factor $(\alpha\delta-\beta\gamma)^{-1}$ and still get the same orbits. We work out the dimension of the space of matrices A øxed by any given choice of $\alpha,\beta,\gamma,\delta$ and λ,μ,ν,ξ. It turns out that it only depends on the conjugacy classes of then matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$. The calculations below show that the number of orbits of A (and hence the number of algebras) is 550 when $p=3$ and

$$\begin{aligned} p^5+p^4+4p^3+6p^2+18p+19 \text{ if } p &= 1 \bmod 3, \\ p^5+p^4+4p^3+6p^2+16p+17 \text{ if } p &= 2 \bmod 3. \end{aligned}$$

$$\begin{pmatrix} \alpha\lambda & \beta\lambda & \beta\mu & -\alpha\mu \\ \gamma\lambda & \delta\lambda & \delta\mu & -\gamma\mu \\ \gamma\nu & \delta\nu & \delta\xi & -\gamma\xi \\ -\alpha\nu & -\beta\nu & -\beta\xi & \alpha\xi \end{pmatrix}\begin{pmatrix} a & b & c \\ k & l & m \\ u & v & w \\ x & y & z \end{pmatrix} - \begin{pmatrix} a & b & c \\ k & l & m \\ u & v & w \\ x & y & z \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda\mu & \mu^2 \\ \lambda\nu & \lambda\xi+\mu\nu & \mu\xi \\ \nu^2 & 2\nu\xi & \xi^2 \end{pmatrix}$$

$$\begin{pmatrix}
\alpha\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 & 0 \\
\gamma\lambda & 0 & 0 & \delta\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \delta\mu & 0 & 0 & -\gamma\mu & 0 & 0 \\
\gamma\nu & 0 & 0 & \delta\nu & 0 & 0 & \delta\xi-\lambda^2 & -\lambda\nu & -\nu^2 & -\gamma\xi & 0 & 0 \\
-\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & 0 & \alpha\xi-\lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \alpha\lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 \\
0 & \gamma\lambda & 0 & -2\lambda\mu & \delta\lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & +\delta\mu & 0 & 0 & -\gamma\mu & 0 \\
0 & \gamma\nu & 0 & 0 & \delta\nu & 0 & -2\lambda\mu & \delta\xi-\lambda\xi-\mu\nu & -2\nu\xi & 0 & -\gamma\xi & 0 \\
0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & -2\lambda\mu & +\alpha\xi-\lambda\xi-\mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \alpha\lambda-\xi^2 & 0 & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu \\
0 & 0 & \gamma\lambda & -\mu^2 & -\mu\xi & \delta\lambda-\xi^2 & 0 & 0 & \delta\mu & 0 & 0 & -\gamma\mu \\
0 & 0 & \gamma\nu & 0 & 0 & \delta\nu & -\mu^2 & -\mu\xi & \delta\xi-\xi^2 & 0 & 0 & -\gamma\xi \\
0 & 0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & +\alpha\xi-\xi^2
\end{pmatrix}$$

The case $\alpha=\delta=1$, $\beta=\gamma=0$ gives

$$\begin{pmatrix}
\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu & 0 & 0 & \xi-\lambda^2 & -\lambda\nu & -\nu^2 & 0 & 0 & 0 \\
-\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
0 & 0 & 0 & -2\lambda\mu & \lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu & 0 & -2\lambda\mu & \xi-\lambda\xi-\mu\nu & -2\nu\xi & 0 & 0 & 0 \\
0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda\mu & \xi-\lambda\xi-\mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \lambda-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & 0 & 0 & -\mu^2 & -\mu\xi & \lambda-\xi^2 & 0 & 0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \nu & -\mu^2 & -\mu\xi & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 & -\mu^2 & -\mu\xi & \xi-\xi^2
\end{pmatrix}$$

When $\lambda = 0$ and $\nu = 1$ we have

$$\left(\begin{array}{cccccccccccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \xi & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & -1 \\
0 & -\mu & -2\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
0 & 0 & 0 & 0 & -\mu & -2\xi & 0 & \mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \xi-\mu & -2\xi & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\mu & -2\xi \\
-\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & 0 & 0 & -\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\mu^2 & -\mu\xi & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu^2 & -\mu\xi & \xi-\xi^2
\end{array}\right)$$

Determinant: $\mu^4\left(\mu^2-3\mu\xi-\mu-\xi^3\right)^2(\mu+\xi-1)^4$.

We borrow an earlier piece of work with the names changed! We need to compute how many pairs γ, η there are with $\gamma^2-3\eta\gamma-\gamma-\eta^3=0$, $\gamma \neq 0$, $\eta^2+4\gamma$ not a square. We can assume that $\eta \neq 0$, as $\eta = 0$ implies $\gamma = 0$ or 1, and $\eta^2+4\gamma$ is then a square. For there to be a solution we must have $(3\eta+1)^2+4\eta^3$ equal to a square. Now

$$(3\eta+1)^2+4\eta^3=(4\eta+1)(1+\eta)^2.$$

So there is a solution to the equation if $\eta=-1$ or if $4\eta+1=k^2$ for some k. Now in the latter case $\eta=(k^2-1)/4$, and so

$$\gamma^2-3\eta\gamma-\gamma-\eta^3=\gamma^2-\frac{3}{4}\gamma k^2-\frac{1}{4}\gamma-\frac{1}{64}k^6+\frac{3}{64}k^4-\frac{3}{64}k^2+\frac{1}{64}.$$

This polynomial has roots:

$$\left(\begin{array}{c}\frac{3}{8}k^2+\frac{1}{8}+\frac{3}{8}k+\frac{1}{8}k^3 \\ \frac{3}{8}k^2+\frac{1}{8}-\frac{3}{8}k-\frac{1}{8}k^3\end{array}\right).$$

But if $\gamma=\frac{3}{8}k^2+\frac{1}{8}+\frac{3}{8}k+\frac{1}{8}k^3$ then

$$\begin{aligned}\eta^2+4\gamma &= \frac{1}{16}k^4+\frac{11}{8}k^2+\frac{9}{16}+\frac{3}{2}k+\frac{1}{2}k^3 \\ &= \frac{1}{16}(k+3)^2(k+1)^2.\end{aligned}$$

So in this case $\eta^2+4\gamma$ is a square.

If $\eta=-1$ then $\gamma^2-3\eta\gamma-\gamma-\eta^3=\gamma^2+3\gamma-\gamma+1=(\gamma+1)^2$. So we have the possibility

$\eta=\gamma=-1$ dealt with above. In this case $\eta^2+4\gamma=-3$, and for this not to be a square we need $p=2 \bmod 3$.

So the only solution to the determinant being zero when $\xi^2+4\mu$ is not a square is $\mu=-1$, $\xi=-1$ when $p=2 \bmod 3$. Then the matrix becomes

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2 \end{pmatrix}$$

$$\text{Smith normal form: } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{rank: } 8$$

$\lambda = \xi, \mu = 1, \nu = 0$

$$\begin{pmatrix} \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 \\ -2\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 \\ -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 \end{pmatrix}$$

$$
\begin{pmatrix}
\lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 \\
-2\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 \\
-1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2
\end{pmatrix}
$$

$\mu = \nu = 0$

$$
\begin{pmatrix}
\lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda^2 & 0 & 0 \\
0 & \lambda-\lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda-\lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda\xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda\xi & 0 \\
0 & 0 & \lambda-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\xi^2
\end{pmatrix}
$$

$\alpha = 1$
$\beta = 0$
$\gamma = 0$
$\delta = 1$
$\lambda = 0$
$\mu = 1$
$\nu = 0$
$\xi = \lambda$

$$
\begin{pmatrix}
\alpha\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 & 0 \\
\gamma\lambda & 0 & 0 & \delta\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \delta\mu & 0 & 0 & -\gamma\mu & 0 & 0 \\
\gamma\nu & 0 & 0 & \delta\nu & 0 & 0 & \delta\xi-\lambda^2 & -\lambda\nu & -\nu^2 & -\gamma\xi & 0 & 0 \\
-\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & 0 & \alpha\xi-\lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \alpha\lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 \\
0 & \gamma\lambda & 0 & -2\lambda\mu & \delta\lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & +\delta\mu & 0 & 0 & -\gamma\mu & 0 \\
0 & \gamma\nu & 0 & 0 & \delta\nu & 0 & -2\lambda\mu & \delta\xi-\lambda\xi-\mu\nu & -2\nu\xi & 0 & -\gamma\xi & 0 \\
0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & -2\lambda\mu & +\alpha\xi-\lambda\xi-\mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \alpha\lambda-\xi^2 & 0 & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu \\
0 & 0 & \gamma\lambda & -\mu^2 & -\mu\xi & \delta\lambda-\xi^2 & 0 & 0 & \delta\mu & 0 & 0 & -\gamma\mu \\
0 & 0 & \gamma\nu & 0 & 0 & \delta\nu & -\mu^2 & -\mu\xi & \delta\xi-\xi^2 & 0 & 0 & -\gamma\xi \\
0 & 0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & +\alpha\xi-\xi^2
\end{pmatrix}
$$

when $\alpha = 1$, $\beta = \gamma = 0$, $\delta \neq 1$ we have

$$
\begin{pmatrix}
\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & \delta\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \delta\mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta\nu & 0 & 0 & \delta\xi-\lambda^2 & -\lambda\nu & -\nu^2 & 0 & 0 & 0 \\
-\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
0 & 0 & 0 & -2\lambda\mu & \delta\lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \delta\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta\nu & 0 & -2\lambda\mu & \delta\xi-\lambda\xi-\mu\nu & -2\nu\xi & 0 & 0 & 0 \\
0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda\mu & \xi-\lambda\xi-\mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \lambda-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & 0 & 0 & -\mu^2 & -\mu\xi & \delta\lambda-\xi^2 & 0 & 0 & \delta\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta\nu & -\mu^2 & -\mu\xi & \delta\xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & -\nu & 0 & 0 & 0 & 0 & 0 & 0 & -\mu^2 & -\mu\xi & \xi-\xi^2
\end{pmatrix}
$$

and if we also set $\mu = \nu = 0$ we have

$$\begin{pmatrix}
\lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta\lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi - \lambda^2 & 0 & 0 \\
0 & \lambda - \lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta\lambda - \lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \lambda\xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi - \lambda\xi & 0 \\
0 & 0 & \lambda - \xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta\lambda - \xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi - \xi^2
\end{pmatrix}$$

If we set $\alpha = 1$, $\beta = \gamma = 0$, $\lambda = \xi$, $\mu = 1$, $\nu = 0$ we get

$$\begin{pmatrix}
\lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \delta\lambda - \lambda^2 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta\lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda - \lambda^2 & 0 & 0 \\
-2\lambda & \lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2\lambda & \delta\lambda - \lambda^2 & 0 & 0 & \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \delta\lambda - \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda - \lambda^2 & 0 \\
-1 & -\lambda & \lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -\lambda & \delta\lambda - \lambda^2 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \delta\lambda - \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda - \lambda^2
\end{pmatrix}$$

The determinant is $\lambda^{12} (\lambda - 1)^6 (\delta - \lambda)^6$ so we only get any nullity if $\lambda = 1$ or $\lambda = \delta$. Try $\lambda = 1$. We get

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \delta - 1 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta - 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2 & \delta - 1 & 0 & 0 & \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & \delta - 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & \delta - 1 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & \delta - 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0
\end{pmatrix}$$

Note $\delta \neq 1$. This has rank 10.
Now try $\lambda = \delta$.

$$\begin{pmatrix}
\delta - \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta - \delta^2 & 0 & 0 \\
-2\delta & \delta - \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2\delta & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\delta & \delta - \delta^2 & 0 \\
-1 & -\delta & \delta - \delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -\delta & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -\delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\delta & \delta - \delta^2
\end{pmatrix}$$

Again, $\delta \neq 1$. This also has rank 10.
Now try $\lambda = 0$, $\nu = 1$, where $\xi^2 + 4\mu$ is not a square.

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \delta\mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & \delta\xi & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & -1 \\
0 & -\mu & -2\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
0 & 0 & 0 & 0 & -\mu & -2\xi & 0 & \delta\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & \delta\xi-\mu & -2\xi & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\mu & -2\xi \\
-\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & 0 & 0 & -\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & \delta\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & -\mu^2 & -\mu\xi & \delta\xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu^2 & -\mu\xi & \xi-\xi^2
\end{pmatrix}$$

The determinant equals$-\mu^4\left(\mu^2-3\mu\xi-\mu-\xi^3\right)\left(-1+\mu+\xi\right)^2\left(\delta^2\mu-\mu^2+3\mu\xi\delta+\delta\xi^3\right)\left(\delta^2-\mu-\delta\xi\right)^2$

The only solutions for a zero determinant are when either of the cubics are zero (given that $\xi^2+4\mu$ is not a square).

The ørst has a solution $\xi=-1$, $\mu=-1$ when $=2 \bmod 3$. Then we have

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\delta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & -\delta & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & -\delta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & -\delta+1 & 2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & -1 & -1 & -\delta-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -2
\end{pmatrix}$$

Smith normal form: $$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\delta^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+\delta^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 0 & -1 & -\delta & 0 & 0 \\
\delta & 0 & 0 & -\delta & 0 & -1 \\
0 & 1 & 2 & 0 & -\delta & 0 \\
0 & \delta & 0 & 0 & -\delta+1 & 2 \\
-1 & -1 & -1 & 0 & 0 & -\delta \\
0 & 0 & \delta & -1 & -1 & -\delta-1
\end{pmatrix}$$

Determinant: $-1+2\delta^3-\delta^6=-\left(\delta-1\right)^2\left(\delta^2+\delta+1\right)^2$.

We also need to ønd the roots of the other cubic $\delta^2\mu-\mu^2+3\mu\xi\delta+\delta\xi^3$.

We borrow an earlier piece of work with the names changed! We need to compute how many pairs γ,η there are with $\gamma^2-3\eta\gamma\delta-\delta^2\gamma-\delta\eta^3=0$, $\gamma\neq 0$, $\eta^2+4\gamma$ not a square. We can assume that $\eta\neq 0$, as $\eta=0$ implies $\gamma=0$ or δ^2, and $\eta^2+4\gamma$ is then a square. For there to be a solution we must have $(3\eta\delta+\delta^2)^2+4\eta^3\delta$ equal to a square. Now

$$(3\eta\delta+\delta^2)^2+4\eta^3\delta=\delta(\delta+4\eta)(\delta+\eta)^2$$

So there is a solution to the equation if $\eta=-\delta$ or if $\delta(4\eta+\delta)=k^2$ for some k. Now in the latter case $\eta=(k^2\delta^{-1}-\delta)/4$, and so

$$\begin{aligned} & \gamma^2-3\eta\gamma\delta-\delta^2\gamma-\delta\eta^3 \\ = \quad & -\frac{1}{64\delta^2}\left(-64\gamma^2\delta^2+48\delta^2\gamma k^2+16\delta^4\gamma+k^6-3k^4\delta^2+3k^2\delta^4-\delta^6\right)\end{aligned}$$

This polynomial has roots

$$\left(\begin{array}{c} -\frac{1}{128\delta^2}\left(-48k^2\delta^2-16\delta^4+16k^3\delta+48k\delta^3\right) \\ -\frac{1}{128\delta^2}\left(-48k^2\delta^2-16\delta^4-16k^3\delta-48k\delta^3\right)\end{array}\right)$$

But if $\gamma=-\frac{1}{128\delta^2}\left(-48k^2\delta^2-16\delta^4+16k^3\delta+48k\delta^3\right)$ then

$$\begin{aligned}\eta^2+4\gamma &= \frac{1}{16}\frac{k^4+22k^2\delta^2+9\delta^4-8k^3\delta-24k\delta^3}{\delta^2} \\ &= \frac{1}{16}\left(k-\delta\right)^2\frac{\left(k-3\delta\right)^2}{\delta^2}\end{aligned}$$

So in this case $\eta^2+4\gamma$ is a square.

If $\eta=-\delta$ then $\gamma^2+2\gamma\delta^2+\delta^4=(\gamma+\delta^2)^2$. So we have the possibility $\eta=-\delta$, $\gamma=-\delta^2$. In this case $\eta^2+4\gamma=-3\delta^2$, and for this not to be a square we need $p=2\bmod 3$. So, reverting to our usual variable names, let $\xi=-\delta$, $\mu=-\delta^2$. We then have the following matrix

$$\left(\begin{array}{cccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&\delta^{-2}&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right)\times$$

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & \delta^2 & 2\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^2 & 0 \\
0 & 0 & 0 & 0 & \delta^2 & 2\delta & 0 & -\delta^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 2\delta & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\
-\delta^2 & -\delta & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\delta^4 & -\delta^3 & -\delta^2 & 0 & 0 & -\delta^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & -\delta^4 & -\delta^3 & -2\delta^2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^3 & -\delta^3 & -\delta^2-\delta
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \times
$$

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & \delta^2 & 2\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^2 & 0 \\
0 & 0 & 0 & 0 & \delta^2 & 2\delta & 0 & -\delta^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 2\delta & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\
-\delta^2 & -\delta & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -\delta^2 & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta & -\delta^4 & -\delta^3 & -2\delta^2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^3 & -\delta^3 & -\delta^2-\delta
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \times
$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & \delta & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & \delta^2 & 2\delta & 0 & -\delta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\ -\delta^2 & -\delta & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\delta^2 & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^3 & -\delta^3 & -\delta^2-\delta \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\ \delta & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta^2 & 2\delta & 0 & -\delta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\ -\delta & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^2 & 0 & \delta^2+1 \\ 0 & 0 & -\delta^2 & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^3 & -\delta^3 & -\delta^2-\delta \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta^{-1} \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & 0 & 0 & 0 & 2\delta & \delta & 0 \\ 0 & 0 & \delta & 2 & 0 & -\delta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\ -\delta & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 0 & \delta^2+1 \\ 0 & -\delta^2 & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta-\delta^3 & -\delta^3 & -\delta^2-\delta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{\delta} & 0 & -\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 2\delta \\ -\delta & 0 & 0 & 0 & 0 & 0 & 0 & -\delta+\delta^2 & 0 & \delta^2+1 \\ 0 & -\delta^2 & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\delta^2 & 0 & -1 & 0 & 0 & 0 \\ 0 & \delta & 2 & 0 & -\delta^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ -\delta & -\delta & -1 & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & -\delta^3 & 0 & 0 & 0 & 0 & 0 \\ \delta & 2 & 0 & -\delta^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & 0 & 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ -\delta & -1 & -\delta^3 & 0 & -2\delta & 0 & 0 & 0 \\ 0 & 1 & -\delta^3 & -\delta^2 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^{-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \delta & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & 0 & 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ -\delta & -1 & -1 & 0 & -2\delta & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\delta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ -1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -2\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 2 & 1-\delta+\delta^2 & 2\delta \\ 0 & 0 & \delta+\delta^2 & \delta & \delta^2+1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

Take out nullity 2.

$$\begin{pmatrix} 2 & 1-\delta+\delta^2 & 2\delta \\ \delta+\delta^2 & \delta & \delta^2+1 \\ -1-\delta^2 & -\delta^2 & -\delta-1 \end{pmatrix}$$

Determinant: $-1+2\delta^3-\delta^6 = -(\delta-1)^2\left(\delta^2+\delta+1\right)^2$

So we get rank 10 unless δ is a cube root of unity (not 1), in which case we get rank 8.

$$\begin{pmatrix} 1 & -\delta & \delta \\ 0 & 0 & 0 \\ \delta & -\delta^2 & \delta^2 \end{pmatrix}$$

Now consider the case when $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\delta = 1$. Then we have

$$
\begin{pmatrix}
\lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \lambda & 0 & 0 & \mu & 0 & 0 & -\mu & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & -\lambda\nu & -\nu^2 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu & 0 & 0 & \xi-\lambda^2 & -\lambda\nu & -\nu^2 & 0 & 0 & 0 \\
-\nu & 0 & 0 & -\nu & 0 & 0 & -\xi & 0 & 0 & \xi-\lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \lambda & 0 & 0 & \mu & 0 & 0 & -\mu & 0 \\
0 & 0 & 0 & -2\lambda\mu & \lambda-\lambda\xi-\mu\nu & -2\nu\xi & 0 & \mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu & 0 & -2\lambda\mu & \xi-\lambda\xi-\mu\nu & -2\nu\xi & 0 & 0 & 0 \\
0 & -\nu & 0 & 0 & -\nu & 0 & 0 & -\xi & 0 & -2\lambda\mu & \xi-\lambda\xi-\mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \lambda-\xi^2 & 0 & 0 & \lambda & 0 & 0 & \mu & 0 & 0 & -\mu \\
0 & 0 & 0 & -\mu^2 & -\mu\xi & \lambda-\xi^2 & 0 & 0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \nu & -\mu^2 & -\mu\xi & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & -\nu & 0 & 0 & -\nu & 0 & 0 & -\xi & -\mu^2 & -\mu\xi & \xi-\xi^2
\end{pmatrix}
$$

When $\mu = 0$, $\nu = 0$ we have

$$
\begin{pmatrix}
\lambda-\lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & \xi-\lambda^2 & 0 & 0 \\
0 & \lambda-\lambda\xi & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda-\lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\lambda\xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & \xi-\lambda\xi & 0 \\
0 & 0 & \lambda-\xi^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & \xi-\xi^2
\end{pmatrix}
$$

Determinant:$\lambda^4\xi^4\left(\xi-1\right)^4\left(\lambda-1\right)^4\left(\lambda-\xi^2\right)^2\left(\lambda^2-\xi\right)^2$

The matrix has rank 12 unless this determinant is zero. $\lambda = 1$ gives

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \xi-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & \xi-1 & 0 & 0 \\
0 & 1-\xi & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\xi^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1-\xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi-\xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi & 0 & 0 & \xi-\xi^2
\end{pmatrix}
$$

If $\lambda = 1$, $\xi = 1$ we get $\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}$ which has rank 6.

Also $\lambda = 1$, $\xi = -1$ gives $\begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-2&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&-2&0&0 \\ 0&2&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-2&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&-2 \end{pmatrix}$

Smith normal form: $\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&4&0&0&0&0&0 \\ 0&0&0&0&0&0&0&4&0&0&0&0 \\ 0&0&0&0&0&0&0&0&4&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$ which has rank 9.

Similarly $\xi = 1$, $\lambda \neq \pm 1$ gives rank 10.
Let $\lambda, \xi \neq 1$, $\lambda = \xi^2$

$$\begin{pmatrix} \xi^2-\xi^4&0&0&\xi^2&0&0&0&0&0&0&0&0 \\ 0&0&0&\xi^2-\xi^4&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\xi-\xi^4&0&0&0&0&0 \\ 0&0&0&0&0&0&-\xi&0&0&\xi-\xi^4&0&0 \\ 0&\xi^2-\xi^3&0&0&\xi^2&0&0&0&0&0&0&0 \\ 0&0&0&0&\xi^2-\xi^3&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&\xi-\xi^3&0&0&0&0 \\ 0&0&0&0&0&0&0&-\xi&0&0&\xi-\xi^3&0 \\ 0&0&0&0&0&\xi^2&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&\xi-\xi^2&0&0&0 \\ 0&0&0&0&0&0&0&0&-\xi&0&0&\xi-\xi^2 \end{pmatrix}$$

So the rank is 11 if $\xi^3 \neq 1$ and 10 if $\xi^3 = 1$. (In this case $\xi = \lambda^2$ as well.)
Finally let $\xi = \lambda^2$, with $\lambda^3 \neq 1$. This gives rank 11 again.

$$\begin{pmatrix}
\lambda-\lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda-\lambda^3 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda-\lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2-\lambda^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda^2 & 0 & 0 & \lambda^2-\lambda^3 & 0 \\
0 & 0 & \lambda-\lambda^4 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda-\lambda^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2-\lambda^4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda^2 & 0 & 0 & \lambda^2-\lambda^4
\end{pmatrix}$$

Now consider the situation when $\xi=\lambda$, $\mu=1$, $\nu=0$. Then we have

$$\begin{pmatrix}
\lambda-\lambda^2 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda-\lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 0 & \lambda-\lambda^2 & 0 & 0 \\
-2\lambda & \lambda-\lambda^2 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda-\lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & -2\lambda & \lambda-\lambda^2 & 0 \\
-1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda-\lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & -1 & -\lambda & \lambda-\lambda^2
\end{pmatrix}$$

determinant: $(\lambda-1)^{12}$

So we get rank 12 unless $\lambda=1$. Then we have

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0
\end{pmatrix}$$

Smith normal form: $\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$

Finally let $\lambda = 0$, $\nu = 1$, $\xi^2 + 4\mu$ not a square. We then have

$$\begin{pmatrix} 0&0&-1&0&0&0&\mu&0&0&-\mu&0&0\\ 0&0&0&0&0&-1&\mu&0&0&0&0&0\\ 0&0&0&1&0&0&\xi&0&-1&0&0&0\\ -1&0&0&-1&0&0&-\xi&0&0&\xi&0&-1\\ 0&-\mu&-2\xi&0&0&0&0&\mu&0&0&-\mu&0\\ 0&0&0&0&-\mu&-2\xi&0&\mu&0&0&0&0\\ 0&0&0&0&1&0&0&\xi-\mu&-2\xi&0&0&0\\ 0&-1&0&0&-1&0&0&-\xi&0&0&\xi-\mu&-2\xi\\ -\mu^2&-\mu\xi&-\xi^2&0&0&0&0&0&\mu&0&0&-\mu\\ 0&0&0&-\mu^2&-\mu\xi&-\xi^2&0&0&\mu&0&0&0\\ 0&0&0&0&0&1&-\mu^2&-\mu\xi&\xi-\xi^2&0&0&0\\ 0&0&-1&0&0&-1&0&0&-\xi&-\mu^2&-\mu\xi&\xi-\xi^2 \end{pmatrix}$$

Determinant: $\mu^4\left(\mu^2 - 3\mu\xi - \mu - \xi^3\right)^2 (\mu + \xi - 1)^4$. So as before we get rank 12 unless $\mu = -1$, $\xi = -1$ and $p = 2 \bmod 3$. We then have

$$\begin{pmatrix} 0&0&-1&0&0&0&-1&0&0&1&0&0\\ 0&0&0&0&0&-1&-1&0&0&0&0&0\\ 0&0&0&1&0&0&-1&0&-1&0&0&0\\ -1&0&0&-1&0&0&1&0&0&-1&0&-1\\ 0&1&2&0&0&0&0&-1&0&0&1&0\\ 0&0&0&0&1&2&0&-1&0&0&0&0\\ 0&0&0&0&1&0&0&0&2&0&0&0\\ 0&-1&0&0&-1&0&0&1&0&0&0&2\\ -1&-1&-1&0&0&0&0&0&-1&0&0&1\\ 0&0&0&-1&-1&-1&0&0&-1&0&0&0\\ 0&0&0&0&0&1&-1&-1&-2&0&0&0\\ 0&0&-1&0&0&-1&0&0&1&-1&-1&-2 \end{pmatrix}$$

Smith normal form: $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Finally consider the case when $\alpha = 0$, $\gamma = 1$, and $\delta^2 + 4\beta$ is not a square. We then have

$$\begin{pmatrix} -\lambda^2 & -\lambda\nu & -\nu^2 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & \delta\lambda - \lambda^2 & -\lambda\nu & -\nu^2 & \delta\mu & 0 & 0 & -\mu & 0 & 0 \\ \nu & 0 & 0 & \delta\nu & 0 & 0 & \delta\xi - \lambda^2 & -\lambda\nu & -\nu^2 & -\xi & 0 & 0 \\ 0 & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & 0 & -\lambda^2 & -\lambda\nu & -\nu^2 \\ -2\lambda\mu & -\lambda\xi - \mu\nu & -2\nu\xi & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -2\lambda\mu & \delta\lambda - \lambda\xi - \mu\nu & -2\nu\xi & 0 & \delta\mu & 0 & 0 & -\mu & 0 \\ 0 & \nu & 0 & 0 & \delta\nu & 0 & -2\lambda\mu & \delta\xi - \lambda\xi - \mu\nu & -2\nu\xi & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & -2\lambda\mu & -\lambda\xi - \mu\nu & -2\nu\xi \\ -\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\mu^2 & -\mu\xi & \delta\lambda - \xi^2 & 0 & 0 & \delta\mu & 0 & 0 & -\mu \\ 0 & 0 & \nu & 0 & 0 & \delta\nu & -\mu^2 & -\mu\xi & \delta\xi - \xi^2 & 0 & 0 & -\xi \\ 0 & 0 & 0 & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & -\xi^2 \end{pmatrix}$$

Consider the case when $\mu = 0$, $\nu = 0$. This gives

$$\begin{pmatrix} -\lambda^2 & 0 & 0 & \beta\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & \delta\lambda - \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \lambda^2 & 0 & 0 & -\xi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta\xi & 0 & 0 & -\lambda^2 & 0 & 0 \\ 0 & -\lambda\xi & 0 & 0 & \beta\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & \delta\lambda - \lambda\xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \lambda\xi & 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta\xi & 0 & 0 & -\lambda\xi & 0 \\ 0 & 0 & -\xi^2 & 0 & 0 & \beta\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & \delta\lambda - \xi^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta\xi - \xi^2 & 0 & 0 & -\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta\xi & 0 & 0 & -\xi^2 \end{pmatrix}$$

Determinant: $\xi^4\lambda^4\left(\xi\delta - \xi^2 + \beta\right)^2\left(\beta + \lambda\delta - \lambda^2\right)^2\left(\lambda^2\xi\delta - \lambda^4 + \xi^2\beta\right)\left(\lambda\xi^2\delta + \lambda^2\beta - \xi^4\right)$

None of these polynomials can have roots!

Next consider $\xi = \lambda$, $\mu = 1$, $\nu = 0$. This gives

$$\begin{pmatrix} -\lambda^2 & 0 & 0 & \lambda\beta & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & \lambda\delta - \lambda^2 & 0 & 0 & \delta & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda\delta - \lambda^2 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda\beta & 0 & 0 & -\lambda^2 & 0 & 0 \\ -2\lambda & -\lambda^2 & 0 & 0 & \lambda\beta & 0 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -2\lambda & \lambda\delta - \lambda^2 & 0 & 0 & \delta & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & \lambda\delta - \lambda^2 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda\beta & 0 & -2\lambda & -\lambda^2 & 0 \\ -1 & -\lambda & -\lambda^2 & 0 & 0 & \lambda\beta & 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \lambda & -1 & -\lambda & \lambda\delta - \lambda^2 & 0 & 0 & \delta & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\lambda & \lambda\delta - \lambda^2 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda\beta & -1 & -\lambda & -\lambda^2 \end{pmatrix}$$

Determinant: $\lambda^{12}\left(\beta+\lambda\delta-\lambda^2\right)^6$, which has no roots. So all these matrices have rank 12.

Finally consider the case when $\lambda=0$, $\nu=1$ and $\xi^2+4\mu$ is not a square. We then have

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & \delta\mu & 0 & 0 & -\mu & 0 & 0\\
1 & 0 & 0 & \delta & 0 & 0 & \xi\delta & 0 & -1 & -\xi & 0 & 0\\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1\\
0 & -\mu & -2\xi & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\mu & -2\xi & 0 & \delta\mu & 0 & 0 & -\mu & 0\\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \xi\delta-\mu & -2\xi & 0 & -\xi & 0\\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & -\mu & -2\xi\\
-\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0\\
0 & 0 & 0 & -\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & \delta\mu & 0 & 0 & -\mu\\
0 & 0 & 1 & 0 & 0 & \delta & -\mu^2 & -\mu\xi & \xi\delta-\xi^2 & 0 & 0 & -\xi\\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & -\xi^2
\end{pmatrix}$$

Determinant:$-\mu^4\left(-\beta^2\mu^2+2\mu^3\beta+3\delta\mu^2\beta\xi+9\mu^2\xi^2\beta+\delta\mu\xi^3\beta+6\mu\xi^4\beta+\xi^6\beta+\delta^2\mu^3-\mu^4+3\xi\delta\mu^3+\delta\mu^2\xi^3\right)$
$\times\left(\delta^2\mu-\mu^2-\delta\mu\xi+2\beta\mu-\delta\beta\xi+\xi^2\beta-\beta^2\right)^2$

What a bugger!

$-\delta^2\mu+\mu^2+\delta\mu\xi-2\beta\mu+\delta\beta\xi-\xi^2\beta+\beta^2$

$(-\delta^2+\delta\xi-2\beta)^2-4(\delta\beta\xi-\xi^2\beta+\beta^2)=$

$\delta^4-2\xi\delta^3+4\beta\delta^2+\xi^2\delta^2-8\delta\beta\xi+4\xi^2\beta=(\delta-\xi)^2\left(\delta^2+4\beta\right)$

So the second polynomial has no roots unless $\delta=\xi$.

$-\beta^2\mu^2+2\mu^3\beta+3\delta\mu^2\beta\xi+9\mu^2\xi^2\beta+\delta\mu\xi^3\beta+6\mu\xi^4\beta+\xi^6\beta+\delta^2\mu^3-\mu^4+3\xi\delta\mu^3+\delta\mu^2\xi^3$

This is quadratic in β.

$(2\mu^3+3\delta\mu^2\xi+9\mu^2\xi^2+\delta\mu\xi^3+6\mu\xi^4+\xi^6)^2+4\mu^2(\delta^2\mu^3-\mu^4+3\xi\delta\mu^3+\delta\mu^2\xi^3)=$
$24\mu^5\delta\xi+\xi^{12}+36\mu^5\xi^2+105\mu^4\xi^4+112\mu^3\xi^6+\delta^2\mu^2\xi^6+2\delta\mu\xi^9+54\mu^2\xi^8+12\mu\xi^{10}+18\delta\mu^2\xi^7$
$+6\delta^2\mu^3\xi^4+54\delta\mu^3\xi^5+62\mu^4\delta\xi^3+9\delta^2\mu^4\xi^2+4\mu^5\delta^2=$
$\left(4\mu+\xi^2\right)\left(\mu+\xi^2\right)^2\left(3\mu\xi+\delta\mu+\xi^3\right)^2$

So the ørst polynomial has no roots either unless $\mu=-\xi^2$ and -3 is not a square ($p=2 \bmod 3$), or $3\mu\xi+\delta\mu+\xi^3=0$.

Consider the case when $\delta=\xi$. Then we have

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & \mu\xi & 0 & 0 & -\mu & 0 & 0\\
1 & 0 & 0 & \xi & 0 & 0 & \xi^2 & 0 & -1 & -\xi & 0 & 0\\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1\\
0 & -\mu & -2\xi & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\mu & -2\xi & 0 & \mu\xi & 0 & 0 & -\mu & 0\\
0 & 1 & 0 & 0 & \xi & 0 & 0 & \xi^2-\mu & -2\xi & 0 & -\xi & 0\\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & -\mu & -2\xi\\
-\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0\\
0 & 0 & 0 & -\mu^2 & -\mu\xi & -\xi^2 & 0 & 0 & \mu\xi & 0 & 0 & -\mu\\
0 & 0 & 1 & 0 & 0 & \xi & -\mu^2 & -\mu\xi & 0 & 0 & 0 & -\xi\\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & -\xi^2
\end{pmatrix}$$

Determinant: $\mu^4\left(\beta-\mu\right)^4\left(\beta^2\mu^2-\beta\xi^6-12\xi^2\mu^2\beta-7\beta\xi^4\mu-2\beta\mu^3-4\mu^3\xi^2+\mu^4-\mu^2\xi^4\right)$.

Again, this has rank 12 unless $\beta=\mu$ or

$(-\xi^6-12\xi^2\mu^2-7\xi^4\mu-2\mu^3)^2-4\mu^2(-4\mu^3\xi^2+\mu^4-\mu^2\xi^4)$ is a square. Now this equals $\xi^2\left(\mu+\xi^2\right)^2\left(4\mu+\xi^2\right)^3$ and so can only be a square if $\xi=0$ or $\mu=-\xi^2$. Let us consider the situation when $\xi=0$. Then we have

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -\mu & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -\mu & 0 & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 & 0 & -\mu & 0 \\
-\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\mu & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & 0 & 1 & 0 & 0 & 0 & -\mu^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & 0 & -\mu^2 & 0 & 0
\end{pmatrix}$$

Determinant: $\mu^6 \left(\beta-\mu\right)^6$. So again we get rank 12 unless $\mu = \beta$. Next consider the situation when $\mu = -\xi^2$. This gives

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & -\xi^2\beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\xi^3 & 0 & 0 & \xi^2 & 0 & 0 \\
1 & 0 & 0 & \xi & 0 & 0 & \xi^2 & 0 & -1 & -\xi & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1 \\
0 & \xi^2 & -2\xi & 0 & 0 & 0 & 0 & -\xi^2\beta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi^2 & -2\xi & 0 & -\xi^3 & 0 & 0 & \xi^2 & 0 \\
0 & 1 & 0 & 0 & \xi & 0 & 0 & 2\xi^2 & -2\xi & 0 & -\xi & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & \xi^2 & -2\xi \\
-\xi^4 & \xi^3 & -\xi^2 & 0 & 0 & 0 & 0 & 0 & -\xi^2\beta & 0 & 0 & 0 \\
0 & 0 & 0 & -\xi^4 & \xi^3 & -\xi^2 & 0 & 0 & -\xi^3 & 0 & 0 & \xi^2 \\
0 & 0 & 1 & 0 & 0 & \xi & -\xi^4 & \xi^3 & 0 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\xi^4 & \xi^3 & -\xi^2
\end{pmatrix}$$

Determinant: $\xi^{12}\left(\beta-2\xi^2\right)^2\left(\beta+\xi^2\right)^4$. Again, $\mu = \beta$ arises as a possibility. The other possibility is $\beta = 2\xi^2$ but this gives $\delta^2 + 4\beta = 9\xi^2$, which is a square. So the case $\delta = \xi$ only gives rise to something of rank less than 12 when $\mu = \beta$.

Now consider the situation when $\mu = -\xi^2$ and $p = 2 \bmod 3$.

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & -\beta\xi^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\delta\xi^2 & 0 & 0 & \xi^2 & 0 & 0 \\
1 & 0 & 0 & \delta & 0 & 0 & \delta\xi & 0 & -1 & -\xi & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1 \\
0 & \xi^2 & -2\xi & 0 & 0 & 0 & 0 & -\beta\xi^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi^2 & -2\xi & 0 & -\delta\xi^2 & 0 & 0 & \xi^2 & 0 \\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \delta\xi+\xi^2 & -2\xi & 0 & -\xi & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & \xi^2 & -2\xi \\
-\xi^4 & \xi^3 & -\xi^2 & 0 & 0 & 0 & 0 & 0 & -\beta\xi^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\xi^4 & \xi^3 & -\xi^2 & 0 & 0 & -\delta\xi^2 & 0 & 0 & \xi^2 \\
0 & 0 & 1 & 0 & 0 & \delta & -\xi^4 & \xi^3 & \delta\xi-\xi^2 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\xi^4 & \xi^3 & -\xi^2
\end{pmatrix}$$

Determinant: $\xi^{12}\left(\beta-\xi^2-\delta\xi\right)^2\left(\beta^2+\beta\delta\xi+\beta\xi^2-\xi^3\delta+\xi^4+\delta^2\xi^2\right)^2$. The determinant can only be zero if

$$(\delta\xi+\xi^2)^2 - 4(-\xi^3\delta+\xi^4+\delta^2\xi^2) = -3\xi^2\left(\delta-\xi\right)^2$$

is square, which since -3 is not a square requires $\xi = \delta$, which we have already dealt with.

$$\begin{pmatrix}
\alpha\lambda - \lambda^2 & -\lambda\nu & -\nu^2 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 & 0 \\
\gamma\lambda & 0 & 0 & \delta\lambda - \lambda^2 & -\lambda\nu & -\nu^2 & \delta\mu & 0 & 0 & -\gamma\mu & 0 & 0 \\
\gamma\nu & 0 & 0 & \delta\nu & 0 & 0 & \delta\xi - \lambda^2 & -\lambda\nu & -\nu^2 & -\gamma\xi & 0 & 0 \\
-\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & 0 & \alpha\xi - \lambda^2 & -\lambda\nu & -\nu^2 \\
-2\lambda\mu & \alpha\lambda - \lambda\xi - \mu\nu & -2\nu\xi & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu & 0 \\
0 & \gamma\lambda & 0 & -2\lambda\mu & \delta\lambda - \lambda\xi - \mu\nu & -2\nu\xi & 0 & +\delta\mu & 0 & 0 & -\gamma\mu & 0 \\
0 & \gamma\nu & 0 & 0 & \delta\nu & 0 & -2\lambda\mu & \delta\xi - \lambda\xi - \mu\nu & -2\nu\xi & 0 & -\gamma\xi & 0 \\
0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & 0 & -2\lambda\mu & +\alpha\xi - \lambda\xi - \mu\nu & -2\nu\xi \\
-\mu^2 & -\mu\xi & \alpha\lambda - \xi^2 & 0 & 0 & \beta\lambda & 0 & 0 & \beta\mu & 0 & 0 & -\alpha\mu \\
0 & 0 & \gamma\lambda & -\mu^2 & -\mu\xi & \delta\lambda - \xi^2 & 0 & 0 & \delta\mu & 0 & 0 & -\gamma\mu \\
0 & 0 & \gamma\nu & 0 & 0 & \delta\nu & -\mu^2 & -\mu\xi & \delta\xi - \xi^2 & 0 & 0 & -\gamma\xi \\
0 & 0 & -\alpha\nu & 0 & 0 & -\beta\nu & 0 & 0 & -\beta\xi & -\mu^2 & -\mu\xi & +\alpha\xi - \xi^2
\end{pmatrix}$$

$\lambda = 0$,
$\nu = 1$,
$\alpha = 0$,
$\gamma = 1$,
$\mu = \beta$,
gives

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \delta\beta & 0 & 0 & -\beta & 0 & 0 \\
1 & 0 & 0 & \delta & 0 & 0 & \delta\xi & 0 & -1 & -\xi & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1 \\
0 & -\beta & -2\xi & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & -2\xi & 0 & \delta\beta & 0 & 0 & -\beta & 0 \\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \delta\xi - \beta & -2\xi & 0 & -\xi & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & -\beta & -2\xi \\
-\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & \delta\beta & 0 & 0 & -\beta \\
0 & 0 & 1 & 0 & 0 & \delta & -\beta^2 & -\beta\xi & \delta\xi - \xi^2 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\beta^2 & -\beta\xi & -\xi^2
\end{pmatrix}$$

Determinant: $-\beta^7 (\xi - \delta)^4 \left(3\beta\xi + \xi^3 + \delta\beta\right)^2$
Try $\xi = \delta$. We get

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \delta\beta & 0 & 0 & -\beta & 0 & 0 \\
1 & 0 & 0 & \delta & 0 & 0 & \delta^2 & 0 & -1 & -\delta & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & 0 & 0 & 0 & 0 & -1 \\
0 & -\beta & -2\delta & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & -2\delta & 0 & \delta\beta & 0 & 0 & -\beta & 0 \\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \delta^2 - \beta & -2\delta & 0 & -\delta & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & 0 & 0 & -\beta & -2\delta \\
-\beta^2 & -\delta\beta & -\delta^2 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\delta\beta & -\delta^2 & 0 & 0 & \delta\beta & 0 & 0 & -\beta \\
0 & 0 & 1 & 0 & 0 & \delta & -\beta^2 & -\delta\beta & 0 & 0 & 0 & -\delta \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & -\beta^2 & -\delta\beta & -\delta^2
\end{pmatrix}$$

Consider the case when $\delta = 0$.

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}
$$

Smith normal form: $\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$. So the rank is 6

Now consider

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \delta\beta & 0 & 0 & -\beta & 0 & 0 \\
1 & 0 & 0 & \delta & 0 & 0 & \delta^2 & 0 & -1 & -\delta & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & 0 & 0 & 0 & 0 & -1 \\
0 & -\beta & -2\delta & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & -2\delta & 0 & \delta\beta & 0 & 0 & -\beta & 0 \\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \delta^2-\beta & -2\delta & 0 & -\delta & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & 0 & 0 & -\beta & -2\delta \\
-\beta^2 & -\delta\beta & -\delta^2 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\delta\beta & -\delta^2 & 0 & 0 & \delta\beta & 0 & 0 & -\beta \\
0 & 0 & 1 & 0 & 0 & \delta & -\beta^2 & -\delta\beta & 0 & 0 & 0 & -\delta \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\delta\beta & -\beta^2 & -\delta\beta & -\delta^2
\end{pmatrix}
$$

when $\delta \neq 0$. We can simplify the situation by substituting $x\beta$ for δ. Set $\delta = x\beta$. We get

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \times$$

$$\begin{pmatrix} 0&0&-1&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&x&0&0&-1&0&0\\ 1&0&0&x&0&0&x^2&0&-1&-x&0&0\\ 0&0&0&-1&0&0&-x&0&0&0&0&-1\\ 0&-1&-2x&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&-2x\beta&0&x\beta&0&0&-1&0\\ 0&1&0&0&x&0&0&x^2\beta-1&-2x\beta&0&-x&0\\ 0&0&0&0&-1&0&0&-x\beta&0&0&-1&-2x\beta\\ -1&-x&-x^2&0&0&0&0&0&1&0&0&0\\ 0&0&0&-1&-x&-x^2\beta&0&0&x\beta&0&0&-1\\ 0&0&1&0&0&x\beta&-1&-x\beta&0&0&0&-x\beta\\ 0&0&0&0&0&-1&0&0&-x\beta&-1&-x&-x^2\beta \end{pmatrix}$$

$$\begin{pmatrix} 0&-1&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&-1&x&0&0&-1&0&0\\ 0&0&-1&0&0&-x&0&0&0&0&-1\\ -1&-2x&0&0&0&0&1&0&0&0&0\\ 0&0&0&-1&-2x\beta&0&x\beta&0&0&-1&0\\ 1&0&0&x&0&0&x^2\beta-1&-2x&0&-x&0\\ 0&0&0&-1&0&0&-x\beta&0&0&-1&-2x\beta\\ -x&-x^2&x&0&0&x^2&0&0&-x&0&0\\ 0&0&-1&-x&-x^2\beta&0&0&x&0&0&-1\\ 0&1&0&0&x\beta&-1&-x\beta&0&0&0&-x\beta\\ 0&0&0&0&-1&0&0&-x&-1&-x&-x^2\beta \end{pmatrix}$$

$$\times \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&0&-1&1&0&0&-1&0&0\\ 0&-1&0&0&-1&0&0&0&0&-1\\ -1&0&0&0&-2&1&0&0&0&0\\ 0&0&-1&-2x\beta&0&x\beta&0&0&-1&0\\ 1&0&x&0&0&x^2\beta-1&-2&0&-x&0\\ 0&0&-1&0&0&-x\beta&0&0&-1&-2x\beta\\ -1&1&0&0&0&0&0&-1&0&0\\ 0&-1&-x&-x^2\beta&0&0&1&0&0&-1\\ 0&0&0&x\beta&0&-x\beta&0&0&0&-x\beta\\ 0&0&0&-1&0&0&-1&-1&-x&-x^2\beta \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0\\ 0&-1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&-1&1&0&0&-1&0&0\\ -1&0&0&-1&0&0&0&0&-1\\ 0&-1&-2x\beta&0&x\beta&0&0&-1&0\\ 0&x&0&-2&x^2\beta&-2&0&-x&0\\ 0&-1&0&0&-x\beta&0&0&-1&-2x\beta\\ 1&0&0&2&-1&0&-1&0&0\\ -1&-x&-x^2\beta&0&0&1&0&0&-1\\ 0&0&x\beta&0&-x\beta&0&0&0&-x\beta\\ 0&0&-1&0&0&-1&-1&-x&-x^2\beta \end{pmatrix}$$

$$\begin{pmatrix} 0&-1&1&0&0&-1&0&0\\ -1&-2x\beta&0&x\beta&0&0&-1&0\\ x&0&-2&x^2\beta&-2&0&-x&0\\ -1&0&0&-x\beta&0&0&-1&-2x\beta\\ 0&0&1&-1&0&-1&0&-1\\ -x&-x^2\beta&1&0&1&0&0&0\\ 0&x\beta&0&-x\beta&0&0&0&-x\beta\\ 0&-1&0&0&-1&-1&-x&-x^2\beta \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&1&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} -1&-2x\beta&x\beta&0&2x\beta&-1&0\\ x&-2&x^2\beta&-2&0&-x&0\\ -1&0&-x\beta&0&0&-1&-2x\beta\\ 0&1&-1&0&-1&0&-1\\ -x&-x^2\beta+1&0&1&x^2\beta&0&0\\ 0&0&0&0&0&0&0\\ 0&-1&0&-1&0&-x&-x^2\beta \end{pmatrix} \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&1&0&1&0&1\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0&0&0\\ x&1&0&0&0&0\\ 0&0&1&0&0&0\\ -x&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} -1&-x\beta&0&0&-1&-2x\beta\\ x&-2+x^2\beta&-2&-2&-x&-2\\ 0&0&0&0&0&0\\ -x&-x^2\beta+1&1&1&0&-x^2\beta+1\\ 0&0&0&0&0&0\\ 0&-1&-1&-1&-x&-1-x^2\beta \end{pmatrix}$$

$$\begin{pmatrix} 1&0&2&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&1&0&1 \end{pmatrix} \begin{pmatrix} -2&-2&-2&-2x&-2x^2\beta-2\\ 0&0&0&0&0\\ 1&1&1&x&x^2\beta+1\\ 0&0&0&0&0\\ -1&-1&-1&-x&-1-x^2\beta \end{pmatrix}$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & x & x^2\beta+1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

So the rank is 8.
Finally, consider the matrix when $3\beta\xi+\xi^3+\delta\beta=0$. Now we have $\delta=-3\xi-\beta^{-1}\xi^3$

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \delta\beta & 0 & 0 & -\beta & 0 & 0 \\
1 & 0 & 0 & \delta & 0 & 0 & \delta\xi & 0 & -1 & -\xi & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1 \\
0 & -\beta & -2\xi & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & -2\xi & 0 & \delta\beta & 0 & 0 & -\beta & 0 \\
0 & 1 & 0 & 0 & \delta & 0 & 0 & \delta\xi-\beta & -2\xi & 0 & -\xi & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & -\beta & -2\xi \\
-\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & \delta\beta & 0 & 0 & -\beta \\
0 & 0 & 1 & 0 & 0 & \delta & -\beta^2 & -\beta\xi & \delta\xi-\xi^2 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\beta^2 & -\beta\xi & -\xi^2
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \times
$$

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\xi\left(3\beta+\xi^2\right) & 0 & 0 & -\beta & 0 & 0 \\
1 & 0 & 0 & -\xi\frac{3\beta+\xi^2}{\beta} & 0 & 0 & -\xi^2\frac{3\beta+\xi^2}{\beta} & 0 & -1 & -\xi & 0 & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & 0 & 0 & -1 \\
0 & -\beta & -2\xi & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta & -2\xi & 0 & -\xi\left(3\beta+\xi^2\right) & 0 & 0 & -\beta & 0 \\
0 & 1 & 0 & 0 & -\xi\frac{3\beta+\xi^2}{\beta} & 0 & 0 & -\frac{3\xi^2\beta+\xi^4+\beta^2}{\beta} & -2\xi & 0 & -\xi & 0 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 & 0 & -\beta & -2\xi \\
-\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 & \beta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & -\xi\left(3\beta+\xi^2\right) & 0 & 0 & -\beta \\
0 & 0 & 1 & 0 & 0 & -\xi\frac{3\beta+\xi^2}{\beta} & -\beta^2 & -\beta\xi & -\xi^2\frac{4\beta+\xi^2}{\beta} & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & -\beta^2 & -\beta\xi & -\xi^2
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & \beta^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -\xi\left(3\beta+\xi^2\right) & 0 \\
\beta & 0 & 0 & -\xi\left(3\beta+\xi^2\right) & 0 & 0 & -\xi^2\left(3\beta+\xi^2\right) & 0 \\
0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi & 0 \\
0 & -\beta & -2\xi & 0 & 0 & 0 & 0 & \beta^2 \\
0 & 0 & 0 & 0 & -\beta & -2\xi & 0 & -\xi\left(3\beta+\xi^2\right) \\
0 & \beta & 0 & 0 & -\xi\left(3\beta+\xi^2\right) & 0 & 0 & -3\xi^2\beta-\xi^4-\beta^2 \\
0 & 0 & 0 & 0 & -\beta & 0 & 0 & -\beta\xi \\
-\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta^2 & -\beta\xi & -\xi^2 & 0 & 0 \\
0 & 0 & \beta & 0 & 0 & -\xi\left(3\beta+\xi^2\right) & -\beta^3 & -\xi\beta^2 \\
0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0
\end{pmatrix}
$$

$$
\text{more matrix run on}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 \\
-\beta & -\beta\xi & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\beta & 0 \\
-2\beta\xi & 0 & -\beta\xi & 0 \\
0 & 0 & -\beta & -2\xi \\
\beta^2 & 0 & 0 & 0 \\
-\xi\left(3\beta+\xi^2\right) & 0 & 0 & -\beta \\
-\xi^2\left(4\beta+\xi^2\right) & 0 & 0 & -\beta\xi \\
-\beta\xi & -\beta^2 & -\beta\xi & -\xi^2
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
\beta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$5\beta^2\xi^2 + \beta\xi^4 + 4\beta^3 = \beta\left(\beta+\xi^2\right)\left(4\beta+\xi^2\right)$
Eventually get the rank is 10. Really!

61.2 Summary of results

We note that if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ together with $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ give rank r then so does any conjugate of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ together with any conjugate of $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$. So we restrict our attention to canonical forms.

61.2.1 Case 1

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ we get the same results (up to scale factors on $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$) for all $k \neq 0$. Take $k = 1$. Then we get rank 12 for all $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ except those speciøed below.

$\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = I$ gives rank 0, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives rank 6, $\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$ $\xi \neq \pm 1$ gives rank 8, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}$ $(\lambda \neq \pm 1)$ gives rank 10 unless $\lambda^3 = 1$, when it gives rank8, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives rank 8, $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ gives rank 8 when $p = 2 \bmod 3$ (so that -3 is not a square).

61.2.2 Case 2

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$ we get the same results (up to scale factors on $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$) for all $k \neq 0$. Take $k = 1$. Then we get rank 12 for all $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ except those speciøed below.

$\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = I$ gives rank 6, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gives rank 9, $\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$ $\xi \neq \pm 1$ gives rank 10, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix}$ $(\lambda \neq \pm 1)$ gives rank 11 unless $\lambda^3 = 1$, when it gives rank 10, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives rank 8, $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ gives rank 10 when $p = 2 \bmod 3$ (so that -3 is not a square).

61.2.3 Case 3

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, where $\alpha \neq \delta$. We get the same results (up to scale factors on $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$) for all $\alpha \neq 0$. Take $\alpha = 1$, $\delta \neq 1$. Then we get rank 12 for all $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ except those speciøed below.

If $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix}$ then we get the nullity of the matrix by adding up the following contributions: add 2 if $\lambda = 1$, add 2 if $\lambda = \delta$, add 1 if $\delta\xi = \lambda^2$, add 1 if $\xi = \lambda^2$, add 2 if $\xi = 1$, add 2 if $\xi = \delta$, add one if $\lambda = \xi^2$.

If $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ we get rank 12 unless $\lambda = 1$ or $\lambda = \delta$, and we get rank 10 in both those cases.

If $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ 1 & \xi \end{pmatrix}$ where $\xi^2 + 4\mu$ is not a square then we get rank 12 unless $p = 2 \bmod 3$ (so that -3 is not a square), in which case $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -\delta^2 \\ 1 & -\delta \end{pmatrix}$ both give rank 10 if $\delta^3 \neq 1$ and rank 8 if $\delta^3 = 1$.

61.2.4 Case 4

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ 1 & \delta \end{pmatrix}$ where $\delta^2 + 4\beta$ is not a square, then we get rank 12 for every $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ in which the characteristic polynomial factorizes. If $\begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ 1 & \xi \end{pmatrix}$

where $\xi^2+4\mu$ is not a square then we get rank 12 unless $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$ (when we get rank 8 if $\delta \neq 0$ and rank 6 if $\delta = 0$, or unless $\beta = \mu$, and $3\mu\xi + \delta\mu + \xi^3 = 0$, in which case (if $\xi \neq 0$) we get rank 10. (Note that if $\xi = 0$ in this last case then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ \nu & \xi \end{pmatrix}$.)

If $3\mu\xi + \delta\mu + \xi^3$ and $\beta = \mu$ then

$$\delta = -\frac{3\mu\xi + \xi^3}{\mu}$$

so that

$$\delta^2 + 4\beta = (\frac{3\mu\xi + \xi^3}{\mu})^2 + 4\mu = \frac{9\xi^2\mu^2 + 6\xi^4\mu + \xi^6 + 4\mu^3}{\mu^2} = (4\mu + \xi^2)\,\frac{(\xi^2+\mu)^2}{\mu^2}$$

is not a square unless $\xi^2 = -\mu$. Now if $\xi^2 = -\mu$ then

$$\xi^2 + 4\mu = -3\xi^2,$$

which can only arise when $p = 2 \bmod 3$. So if $p \neq 2 \bmod 3$ then every matrix $\begin{pmatrix} 0 & \mu \\ 1 & \xi \end{pmatrix}$ with $\xi^2 + 4\mu$ not a square and $\xi \neq 0$ gives rise to a unique matrix $\begin{pmatrix} 0 & \mu \\ 1 & \delta \end{pmatrix}$ with $\delta = -\frac{3\mu\xi+\xi^3}{\mu}$ and $\delta^2 + 4\mu$ not a square. But when $p = 2 \bmod 3$ then we have to exclude the matrices $\begin{pmatrix} 0 & \mu \\ 1 & \xi \end{pmatrix}$ with $\mu = -\xi^2$.

It follows that if $p \neq 2 \bmod 3$ then there are $(p-1)^2/2$ pairs of matrices $\begin{pmatrix} 0 & \beta \\ 1 & \delta \end{pmatrix}$, $\begin{pmatrix} 0 & \mu \\ 1 & \xi \end{pmatrix}$ with $\delta^2 + 4\beta$ and $\xi^2 + 4\mu$ not squares which give rank 10, but if $p = 2 \bmod 3$ then there are only $(p-1)^2/2 - (p-1) = (p-1)(p-3)/2$ such pairs. In addition there are $(p-1)/2$ pairs which give rank 6, and $(p-1)^2/2$ pairs which give rank 8.

Here are the sums of $fix(g)$ for all g in the automorphism group. The ørst applies when $p = 1 \bmod 3$, and the second when $p = 2 \bmod 3$.

$p^{13} - p^{12} + p^{11} + p^{10} + 5p^9 - 10p^8 - 37p^7 + 40p^6 + 50p^5 - 49p^4 - 20p^3 + 19p^2$
$p^{13} - p^{12} + p^{11} + p^{10} + 3p^9 - 8p^8 - 31p^7 + 34p^6 + 44p^5 - 43p^4 - 18p^3 + 17p^2$

$$\frac{p^{13} - p^{12} + p^{11} + p^{10} + 5p^9 - 10p^8 - 37p^7 + 40p^6 + 50p^5 - 49p^4 - 20p^3 + 19p^2}{(p^2-1)^2(p^2-p)^2} = p^5+p^4+4p^3+6p^2+18p+19$$

$$\frac{p^{13} - p^{12} + p^{11} + p^{10} + 3p^9 - 8p^8 - 31p^7 + 34p^6 + 44p^5 - 43p^4 - 18p^3 + 17p^2}{(p^2-1)^2(p^2-p)^2} = p^5+p^4+4p^3+6p^2+16p+17$$

$p^5 + p^4 + 4p^3 + 6p^2 + 18p + 19$
$p^5 + p^4 + 4p^3 + 6p^2 + 16p + 17$
$p = 3$ gives 550 which doesn't øt either of these formulae.
$p = 5$ gives 4497 (O.K.)
$p = 7$ gives 21019 (O.K.)
$p = 11$ gives 181935 (O.K.)
$p = 13$ gives 409909 (O.K.)

$p = 17$ gives 1525053 (O.K.)
$p = 19$ gives 2636383 (O.K.)
$p = 23$ gives 6768411 (O.K.)
$p = 29$ gives 21321513 (O.K.)
$p = 31$ gives 29678179 (O.K.)
$p = 37$ gives 71429629 (O.K.)
$p = 41$ gives 118968405 (O.K.)

61.3 Case 6 in the descendants of 4.1 with L^2 having order p^3

Let L satisfy $da = 0$, $db = \omega ca$, $dc = ba$. Then L^2 is generated by ba, ca, cb and $pL \leq L^2$. It is straightforward to show that all elements in the linear span of a, b, c, d have breadth 3, except for those of the form $\alpha a + \delta d$. Using this we can show that if a', b', c', d' generate L and satisfy the same commutator relations as a, b, c, d then (modulo L^2)

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We let

$$\begin{pmatrix} pa \\ pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

where A is a 4×3 matrix over $\mathbb{Z}_p$. Then under a change of generating set of the form described above we see that

$$A \mapsto \begin{pmatrix} \alpha & 0 & 0 & \delta \\ \pm\lambda & \pm\gamma & \pm\omega\beta & \pm\mu \\ \nu & \beta & \gamma & \xi \\ \pm\omega\delta & 0 & 0 & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix}^{-1}.$$

We consider the possibilities for pa, pd. They are $0, 0$; $0, cb$; $0, ca$; cb, ca; and p possibilities when pa, pd span $\langle ba, ca \rangle$.

In this last case we can take $pa = ca$ and let $pd = \rho ba + \sigma ca$ with $\rho \neq 0$. Then we have

$$\begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \delta \\ \pm\omega\delta & \pm\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta \end{pmatrix}^{-1}.$$

To ensure that $pa' = c'a'$ we require

$$\delta\rho ba + (\alpha + \delta\sigma)ca = (\alpha\beta - \gamma\delta)ba + (\alpha\gamma - \omega\beta\delta)ca.$$

So we need

$$\begin{aligned} \delta\rho &= \alpha\beta - \gamma\delta, \\ \alpha + \delta\sigma &= \alpha\gamma - \omega\beta\delta, \end{aligned}$$

which implies that $\beta = \frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}$. This gives

$$\begin{pmatrix} 0 & 1 \\ \rho\frac{\delta^2\omega-\alpha^2}{\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\alpha^2} & -\frac{-\rho^2\omega\alpha\delta+\sigma\omega\delta^2+\omega\delta\alpha+\sigma^2\alpha\delta+\sigma\alpha^2}{\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\alpha^2} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ \rho\frac{\delta^2\omega-\alpha^2}{\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\alpha^2} & \frac{-\rho^2\omega\alpha\delta+\sigma\omega\delta^2+\omega\delta\alpha+\sigma^2\alpha\delta+\sigma\alpha^2}{\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\alpha^2} \end{pmatrix}.$$

These orbits are given by dec4.16b. Here is the proof that there are p orbits.

$$A \mapsto \begin{pmatrix} \alpha & \delta \\ \pm\omega\delta & \pm\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \pm(\alpha\gamma-\omega\beta\delta) & \pm(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1}.$$

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} - \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\delta z-x\alpha\gamma+x\omega\beta\delta-y\alpha\beta+y\gamma\delta & \alpha y+\delta t-x\omega\alpha\beta+x\omega\gamma\delta-y\alpha\gamma+y\omega\beta\delta \\ \omega\delta x+\alpha z-z\alpha\gamma+z\omega\beta\delta-t\alpha\beta+t\gamma\delta & \omega\delta y+\alpha t-z\omega\alpha\beta+z\omega\gamma\delta-t\alpha\gamma+t\omega\beta\delta \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha\gamma+\omega\beta\delta & -\alpha\beta+\gamma\delta & \delta & 0 \\ -\omega\alpha\beta+\omega\gamma\delta & \alpha-\alpha\gamma+\omega\beta\delta & 0 & \delta \\ \omega\delta & 0 & \alpha-\alpha\gamma+\omega\beta\delta & -\alpha\beta+\gamma\delta \\ 0 & \omega\delta & -\omega\alpha\beta+\omega\gamma\delta & \alpha-\alpha\gamma+\omega\beta\delta \end{pmatrix}$$

Determinant: $\left(\beta^2\omega-(\gamma-1)^2\right)\left(-\omega\delta^2+\alpha^2\right)$
$\times\left(-\alpha^2+\alpha^2\beta^2\omega+2\alpha^2\gamma-\alpha^2\gamma^2-4\alpha\omega\beta\delta-\omega^2\beta^2\delta^2+\omega\delta^2+\gamma^2\delta^2\omega+2\omega\delta^2\gamma\right)$
Now the discrimant of the last factor, when thought of as a polynomial in δ, is
$(-4\alpha\omega\beta)^2-4(-\alpha^2+\alpha^2\beta^2\omega+2\alpha^2\gamma-\alpha^2\gamma^2)(-\omega^2\beta^2+\omega+\gamma^2\omega+2\omega\gamma) = 4\alpha^2\omega\left(1-\gamma^2+\omega\beta^2\right)^2$
So the last factor can only be zero if $\alpha = 0$ or $1-\gamma^2+\omega\beta^2 = 0$.
If $\alpha = 0$ then the last factor is $-\omega^2\beta^2\delta^2+\omega\delta^2+\gamma^2\delta^2\omega+2\omega\delta^2\gamma = -\omega\delta^2\left(\omega\beta^2-(\gamma+1)^2\right)$
So then the last factor is only zero if $\alpha = 0$, $\beta = 0$ and $\gamma = -1$.
On the other hand, if $1-\gamma^2+\omega\beta^2 = 0$ then the last factor is
$-\alpha^2+\alpha^2(\gamma^2-1)+2\alpha^2\gamma-\alpha^2\gamma^2-4\alpha\omega\beta\delta-\omega(\gamma^2-1)\delta^2+\omega\delta^2+\gamma^2\delta^2\omega+2\omega\delta^2\gamma$
$= -2\alpha^2+2\alpha^2\gamma-4\alpha\omega\beta\delta+2\omega\delta^2+2\omega\delta^2\gamma$
We get zero if $\beta = 0$, $\gamma = -1$ and $\alpha = 0$. For $\gamma \neq -1$ we get zero if

$$\delta = \frac{\alpha\beta}{(1+\gamma)}$$

Try $\beta = 0$, $\gamma = 1$. We get

$$\begin{pmatrix} 0 & \delta & \delta & 0 \\ \omega\delta & 0 & 0 & \delta \\ \omega\delta & 0 & 0 & \delta \\ 0 & \omega\delta & \omega\delta & 0 \end{pmatrix}$$

which has rank 2 if $\delta \neq 0$ and rank 0 if $\delta = 0$.
Try $\beta = 0$, $\gamma = -1$ and $\alpha = 0$. We get

$$\begin{pmatrix} 0 & -\delta & \delta & 0 \\ -\omega\delta & 0 & 0 & \delta \\ \omega\delta & 0 & 0 & -\delta \\ 0 & \omega\delta & -\omega\delta & 0 \end{pmatrix}$$

which has rank 2.

Finally try $\delta = \frac{\alpha\beta}{(1+\gamma)}$ with $\gamma \neq \pm 1$. We get

$$\begin{pmatrix} \alpha\left(1-\gamma^2+\omega\beta^2\right) & -\alpha\beta & \alpha\beta & 0 \\ -\omega\alpha\beta & \alpha\left(1-\gamma^2+\omega\beta^2\right) & 0 & \alpha\beta \\ \omega\alpha\beta & 0 & \alpha\left(1-\gamma^2+\omega\beta^2\right) & -\alpha\beta \\ 0 & \omega\alpha\beta & -\omega\alpha\beta & \alpha\left(1-\gamma^2+\omega\beta^2\right) \end{pmatrix}$$

Determinant: $\alpha^4\left(2\gamma-1-\gamma^2+\omega\beta^2\right)\left(\omega\beta^2-1-\gamma^2-2\gamma\right)\left(1-\gamma^2+\omega\beta^2\right)^2$

Only the last braket can be zero since $\alpha, \beta \neq 0$, and when this bracket is zero we get rank 2.

So the contribution to Burnside's Lemma is

$p(p-1)(p^2-1)+(p-1)(p^4-1)+(p-1)(p^2-1)+(p-1)^2(p^2-1)+(p^2-1)^2$

$=(p+2)\left(p-1\right)^2\left(p+1\right)^2$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}-\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}$$

$$=\begin{pmatrix} \alpha x+\delta z+x\alpha\gamma-x\omega\beta\delta-\alpha y\beta+y\gamma\delta & \alpha y+\delta t+x\omega\alpha\beta-x\omega\gamma\delta-\alpha y\gamma+y\omega\beta\delta \\ -\omega\delta x-\alpha z+z\alpha\gamma-z\omega\beta\delta-t\alpha\beta+t\gamma\delta & -\omega\delta y-\alpha t+z\omega\alpha\beta-z\omega\gamma\delta-t\alpha\gamma+t\omega\beta\delta \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha\gamma-\omega\beta\delta & -\alpha\beta+\gamma\delta & \delta & 0 \\ \omega\alpha\beta-\omega\gamma\delta & \alpha-\alpha\gamma+\omega\beta\delta & 0 & \delta \\ -\omega\delta & 0 & -\alpha+\alpha\gamma-\omega\beta\delta & -\alpha\beta+\gamma\delta \\ 0 & -\omega\delta & \omega\alpha\beta-\omega\gamma\delta & -\alpha-\alpha\gamma+\omega\beta\delta \end{pmatrix}$$

Determinant: $\left(1-\gamma^2+\omega\beta^2\right)^2\left(\omega\delta^2-\alpha^2\right)^2$

So the rank is four unless $1-\gamma^2+\omega\beta^2=0$ (which happens $p+1$ times).

Consider the case when $\delta=0$. We get

$$\begin{pmatrix} \alpha+\alpha\gamma & -\alpha\beta & 0 & 0 \\ \omega\alpha\beta & -\alpha\gamma+\alpha & 0 & 0 \\ 0 & 0 & -\alpha+\alpha\gamma & -\alpha\beta \\ 0 & 0 & \omega\alpha\beta & -\alpha-\alpha\gamma \end{pmatrix}$$

$\begin{pmatrix} \alpha+\alpha\gamma & -\alpha\beta \\ \omega\alpha\beta & -\alpha\gamma+\alpha \end{pmatrix}$ Determinant: $\alpha^2-\alpha^2\gamma^2+\omega\alpha^2\beta^2$

$\begin{pmatrix} -\alpha+\alpha\gamma & -\alpha\beta \\ \omega\alpha\beta & -\alpha-\alpha\gamma \end{pmatrix}$ Determinant: $\alpha^2-\alpha^2\gamma^2+\omega\alpha^2\beta^2$

So the rank is if $\delta=0$ and $1-\gamma^2+\omega\beta^2=0$.

If $\delta \neq 0$ then we can take $\delta=1$, and we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha-\alpha\gamma+\omega\beta & \alpha\beta-\gamma & 1 & 0 \\ -\omega\alpha\beta+\omega\gamma & \alpha+\alpha\gamma-\omega\beta & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha+\alpha\gamma-\omega\beta & -\alpha\beta+\gamma & 1 & 0 \\ \omega\alpha\beta-\omega\gamma & \alpha-\alpha\gamma+\omega\beta & 0 & 1 \\ -\omega & 0 & -\alpha+\alpha\gamma-\omega\beta & -\alpha\beta+\gamma \\ 0 & -\omega & \omega\alpha\beta-\omega\gamma & -\alpha-\alpha\gamma+\omega\beta \end{pmatrix}$$

$$=\begin{pmatrix} \alpha+\alpha\gamma-\omega\beta & -\alpha\beta+\gamma & 1 & 0 \\ \omega\alpha\beta-\omega\gamma & \alpha-\alpha\gamma+\omega\beta & 0 & 1 \\ \alpha^2-\alpha^2\gamma^2-\omega^2\beta^2+\omega\alpha^2\beta^2+\omega\gamma^2-\omega & 0 & 0 & 0 \\ 0 & \alpha^2-\alpha^2\gamma^2-\omega^2\beta^2+\omega\alpha^2\beta^2+\omega\gamma^2-\omega & 0 & 0 \end{pmatrix}$$

Now $\omega\beta^2 = (\gamma^2 - 1)$, so

$$\alpha^2 - \alpha^2\gamma^2 - \omega^2\beta^2 + \omega\alpha^2\beta^2 + \omega\gamma^2 - \omega = 0$$

and we have rank 2.

So the contribution to Burnside's Lemma is $(p+1)(p^2-1)(p^2-1) + (p^2-1)^2$
$= (p+2)\left(p-1\right)^2\left(p+1\right)^2$

Thus there are $p+2$ orbits, which include one orbit with the matrix equal to 0, and one with the matrix of rank 1.

So there are p orbits of matrices with rank 2.

61.4 $pa = pd = 0$

There are $p+4$ algebras with $pa = pd = 0$. One possibility is $pb = pc = 0$ also. And if pb, pc span a one dimensional space then we may suppose that $pb = 0$ and that $pc = ca$ or cb. Suppose however that pb, pc are linearly independent. If they do not both lie in the subspace spanned by ba, ca then we may suppose that $pb = cb$ and that $pc = ca$. If pb, pc both lie in the subspace spanned by ba, ca then we can suppose that $pb = ca$. So let $pb = ca$ and let $pc = \rho ba + \sigma ca$. If we consider a change of generating set of the form above we have

$$\begin{aligned} pb' &= \pm(\gamma pb + \omega\beta pc) = \pm(\omega\beta\rho ba + (\gamma + \omega\beta\sigma)ca), \\ pc' &= \beta pb + \gamma pc = \gamma\rho ba + (\beta + \gamma\sigma)ca \end{aligned}$$

and

$$\begin{aligned} b'a' &= \pm((\alpha\gamma - \omega\beta\delta)ba + (\omega\alpha\beta - \omega\gamma\delta)ca), \\ c'a' &= (\alpha\beta - \gamma\delta)ba + (\alpha\gamma - \omega\beta\delta)ca. \end{aligned}$$

To ensure that $pb' = c'a'$ we require that

$$\begin{aligned} \pm\omega\beta\rho &= \alpha\beta - \gamma\delta, \\ \pm(\gamma + \omega\beta\sigma) &= \alpha\gamma - \omega\beta\delta. \end{aligned}$$

This implies that

$$\begin{aligned} \alpha(-\gamma^2 + \omega\beta^2) &= \pm(\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma), \\ \delta(-\gamma^2 + \omega\beta^2) &= \pm(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma). \end{aligned}$$

$$(-\gamma^2 + \omega\beta^2) \begin{pmatrix} \gamma & \omega\beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \times$$
$$\begin{pmatrix} (\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\gamma - \omega\beta(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) & \omega(\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\beta - \omega\gamma(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) \\ (\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\beta - \gamma(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) & (\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\gamma - \omega\beta(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 0 & 1 \\ (-\gamma^2 + \omega\beta^2)\frac{\rho}{\omega^3\beta^2\rho^2 - \beta^2\sigma^2\omega^2 - 2\omega\beta\sigma\gamma - \gamma^2} & \frac{\gamma\rho^2\beta\omega^2 - \omega\beta^2\sigma - \omega\beta\sigma^2\gamma - \beta\gamma - \gamma^2\sigma}{\omega^3\beta^2\rho^2 - \beta^2\sigma^2\omega^2 - 2\omega\beta\sigma\gamma - \gamma^2} \end{pmatrix}$$

$$(-\gamma^2 + \omega\beta^2) \begin{pmatrix} -\gamma & -\omega\beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \times$$
$$\begin{pmatrix} (\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\gamma - \omega\beta(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) & \omega(\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\beta - \omega\gamma(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) \\ -(\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\beta + \gamma(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) & -(\omega^2\beta^2\rho - \gamma^2 - \omega\beta\gamma\sigma)\gamma + \omega\beta(\omega\beta\gamma\rho - \beta\gamma - \omega\beta^2\sigma) \end{pmatrix}^{-}$$

$$= \begin{pmatrix} 0 & 1 \\ (-\gamma^2+\omega\beta^2)\frac{\rho}{\omega^3\beta^2\rho^2-\beta^2\sigma^2\omega^2-2\omega\beta\gamma\sigma-\gamma^2} & \frac{\omega\beta^2\sigma-\gamma\rho^2\beta\omega^2+\beta\omega\gamma\sigma^2+\beta\gamma+\gamma^2\sigma}{\omega^3\beta^2\rho^2-\beta^2\sigma^2\omega^2-2\omega\beta\gamma\sigma-\gamma^2} \end{pmatrix}$$

There are p orbits here: the proof is similar to the case above, with pa, pd spanning ba, ca. So we have $p+4$ algebras of this sort.

61.5 $pa = 0, pd = cb$

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We need to consider change of generating set with $\delta = 0$ to ensure $pa' = 0$. And to ensure that $pd' = c'b'$ we require $\alpha = \gamma^2 - \omega\beta^2$. We can choose μ, ξ so that $pb', pc' \in \langle ba, ca\rangle$. So we can suppose that

$$\begin{pmatrix} pb \\ pc \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

where A is a 2×2 matrix over $\mathbb{Z}_p$, and we consider transformations of A of the form

$$A \mapsto (\gamma^2 - \omega\beta^2)^{-1} \begin{pmatrix} \pm\gamma & \pm\omega\beta \\ \beta & \gamma \end{pmatrix} A \begin{pmatrix} \pm\gamma & \pm\omega\beta \\ \beta & \gamma \end{pmatrix}^{-1}.$$

We compute these orbits in dec4.16c. For $p = 3$ there are 16 orbits, for $p = 5$ there are 35, for $p = 7$ there are 62, for $p = 11$ there are 140, for $p = 13$ there are 191.

One possibility is that $A = 0$. And if pb, pc are linearly dependent then we may suppose that $pb = 0$. This restricts us to transformations of the form above with $\beta = 0$, so we have

$$\begin{pmatrix} 0 & 0 \\ r & s \end{pmatrix} \mapsto \gamma^{-2} \begin{pmatrix} 0 & 0 \\ \pm r & s \end{pmatrix}.$$

If $r = 0$ there are two possibilities, taking $s = 1, \omega$. If $r \neq 0$ then we have to distinguish between the cases $p = 1 \bmod 4$ and $p = 3 \bmod 4$. If $p = 1 \bmod 4$ then -1 is a square, and so we can take $r = 1, \omega$. For each choice of r we can take $\gamma^2 = \pm 1$ and so we can take $0 \leq s \leq (p-1)/2$. This gives $p+1$ algebras. If $p = 3 \bmod 4$ then -1 is not a square. So we can take $r = 1$, which forces $\gamma^2 = 1$, giving p algebras.

So if pb, pc are not linearly independent then there are a total of $p+4$ algebras when $p = 1 \bmod 4$ and $p+3$ when $p = 3 \bmod 4$ (including the case $pb = pc = 0$).

If pb, pc are linearly independent then A has rank 2, and so A can only be øxed when $\gamma^2 - \omega\beta^2 = \pm 1$. We consider

$$\begin{pmatrix} \gamma & \omega\beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} - (\gamma^2 - \omega\beta^2) \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \gamma & \omega\beta \\ \beta & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma x + \omega\beta z - \gamma^3 x + \gamma x\omega\beta^2 - \beta y\gamma^2 + \beta^3 y\omega & \gamma y + \omega\beta t - x\omega\beta\gamma^2 + x\omega^2\beta^3 - \gamma^3 y + \gamma y\omega\beta^2 \\ \beta x + \gamma z - \gamma^3 z + \gamma z\omega\beta^2 - t\beta\gamma^2 + t\beta^3\omega & \beta y + \gamma t - \omega\beta z\gamma^2 + \omega^2\beta^3 z - \gamma^3 t + \gamma t\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \gamma-(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta & \omega\beta & 0 \\ -(\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma & 0 & \omega\beta \\ \beta & 0 & \gamma-(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta \\ 0 & \beta & -(\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma \end{pmatrix}$$

and

$$\begin{pmatrix} -\gamma & -\omega\beta \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}-(\gamma^2-\omega\beta^2)\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} -\gamma & -\omega\beta \\ \beta & \gamma \end{pmatrix}$$

$$=\begin{pmatrix} -\gamma x-\omega\beta z+\gamma^3x-\gamma x\omega\beta^2-\beta y\gamma^2+\beta^3y\omega & -\gamma y-\omega\beta t+x\omega\beta\gamma^2-x\omega^2\beta^3-\gamma^3y+\gamma y\omega\beta^2 \\ \beta x+\gamma z+\gamma^3z-\gamma z\omega\beta^2-t\beta\gamma^2+t\beta^3\omega & \beta y+\gamma t+\omega\beta z\gamma^2-\omega^2\beta^3z-\gamma^3t+\gamma t\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} -\gamma+(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta & -\omega\beta & 0 \\ (\gamma^2-\omega\beta^2)\omega\beta & -\gamma-(\gamma^2-\omega\beta^2)\gamma & 0 & -\omega\beta \\ \beta & 0 & \gamma+(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta \\ 0 & \beta & (\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma \end{pmatrix}$$

First consider the rank of

$$\begin{pmatrix} \gamma-(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta & \omega\beta & 0 \\ -(\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma & 0 & \omega\beta \\ \beta & 0 & \gamma-(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta \\ 0 & \beta & -(\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma \end{pmatrix}$$

Determinant: $\left(\omega\beta^2-(\gamma-1)^2\right)\left(\omega\beta^2-(\gamma+1)^2\right)\left(\omega\beta^2+1-\gamma^2\right)^2\left(\omega\beta^2-\gamma^2\right)^2$

So the rank is 4 unless $\beta=0$ and $\gamma=\pm1$ or $\gamma^2-\omega\beta^2=1$.

First consider the case when $\beta=0$ and $\gamma=\pm1$. We get the zero matrix, with rank 0, in both these cases.

Next consider the case when $\gamma^2-\omega\beta^2=1$. Note that the case $\beta=0$ has already been covered. We have

$$\begin{pmatrix} 0 & -\beta & \omega\beta & 0 \\ -\omega\beta & 0 & 0 & \omega\beta \\ \beta & 0 & 0 & -\beta \\ 0 & \beta & -\omega\beta & 0 \end{pmatrix}$$

which has rank 2.

So the contribution to Burnsides Lemma from these is $2(p^4-1)+k(p^2-1)+(p^2-1)$, where k is the number of pairs β,γ with $\beta\neq0$ and $\gamma^2-\omega\beta^2=1$.

Next, consider the rank of

$$\begin{pmatrix} -\gamma+(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta & -\omega\beta & 0 \\ (\gamma^2-\omega\beta^2)\omega\beta & -\gamma-(\gamma^2-\omega\beta^2)\gamma & 0 & -\omega\beta \\ \beta & 0 & \gamma+(\gamma^2-\omega\beta^2)\gamma & -(\gamma^2-\omega\beta^2)\beta \\ 0 & \beta & (\gamma^2-\omega\beta^2)\omega\beta & \gamma-(\gamma^2-\omega\beta^2)\gamma \end{pmatrix}$$

Determinant: $\left(\omega\beta^2+1-\gamma^2\right)^2\left(\omega\beta^2-\gamma^2\right)^2\left(-1+\omega\beta^2-\gamma^2\right)^2$

So the rank is 4 unless $\gamma^2-\omega\beta^2=\pm1$.

First consider the case when $\gamma^2-\omega\beta^2=1$. Then we have

$$\begin{pmatrix} 0 & -\beta & -\omega\beta & 0 \\ \omega\beta & -2\gamma & 0 & -\omega\beta \\ \beta & 0 & 2\gamma & -\beta \\ 0 & \beta & \omega\beta & 0 \end{pmatrix}.$$

Adding row 1 to row 4 we have

$$\begin{pmatrix} 0 & -\beta & -\omega\beta & 0 \\ \omega\beta & -2\gamma & 0 & -\omega\beta \\ \beta & 0 & 2\gamma & -\beta \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subtracting ω times row 3 from row 2 we have

$$\begin{pmatrix} 0 & -\beta & -\omega\beta & 0 \\ 0 & -2\gamma & -2\omega\gamma & 0 \\ \beta & 0 & 2\gamma & -\beta \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 2, since at least one of β, γ must be non-zero.

Now consider the case when $\gamma^2 - \omega\beta^2 = -1$. Then we have

$$\begin{pmatrix} -2\gamma & \beta & -\omega\beta & 0 \\ -\omega\beta & 0 & 0 & -\omega\beta \\ \beta & 0 & 0 & \beta \\ 0 & \beta & -\omega\beta & 2\gamma \end{pmatrix}$$

If $\beta = 0$ this has rank 2. If $\beta \neq 0$ it also has rank 2.

So the contribution to Burnsides Lemma from these is $l(p^2-1) + (p^2-1)$, where l is the number of pairs β, γ with $\gamma^2 - \omega\beta^2 = \pm 1$.

So the total number of orbits is

$$\frac{2(p^4-1) + (k+l+2)(p^2-1)}{2(p^2-1)}$$

where k is the number of pairs β, γ with $\beta \neq 0$ and $\gamma^2 - \omega\beta^2 = 1$ and l is the number of pairs β, γ with $\gamma^2 - \omega\beta^2 = \pm 1$. (The values of k, l are computed in dec4.16d.

Now $\gamma^2 - \omega\beta^2$ takes on all possible values in $\mathbb{Z}_p$, though it can only take the value 0 when $\beta = \gamma = 0$. All other values occur $p+1$ times. So $k = p-1$ and $l = 2p+2$. So the number of algebras is $p^2 + (3p+5)/2$.

61.6 $\quad pa = 0, pd = ca$

To ensure that $pa = 0$ we consider a', b', c', d' as above with $\delta = 0$. And to ensure that $pd = ca$ we require $\beta = 0$ and $\gamma = \pm 1$

$$\begin{aligned} a' &= \alpha a, \\ b' &= \pm(\lambda a \pm b + \mu d), \\ c' &= \nu a + \pm c + \xi d, \\ d' &= \pm\alpha d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \pm\alpha & 0 \\ -\nu - \mu & \lambda + \omega\xi & \pm 1 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We let

$$\begin{pmatrix} pb \\ pc \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

where A is a 3×3 matrix over $\mathbb{Z}_p$ of the form

$$\begin{pmatrix} u & v & w \\ x & y & z \\ 0 & 1 & 0 \end{pmatrix}$$

Then under a change of generating set of the form described above we see that

$$\begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & \xi \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} u & v & w \\ x & y & z \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ -\nu & \lambda & 1 \end{pmatrix}^{-1} = \alpha^{-1} \begin{pmatrix} u + w\nu & v + \mu - w\lambda & \alpha w \\ x + z\nu & y + \xi - z\lambda & \alpha z \\ 0 & \alpha & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\mu \\ 0 & -1 & \xi \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} u & v & w \\ x & y & z \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ -\nu & \lambda & -1 \end{pmatrix}^{-1} = \alpha^{-1} \begin{pmatrix} u - w\nu & -v + \mu - w\lambda & -\alpha w \\ -x + z\nu & y - \xi + z\lambda & \alpha z \\ 0 & \alpha & 0 \end{pmatrix}$$

We replace $\lambda + \omega\xi$ by λ, and $\nu + \mu$ by ν above.
Clearly we can take $v = y = 0$ if we set $\mu = w\lambda$ and $\xi = z\lambda$.

$$\begin{pmatrix} 1 & 0 & w\lambda \\ 0 & 1 & z\lambda \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} u & 0 & w \\ x & 0 & z \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ -\nu & \lambda & 1 \end{pmatrix}^{-1} = \alpha^{-1} \begin{pmatrix} u + w\nu & 0 & \alpha w \\ x + z\nu & 0 & \alpha z \\ 0 & \alpha & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -w\lambda \\ 0 & -1 & z\lambda \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} u & 0 & w \\ x & 0 & z \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ -\nu & \lambda & -1 \end{pmatrix}^{-1} = \alpha^{-1} \begin{pmatrix} u - w\nu & 0 & -\alpha w \\ -x + z\nu & 0 & \alpha z \\ 0 & \alpha & 0 \end{pmatrix}.$$

If $w = z = 0$ then u, x transform to $\alpha^{-1}u, \pm\alpha^{-1}x$ so we can take $u, x = 0, 0;\ 0, 1;\ 1, x$ with $0 \leq x \leq (p-1)/2$, giving $(p+5)/2$ possibilities.

If $w = 0$, $z \neq 0$ we can take $x = 0$ and $u = 0, 1$ giving $2(p-1)$ possibilities.

If $w \neq 0$ we can take $u = 0$, $0 < w \leq (p-1)/2$, $x = 0, 1$ giving $(p-1)p$ possibilities.

So the total number of algebras here is $p^2 + (3p+1)/2$. Dec4.16e computes these.

61.7 $pa = cb, pd = ca$

Now consider the case when $pa = cb$, $pd = ca$. We consider possible generators a', b', c', d' as above. If $d' = \pm(\omega\delta a + \alpha d)$ then

$$pd' = \pm(\alpha ca + \omega\delta cb)$$

and for this to equal $c'a'$ we require $\delta = 0$. So, taking $\delta = 0$ we have

$$\begin{aligned} pa' &= \alpha pa = \alpha cb \\ pd' &= \pm\alpha pd = \pm\alpha ca. \end{aligned}$$

To ensure that $pd' = c'a'$ we require $\beta = 0$, $\gamma = \pm 1$. So we have

$$\begin{aligned} a' &= \alpha a, \\ b' &= \pm(\lambda a \pm b + \mu d), \\ c' &= \nu a \pm c + \xi d, \\ d' &= \pm\alpha d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \pm\alpha & 0 \\ -\nu-\mu & \lambda+\omega\xi & \pm 1 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

Now, to ensure that $pa' = c'b'$ we require $\alpha = \pm 1$, $\nu = -\mu$, $\lambda = -\omega\xi$. So we have

$$\begin{aligned} a' &= \pm a, \\ b' &= \pm(-\omega\xi a \pm b + \mu d), \\ c' &= -\mu a \pm c + \xi d, \\ d' &= d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

We can choose μ, ξ so that pb' is a scalar multiple of ba, but if we do this then we require $\mu = \xi = 0$. So assuming that $pb = \rho ba$ and $pc = \sigma ba + \varsigma ca + \tau cb$ we can consider a', b', c', d' of the form

$$\begin{aligned} a' &= \pm a, \\ b' &= b, \\ c' &= \pm c, \\ d' &= d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix},$$

giving

$$\begin{aligned} pb' &= \rho ba = \pm\rho b'a', \\ pc' &= \pm(\sigma ba + \varsigma ca + \tau cb) = \sigma b'a' \pm \varsigma ca + \tau cb. \end{aligned}$$

So if $\rho = 0$ we can take $0 \leq \sigma, \tau < p$, $0 \leq \varsigma \leq (p-1)/2$ and if $\rho \neq 0$ then we can take $0 < \rho \leq (p-1)/2$ and $0 \leq \sigma, \varsigma, \tau < p$. So we have $\frac{1}{2}p^2 + \frac{1}{2}p^4$ algebras here.

61.8 $pa = ca$ and $pd = \rho ba + \sigma ca$ with $\rho \neq 0$

Let $pa = ca$ and let $pd = \rho ba + \sigma ca$ with $\rho \neq 0$. We consider a', b', c', d' of the form

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

To ensure that $pa' = c'a'$ and $pd' = \rho b'a' + \sigma c'a'$ we require that $\beta = \frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}$, $\rho^2\delta^2\omega - \sigma^2\delta^2 - 2\sigma\alpha\delta - \alpha^2 = \delta^2\omega - \alpha^2$, $-\rho^2\omega\alpha\delta + \sigma\omega\delta^2 + \omega\delta\alpha + \sigma^2\alpha\delta + \sigma\alpha^2 = \mp\sigma(\delta^2\omega - \alpha^2)$.

If pa, pb, pc, pd span a space of dimension two then we can take

$$\begin{aligned} a' &= a, \\ b' &= \lambda a + b + \mu d, \\ c' &= \nu a + c + \xi d, \\ d' &= \alpha d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pm(-\nu-\mu) & \pm(\lambda+\omega\xi) & \pm\gamma^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

so that $pb' = pc' = 0$. So we have p algebras of this kind.

If pa, pb, pc, pd span a space of dimension 3 then we can take a', b', c', d' of the same form so that pb', pc' are linearly dependent. Then taking

$$\begin{aligned} a' &= a, \\ b' &= \gamma b + \omega\beta c, \\ c' &= \beta b + \gamma c, \\ d' &= d \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \gamma & \omega\beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \pm(\gamma^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}$$

we can take $pb' = 0$, though this ıscramblesȷ pa', pd', though we still have pa', pd' as linear combinations of $b'a'$, $c'a'$.

So we assume that pa, pd span $\langle ba, ca\rangle$, that $pb = 0$ and that pa, pc, pd are linearly independent. If we pick a', b', c', d' as above so that pa', pd' span $\langle b'a', c'a'\rangle$, so that $pb' = 0$ and that pa', pc', pd' are linearly independant then we require $\beta = \lambda = \mu = 0$ giving

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm\gamma b, \\ c' &= \nu a + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm\alpha\gamma & \pm(-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \pm(-\gamma\nu) & \pm\omega\gamma\xi & \pm\gamma^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

First we consider the possibilities for pa, pd. We let

$$\begin{pmatrix} pa \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}$$

for some non-singular 2×2 matrix over $\mathbb{Z}_p$. We then have

$$A \mapsto \gamma^{-1} \begin{pmatrix} \alpha & \delta \\ \pm\omega\delta & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm\alpha & \pm(-\omega\delta) \\ -\delta & \alpha \end{pmatrix}^{-1}.$$

We try a Burnside Lemma count.

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} - \gamma\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & -\omega\delta \\ -\delta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x + \delta z - \gamma x\alpha + \gamma y\delta & \alpha y + \delta t + \gamma x\omega\delta - \gamma y\alpha \\ \omega\delta x + \alpha z - \gamma z\alpha + \gamma t\delta & \omega\delta y + \alpha t + \gamma z\omega\delta - \gamma t\alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \gamma\alpha & \gamma\delta & \delta & 0 \\ \gamma\omega\delta & \alpha - \gamma\alpha & 0 & \delta \\ \omega\delta & 0 & \alpha - \gamma\alpha & \gamma\delta \\ 0 & \omega\delta & \gamma\omega\delta & \alpha - \gamma\alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} - \gamma\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} -\alpha & \omega\delta \\ -\delta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x + \delta z + \gamma x\alpha + \gamma y\delta & \alpha y + \delta t - \gamma x\omega\delta - \gamma y\alpha \\ -\omega\delta x - \alpha z + \gamma z\alpha + \gamma t\delta & -\omega\delta y - \alpha t - \gamma z\omega\delta - \gamma t\alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha + \gamma\alpha & \gamma\delta & \delta & 0 \\ -\gamma\omega\delta & \alpha - \gamma\alpha & 0 & \delta \\ -\omega\delta & 0 & -\alpha + \gamma\alpha & \gamma\delta \\ 0 & -\omega\delta & -\gamma\omega\delta & -\alpha - \gamma\alpha \end{pmatrix}$$

First consider

$$\begin{pmatrix} \alpha - \gamma\alpha & \gamma\delta & \delta & 0 \\ \gamma\omega\delta & \alpha - \gamma\alpha & 0 & \delta \\ \omega\delta & 0 & \alpha - \gamma\alpha & \gamma\delta \\ 0 & \omega\delta & \gamma\omega\delta & \alpha - \gamma\alpha \end{pmatrix}$$

Determinant: $(\gamma-1)^2\left(-\alpha^2+\omega\delta^2\right)\left(\omega\delta^2(\gamma+1)^2-\alpha^2(\gamma-1)^2\right)$

So we get rank 4 unless $\gamma = 1$ or $\alpha = 0$ and $\gamma = -1$.

First consider the case when $\gamma = 1$. Then we have

$$\begin{pmatrix} 0 & \delta & \delta & 0 \\ \omega\delta & 0 & 0 & \delta \\ \omega\delta & 0 & 0 & \delta \\ 0 & \omega\delta & \omega\delta & 0 \end{pmatrix}$$

which has rank 2 if $\delta \neq 0$ and rank 0 if $\delta = 0$.

Now consider the case when $\alpha = 0$ and $\gamma = -1$. We then have

$$\begin{pmatrix} 0 & -\delta & \delta & 0 \\ -\omega\delta & 0 & 0 & \delta \\ \omega\delta & 0 & 0 & -\delta \\ 0 & \omega\delta & -\omega\delta & 0 \end{pmatrix}$$

which has rank 2 (since we must have $\delta \neq 0$).

So we have

$$\frac{(p-1)p(p^2-1)+(p-1)(p^4-1)+(p-1)(p^2-1)+2(p^2-1)(p-1)}{2(p^2-1)(p-1)} = \frac{1}{2}p^2+\frac{1}{2}p+2$$

possibilities.

$$(\alpha^2-\omega\delta^2)\begin{pmatrix}\alpha & \delta\\ \omega\delta & \alpha\end{pmatrix}\begin{pmatrix}x & y\\ z & t\end{pmatrix}\begin{pmatrix}\alpha & -\omega\delta\\ -\delta & \alpha\end{pmatrix}^{-1}$$
$$=\begin{pmatrix} t\delta^2+\delta\alpha z+\delta\alpha y+\alpha^2 x & \omega z\delta^2+\omega\alpha x\delta+\alpha t\delta+\alpha^2 y\\ \omega y\delta^2+\omega\alpha x\delta+\alpha t\delta+z\alpha^2 & x\delta^2\omega^2+\omega\delta\alpha z+\omega\delta\alpha y+t\alpha^2\end{pmatrix}$$

$$(\alpha^2-\omega\delta^2)\begin{pmatrix}\alpha & \delta\\ -\omega\delta & -\alpha\end{pmatrix}\begin{pmatrix}x & y\\ z & t\end{pmatrix}\begin{pmatrix}-\alpha & \omega\delta\\ -\delta & \alpha\end{pmatrix}^{-1}$$
$$=\begin{pmatrix} -t\delta^2-\delta\alpha z-\delta\alpha y-\alpha^2 x & \omega z\delta^2+\omega\alpha x\delta+\alpha t\delta+\alpha^2 y\\ \omega y\delta^2+\omega\alpha x\delta+\alpha t\delta+z\alpha^2 & -x\delta^2\omega^2-\omega\delta\alpha z-\omega\delta\alpha y-t\alpha^2\end{pmatrix}$$

$\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\delta^2\omega=0.$
So $\delta=0$ or $\delta=\frac{2\sigma\alpha}{\rho^2\omega-\sigma^2-\omega}$.
$\beta=\frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}$, $\gamma=\frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}$,
$\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\alpha^2=\delta^2\omega-\alpha^2$, $-\rho^2\omega\alpha\delta+\sigma\omega\delta^2+\omega\delta\alpha+\sigma^2\alpha\delta+\sigma\alpha^2$
$=\mp\sigma(\delta^2\omega-\alpha^2)$.
Try $\delta=0$
$\frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}=0,$
$\frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}=1$
$-\rho^2\omega\alpha\delta+\sigma\omega\delta^2+\omega\delta\alpha+\sigma^2\alpha\delta+\sigma\alpha^2=\sigma\alpha^2$
So if $\delta=0$ we have $\beta=0$ and $\gamma=1$ and we are in the $+$ situation unless $\sigma=0$.

$$\begin{aligned} a' &= \alpha a,\\ b' &= \lambda a+b+\mu d,\\ c' &= \nu a+c+\xi d,\\ d' &= \alpha d\end{aligned}$$

and
$$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ -\nu-\mu & \lambda+\omega\xi & 1\end{pmatrix}$$

Now try $\delta=\frac{2\sigma\alpha}{\rho^2\omega-\sigma^2-\omega}$
$\frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}=2\left(\rho-1\right)\frac{\sigma}{\rho^2\omega-2\rho\omega+\omega-\sigma^2}$
$\frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}=\frac{\rho^2\omega-2\rho\omega+\omega+\sigma^2}{\rho^2\omega-2\rho\omega+\omega-\sigma^2}$
$\rho^2\delta^2\omega-\sigma^2\delta^2-2\sigma\alpha\delta-\delta^2\omega=0$
$-\rho^2\omega\alpha\delta+\sigma\omega\delta^2+\omega\delta\alpha+\sigma^2\alpha\delta+\sigma\alpha^2$
$=-\sigma\alpha^2\frac{\rho^4\omega^2-2\rho^2\omega\sigma^2-2\rho^2\omega^2+\sigma^4-2\omega\sigma^2+\omega^2}{(\rho^2\omega-\sigma^2-\omega)^2}$
$\sigma(\delta^2\omega-\alpha^2)=-\sigma\alpha^2\frac{\rho^4\omega^2-2\rho^2\omega\sigma^2-2\rho^2\omega^2+\sigma^4-2\omega\sigma^2+\omega^2}{(\rho^2\omega-\sigma^2-\omega)^2}$
So we need $\sigma=0$, or we need to be in the $-$ situation.
The case $\sigma=0$ puts us back in the situation $\delta=0$, $\beta=0$ and $\gamma=1$.

$$\begin{pmatrix}\alpha & \delta\\ \omega\delta & \alpha\end{pmatrix}\begin{pmatrix}0 & 1\\ \rho & 0\end{pmatrix}\begin{pmatrix}(\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta)\\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}0 & 1\\ \rho\frac{-\alpha^2+\omega\delta^2}{-\alpha^2+\rho^2\delta^2\omega} & \omega\delta\alpha\frac{\rho^2-1}{-\alpha^2+\rho^2\delta^2\omega}\end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 0 & 1 \\ \rho\frac{-\alpha^2+\omega\delta^2}{-\alpha^2+\rho^2\delta^2\omega} & -\omega\delta\alpha\frac{\rho^2-1}{-\alpha^2+\rho^2\delta^2\omega} \end{pmatrix}$$

$\beta = \frac{\alpha\delta\rho+\alpha\delta}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2\rho+\alpha^2}{\alpha^2-\omega\delta^2}$

So to stabilize the pair ρ, σ when $\sigma = 0$ we need $\delta = 0$ or $\rho = \pm 1$.

So consider the case $\rho = \pm 1$ and $\sigma = 0$.

$\beta = \frac{\alpha\delta\rho+\alpha\delta}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2\rho+\alpha^2}{\alpha^2-\omega\delta^2}$, and other conditions are automatically satisøed.

$\beta = \frac{\alpha\delta+\alpha\delta}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2+\alpha^2}{\alpha^2-\omega\delta^2}$

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\beta = 0$, $\gamma = 1$

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \delta \\ \pm\omega\delta & \pm\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \begin{pmatrix} \pm(\alpha\gamma-\omega\beta\delta) & \pm(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}^{-1}.$$

$\beta = \frac{\alpha\delta\rho+\alpha\delta+\delta^2\sigma}{\alpha^2-\omega\delta^2}$, $\gamma = \frac{\omega\delta^2\rho+\alpha^2+\alpha\delta\sigma}{\alpha^2-\omega\delta^2}$.

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ -2\rho\delta\sigma & \omega\delta-\delta\sigma^2-\rho^2\omega\delta \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \rho & \sigma \end{pmatrix} \begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & -\omega\delta-2\alpha\sigma-\delta\sigma^2+\rho^2\omega\delta \end{pmatrix}$$

Now $\delta = 0$ gives $\beta = 0$, $\gamma = 1$.

$$\begin{aligned} a' &= \alpha a + \delta d, \\ b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\ c' &= \nu a + \beta b + \gamma c + \xi d, \\ d' &= \pm(\omega\delta a + \alpha d) \end{aligned}$$

and

$$\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma-\omega\beta\delta) & \pm(\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ \pm(\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & \pm(\gamma\lambda-\omega\beta\nu+\omega\gamma\xi-\omega\beta\nu) & \pm(\gamma^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ x & y & z \\ \rho & \sigma & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ (\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & (\gamma\lambda-\omega\beta\nu+\omega\gamma\xi-\omega\beta\nu) & (\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$= \alpha^{-1} \begin{pmatrix} 0 & \alpha & 0 \\ u+\mu\rho+w\nu+w\mu & \lambda+v+\mu\sigma-w\lambda-w\omega\xi & \alpha w \\ x+\xi\rho+z\nu+z\mu & \nu+y+\xi\sigma-z\lambda-z\omega\xi & \alpha z \\ \alpha\rho & \alpha\sigma & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ x & y & z \\ \rho & \sigma & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ -(\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & -(\gamma\lambda-\omega\beta\nu+\omega\gamma\xi-\omega\beta\nu) & -(\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$= \alpha^{-1} \begin{pmatrix} 0 & \alpha & 0 \\ u+\mu\rho+w\nu+w\mu & -\lambda-v-\mu\sigma+w\lambda+w\omega\xi & \alpha w \\ -x-\xi\rho-z\nu-z\mu & \nu+y+\xi\sigma-z\lambda-z\omega\xi & -\alpha z \\ \alpha\rho & -\alpha\sigma & 0 \end{pmatrix}$$

Get x component zero by clever choice of ξ. To keep it zero we require $\xi = \frac{-z\nu-z\mu}{\rho}$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ \lambda & \gamma & \omega\beta & \mu \\ \nu & \beta & \gamma & \xi \\ \omega\delta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ 0 & y & z \\ \rho & \sigma & 0 \end{pmatrix} \begin{pmatrix} (\alpha\gamma-\omega\beta\delta) & (\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ (\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & (\gamma\lambda-\omega\beta\nu+\omega\gamma\xi-\omega\beta\nu) & (\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$= \alpha^{-1}\rho^{-1} \begin{pmatrix} 0 & \alpha\rho & 0 \\ \rho\,(u+\mu\rho+w\nu+w\mu) & \lambda\rho+\rho v+\rho\mu\sigma-w\lambda\rho+w\omega z\nu+w\omega z\mu & \alpha\rho w \\ 0 & \nu\rho+y\rho-\sigma z\nu-\sigma z\mu-z\lambda\rho+\omega z^2\nu+\omega z^2\mu & \alpha\rho z \\ \alpha\rho^2 & \alpha\rho\sigma & 0 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & 0 & 0 & \delta \\ -\lambda & -\gamma & -\omega\beta & -\mu \\ \nu & \beta & \gamma & \xi \\ -\omega\delta & 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ u & v & w \\ 0 & y & z \\ \rho & \sigma & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} -(\alpha\gamma-\omega\beta\delta) & -(\omega\alpha\beta-\omega\gamma\delta) & 0 \\ \alpha\beta-\gamma\delta & \alpha\gamma-\omega\beta\delta & 0 \\ -(\beta\lambda-\gamma\nu+\omega\beta\xi-\gamma\mu) & -(\gamma\lambda-\omega\beta\nu+\omega\gamma\xi-\omega\beta\nu) & -(\gamma^2-\omega\beta^2) \end{pmatrix}^{-1}$$

$$= \alpha^{-1}\rho^{-1} \begin{pmatrix} 0 & \alpha\rho & 0 \\ \rho\,(u+\mu\rho+w\nu+w\mu) & -\lambda\rho-\rho v-\rho\mu\sigma+w\lambda\rho-w\omega z\nu-w\omega z\mu & \alpha\rho w \\ 0 & \nu\rho+y\rho-\sigma z\nu-\sigma z\mu-z\lambda\rho+\omega z^2\nu+\omega z^2\mu & -\alpha\rho z \\ \alpha\rho^2 & -\alpha\rho\sigma & 0 \end{pmatrix}$$

61.9 pa, pd span ba, ca

We start again here with another method.

$$
\begin{aligned}
a' &= \alpha a + \delta d, \\
b' &= \pm(\lambda a + \gamma b + \omega\beta c + \mu d), \\
c' &= \nu a + \beta b + \gamma c + \xi d, \\
d' &= \pm(\omega\delta a + \alpha d)
\end{aligned}
$$

and

$$
\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm(\alpha\gamma - \omega\beta\delta) & \pm(\omega\alpha\beta - \omega\gamma\delta) & 0 \\ \alpha\beta - \gamma\delta & \alpha\gamma - \omega\beta\delta & 0 \\ \pm(\beta\lambda - \gamma\nu + \omega\beta\xi - \gamma\mu) & \pm(\gamma\lambda - \omega\beta\mu + \omega\gamma\xi - \omega\beta\nu) & \pm(\gamma^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.
$$

Consider the case when pa, pd span ba, ca. Then we can choose λ, μ, ν, ξ so that pb', pc' are linearly dependent. And then we can choose $pb' = 0$. So we can assume that $pb = 0$.

First consider the case when pa, pd, pc are linearly dependent. Then we can assume that $pc = 0$, though this restricts us to a', b', c', d' with λ, μ, ν, ξ all zero. The evidence is that there are p orbits (though no proof yet). See dec4.16b.

Next consiser the case when pa, pd, pc are linearly independent, and $pb = 0$. This restricts us to a', b', c', d' with $\lambda = \beta = \mu = 0$.

$$
\begin{aligned}
a' &= \alpha a + \delta d, \\
b' &= \pm\gamma b, \\
c' &= \nu a + \gamma c + \xi d, \\
d' &= \pm(\omega\delta a + \alpha d)
\end{aligned}
$$

and

$$
\begin{pmatrix} b'a' \\ c'a' \\ c'b' \end{pmatrix} = \begin{pmatrix} \pm\alpha\gamma & \pm(-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \pm(-\gamma\nu) & \pm\omega\gamma\xi & \pm\gamma^2 \end{pmatrix} \begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.
$$

Let

$$
\begin{pmatrix} pa \\ pd \end{pmatrix} = A \begin{pmatrix} ba \\ ca \end{pmatrix}.
$$

Then

$$
A \mapsto \begin{pmatrix} \alpha & \delta \\ \pm\omega\delta & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm\alpha\gamma & \pm(-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{pmatrix}^{-1}.
$$

$$
\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \alpha\gamma & (-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{pmatrix}^{-1}
$$

$$
= \gamma^{-1}\left(-\alpha^2 + \omega\delta^2\right)^{-1} \begin{pmatrix} -\alpha^2 x - \alpha\delta z - \delta\alpha y - \delta^2 t & -\omega\delta\alpha x - \omega\delta^2 z - \alpha^2 y - \alpha\delta t \\ -\omega\delta\alpha x - \alpha^2 z - \omega\delta^2 y - \alpha\delta t & -\omega^2\delta^2 x - \omega\delta\alpha z - \alpha\omega\delta y - \alpha^2 t \end{pmatrix}
$$

and

$$
\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} -\alpha\gamma & -(-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{pmatrix}^{-1}
$$

$$= \gamma^{-1}\left(-\alpha^2+\omega\delta^2\right)^{-1}\begin{pmatrix} \alpha^2 x+\alpha\delta z+\delta\alpha y+\delta^2 t & -\omega\delta\alpha x-\omega\delta^2 z-\alpha^2 y-\alpha\delta t \\ -\omega\delta\alpha x-\alpha^2 z-\omega\delta^2 y-\alpha\delta t & \omega^2\delta^2 x+\omega\delta\alpha z+\alpha\omega\delta y+\alpha^2 t \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ \omega\delta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}-\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha\gamma & (-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\delta z-\gamma\alpha x+y\gamma\delta & \alpha y+\delta t+\gamma\omega\delta x-\gamma\alpha y \\ \omega\delta x+\alpha z-\gamma\alpha z+\gamma\delta t & \omega\delta y+\alpha t+z\omega\gamma\delta-\gamma\alpha t \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\gamma\alpha & \gamma\delta & \delta & 0 \\ \gamma\omega\delta & \alpha-\gamma\alpha & 0 & \delta \\ \omega\delta & 0 & \alpha-\gamma\alpha & \gamma\delta \\ 0 & \omega\delta & \omega\gamma\delta & \alpha-\gamma\alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ -\omega\delta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}-\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} -\alpha\gamma & -(-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\delta z+\gamma\alpha x+y\gamma\delta & \alpha y+\delta t-\gamma\omega\delta x-\gamma\alpha y \\ -\omega\delta x-\alpha z+\gamma\alpha z+\gamma\delta t & -\omega\delta y-\alpha t-z\omega\gamma\delta-\gamma\alpha t \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\gamma\alpha & \gamma\delta & \delta & 0 \\ -\gamma\omega\delta & \alpha-\gamma\alpha & 0 & \delta \\ -\omega\delta & 0 & -\alpha+\gamma\alpha & \gamma\delta \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\gamma\alpha \end{pmatrix}$$

First consider the nullity of

$$\begin{pmatrix} \alpha-\gamma\alpha & \gamma\delta & \delta & 0 \\ \gamma\omega\delta & \alpha-\gamma\alpha & 0 & \delta \\ \omega\delta & 0 & \alpha-\gamma\alpha & \gamma\delta \\ 0 & \omega\delta & \omega\gamma\delta & \alpha-\gamma\alpha \end{pmatrix}$$

Determinant: $(\gamma-1)^2\left(-\alpha^2+\omega\delta^2\right)\left(\omega\delta^2(\gamma+1)^2-\alpha^2(\gamma-1)^2\right)$

So the determinant is non-zero unless $\gamma=1$ or $\gamma=-1$ and $\alpha=0$.

First consider $\gamma=1$. We get

$$\begin{pmatrix} 0 & \delta & \delta & 0 \\ \omega\delta & 0 & 0 & \delta \\ \omega\delta & 0 & 0 & \delta \\ 0 & \omega\delta & \omega\delta & 0 \end{pmatrix}$$

which has rank 0 if $\delta=0$ and rank 2 if $\delta\neq 0$.

Now consider $\gamma=-1$ and $\alpha=0$. We get

$$= \begin{pmatrix} 0 & -\delta & \delta & 0 \\ \omega\delta & 0 & 0 & \delta \\ -\omega\delta & 0 & 0 & -\delta \\ 0 & -\omega\delta & \omega\delta & 0 \end{pmatrix}$$

which has rank 2 since $\delta\neq 0$.

So the contribution to Burnside's Lemma from these is $(p-1)(p^4-1)+p(p-1)(p^2-1)+(p-1)(p^2-1)+(p-1)(p^2-1)$.

Next consider the nullity of

$$\begin{pmatrix} \alpha+\gamma\alpha & \gamma\delta & \delta & 0 \\ -\gamma\omega\delta & \alpha-\gamma\alpha & 0 & \delta \\ -\omega\delta & 0 & -\alpha+\gamma\alpha & \gamma\delta \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\gamma\alpha \end{pmatrix}$$

Determinant: $(\gamma-1)^2(\gamma+1)^2\left(-\alpha^2+\delta^2\omega\right)^2$. So we have rank 4 unless $\gamma=\pm1$.

Try $\gamma=1$. We get

$$\begin{pmatrix} 2\alpha & \delta & \delta & 0 \\ -\omega\delta & 0 & 0 & \delta \\ -\omega\delta & 0 & 0 & \delta \\ 0 & -\omega\delta & -\omega\delta & -2\alpha \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha & \delta & 0 \\ -\omega\delta & 0 & \delta \\ 0 & -\omega\delta & -2\alpha \end{pmatrix}$$

Determinant: 0. So the rank is always 2.

Finally try $\gamma=-1$. We get

$$\begin{pmatrix} 0 & -\delta & \delta & 0 \\ \omega\delta & 2\alpha & 0 & \delta \\ -\omega\delta & 0 & -2\alpha & -\delta \\ 0 & -\omega\delta & \omega\delta & 0 \end{pmatrix}$$

Which has rank 2 in every case.

So the contribution to Burnside's Lemma is $2(p^2-1)(p^2-1)+(p-1)(p^2-1)$.

The total number of orbits is $(p^2+3p+6)/2$. However this includes the orbits when A has rank less than 2. There is one orbit with $A=0$. So consider orbits with A of rank 1. We can then take $pa=0$, so that the ørst row of A is zero. This restricts us to

$$A\mapsto\begin{pmatrix} \alpha & 0 \\ 0 & \pm\alpha \end{pmatrix} A \begin{pmatrix} \pm\alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}^{-1}$$

which transforms $\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}$ to $\gamma^{-1}\begin{pmatrix} 0 & 0 \\ z & \pm t \end{pmatrix}$. So we have $(p+3)/2$ orbit representatives $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$ $(0\le t\le(p-1)/2)$. So the number of rank 2 orbits is $\frac{1}{2}p^2+p+\frac{1}{2}$.

But what about pc?

This is where we start again

Let

$$\begin{pmatrix} pa \\ pd \\ pc \end{pmatrix}=\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

Then

$$\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\mapsto\begin{pmatrix} \alpha & \delta & 0 \\ \pm\omega\delta & \pm\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} \pm\alpha\gamma & \pm(-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \pm(-\gamma\nu) & \pm\omega\gamma\xi & \pm\gamma^2 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha & \delta & 0 \\ \omega\delta & \alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}-\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} \alpha\gamma & (-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ (-\gamma\nu) & \omega\gamma\xi & \gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u + \delta w - u\alpha\gamma + v\gamma\delta & \alpha v + \delta x + u\omega\gamma\delta - v\alpha\gamma & 0 \\ \omega\delta u + \alpha w - w\alpha\gamma + x\gamma\delta & \omega\delta v + \alpha x + w\omega\gamma\delta - x\alpha\gamma & 0 \\ \nu u + \xi w + \gamma y - y\alpha\gamma + z\gamma\delta + t\gamma\nu & \nu v + \xi x + \gamma z + y\omega\gamma\delta - z\alpha\gamma - t\omega\gamma\xi & \gamma t - t\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 & 0 \\ \omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 & 0 \\ \omega\delta & 0 & \alpha-\alpha\gamma & \gamma\delta & 0 & 0 & 0 \\ 0 & \omega\delta & \omega\gamma\delta & \alpha-\alpha\gamma & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & \gamma-\alpha\gamma & \gamma\delta & \gamma\nu \\ 0 & \nu & 0 & \xi & \omega\gamma\delta & \gamma-\alpha\gamma & -\omega\gamma\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma-\gamma^2 \end{pmatrix}$$

Determinant: $\gamma^3(\gamma-1)^3\left(-(\alpha-1)^2+\delta^2\omega\right)(\delta^2\omega-\alpha^2)\left(\delta^2\omega(\gamma+1)^2-\alpha^2(\gamma-1)^2\right)$

So the determinant has rank 7 unless $\gamma = 1$, or $\alpha = 1$ and $\delta = 0$, or $\gamma = -1$ and $\alpha = 0$.

First consider the case $\gamma = 1$. We then have

$$\begin{pmatrix} 0 & \delta & \delta & 0 & 0 & 0 & 0 \\ \omega\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ \omega\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & \omega\delta & \omega\delta & 0 & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & 1-\alpha & \delta & \nu \\ 0 & \nu & 0 & \xi & \omega\delta & 1-\alpha & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can reduce this to

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & -\xi & \xi & 0 & 1-\alpha & \delta & \nu \\ -\omega\xi & \nu & 0 & \xi & \omega\delta & 1-\alpha & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\delta \neq 0$ this has rank 4, and if $\delta = 0$ it has rank 2 unless $\alpha = 1$ and $\nu = \xi = 0$, in which case it has rank 0.

Next consider the case $\alpha = 1$ and $\delta = 0$ (and $\gamma \neq 1$). We then have

$$= \begin{pmatrix} 1-\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\gamma & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & 0 & 0 & \gamma\nu \\ 0 & \nu & 0 & \xi & 0 & 0 & -\omega\gamma\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma-\gamma^2 \end{pmatrix}$$

which has rank 5.

And ønally consider the case $\gamma = -1$ and $\alpha = 0$. We obtain

$$\begin{pmatrix} 0 & -\delta & \delta & 0 & 0 & 0 & 0 \\ -\omega\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ \omega\delta & 0 & 0 & -\delta & 0 & 0 & 0 \\ 0 & \omega\delta & -\omega\delta & 0 & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & -1 & -\delta & -\nu \\ 0 & \nu & 0 & \xi & -\omega\delta & -1 & \omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Note that $\delta \neq 0$. So the rank is 5.

So the contribution to Burnside's Lemma from matrices of the ørst type is $(p-1)p^3(p^3-1)+((p-1)p^2-1)(p^5-1)+(p^7-1)+(p-2)p^2(p^2-1)$
$+(p-1)p^2(p^2-1)+(p^2-1)(p-1)p^2$
$=p^7-5p^4-p^6-3p^3+p^8+5p^2+2p^5$
$=p^2\,(p+1)\left(p^3+2p^2+2p+5\right)(p-1)^2$

$$\begin{pmatrix} \alpha & \delta & 0 \\ -\omega\delta & -\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}-\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} -\alpha\gamma & \omega\gamma\delta & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \gamma\nu & -\omega\gamma\xi & -\gamma^2 \end{pmatrix}$$

$$=\begin{pmatrix} \alpha u+\delta w+u\alpha\gamma+v\gamma\delta & \alpha v+\delta x-u\omega\gamma\delta-v\alpha\gamma & 0 \\ -\omega\delta u-\alpha w+w\alpha\gamma+x\gamma\delta & -\omega\delta v-\alpha x-w\omega\gamma\delta-x\alpha\gamma & 0 \\ \nu u+\xi w+\gamma y+y\alpha\gamma+z\gamma\delta-t\gamma\nu & \nu v+\xi x+\gamma z-y\omega\gamma\delta-z\alpha\gamma+t\omega\gamma\xi & \gamma t+t\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta & 0 & 0 & 0 \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & \gamma+\alpha\gamma & \gamma\delta & -\gamma\nu \\ 0 & \nu & 0 & \xi & -\omega\gamma\delta & \gamma-\alpha\gamma & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$$

Determinant: $-\gamma^3\left(\alpha^2-\omega\delta^2-1\right)\left(\alpha^2-\omega\delta^2\right)^2(\gamma-1)^2(\gamma+1)^3$

So the rank is 7 unless $\gamma=\pm1$ or $\alpha^2-\omega\delta^2=1$.

Let $\gamma=1$. We get

$$\begin{pmatrix} 2\alpha & \delta & \delta & 0 & 0 & 0 & 0 \\ -\omega\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & -\omega\delta & -\omega\delta & -2\alpha & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & 1+\alpha & \delta & -\nu \\ 0 & \nu & 0 & \xi & -\omega\delta & 1-\alpha & \omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\alpha & 0 & 0 \\ \nu & -\xi & \xi & 0 & 1+\alpha & \delta \\ \omega\xi & \nu & 0 & \xi & -\omega\delta & 1-\alpha \end{pmatrix}$$

First suppose that $\delta \neq 0$. Then we have

$$\begin{pmatrix} 2\alpha & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & -\xi & \xi & 1+\alpha & \delta \\ \omega\xi & \nu & 0 & -\omega\delta & 1-\alpha \end{pmatrix}\begin{pmatrix} \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2\alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha-1 & \delta \end{pmatrix} \begin{pmatrix} 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu\delta-2\alpha\xi & -\xi & \xi & 1+\alpha & \delta \\ \omega\xi\delta & \nu & 0 & -\omega\delta & -\alpha+1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu\delta-2\alpha\xi & -\xi & \xi & 1+\alpha & \delta \\ \alpha\nu\delta-2\alpha^2\xi-\nu\delta+2\alpha\xi+\delta^2\omega\xi & -\alpha\xi+\xi+\nu\delta & (\alpha-1)\,\xi & \alpha^2-1-\omega\delta^2 & 0 \end{pmatrix}$$

So the rank is 5 unless

$$\alpha\nu\delta-2\alpha^2\xi-\nu\delta+2\alpha\xi+\delta^2\omega\xi=-\alpha\xi+\xi+\nu\delta=\alpha^2-1-\omega\delta^2=0,$$

when we get rank 4.

Now if $\alpha^2-1-\omega\delta^2$ then
$\alpha\nu\delta-2\alpha^2\xi-\nu\delta+2\alpha\xi+\delta^2\omega\xi=$
$\alpha\nu\delta-2\alpha^2\xi-\nu\delta+2\alpha\xi+\delta^2\omega\xi+(\alpha^2-1-\omega\delta^2)\xi=-(\alpha-1)(\alpha\xi-\xi-\nu\delta)$
So this situation arises once for every solution of $\alpha^2-1-\omega\delta^2=0$.
If $\gamma=1$ and $\delta=0$ then we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\xi & \xi & 1+\alpha & 0 \\ \nu & 0 & 0 & 1-\alpha \end{pmatrix}$$

So we get rank 5 unless $\alpha=1$ and $\nu=0$ or $\alpha=-1$ and $\xi=0$, when we get rank 4. This situation arises p times for every solution of $\alpha^2-\omega\delta^2=1$.

Next consider the case when $\gamma=-1$. Then we have

$$\begin{pmatrix} 0 & -\delta & \delta & 0 & 0 & 0 & 0 \\ \omega\delta & 2\alpha & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & -2\alpha & -\delta & 0 & 0 & 0 \\ 0 & -\omega\delta & \omega\delta & 0 & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & -1-\alpha & -\delta & \nu \\ 0 & \nu & 0 & \xi & \omega\delta & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\delta & \delta & 0 & 0 & 0 & 0 \\ \omega\delta & 2\alpha & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & -2\alpha & -\delta & 0 & 0 & 0 \\ 0 & -\omega\delta & \omega\delta & 0 & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & -1-\alpha & -\delta & \nu \\ 0 & \nu & 0 & \xi & \omega\delta & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \delta & 0 & 0 & 0 & 0 \\ \omega\delta & 2\alpha & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & -2\alpha & -2\alpha & -\delta & 0 & 0 & 0 \\ 0 & 0 & \omega\delta & 0 & 0 & 0 & 0 \\ \nu & \xi & \xi & 0 & -1-\alpha & -\delta & \nu \\ 0 & \nu & 0 & \xi & \omega\delta & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 & 0 & 0 \\ \omega\delta & 2\alpha & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & -2\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega\delta & 0 & 0 & 0 & 0 \\ \nu & \xi & \xi & 0 & -1-\alpha & -\delta & \nu \\ 0 & \nu & 0 & \xi & \omega\delta & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

First suppose that $\delta \neq 0$. Then we have rank 4 unless

$$\nu\alpha - \nu - \omega\xi\delta = -\alpha\xi - \xi + \nu\delta = -\alpha^2 + 1 + \omega\delta^2 = 0$$

when we have rank 3. For every solution of $-\alpha^2 + 1 + \omega\delta^2 = 0$ this situation arises p times.

$$\begin{pmatrix} \omega\delta & 2\alpha & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & 0 & -1-\alpha & -\delta & \nu \\ 0 & \nu & \xi & \omega\delta & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & 0 & -1-\alpha & -\delta & 0 \\ -\omega\xi & \nu - 2\alpha\delta^{-1}\xi & \xi & \omega\delta & \alpha-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (\alpha-1) & \delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & -1-\alpha & -\delta & 0 \\ -\omega\xi & \nu - 2\alpha\delta^{-1}\xi & \omega\delta & \alpha-1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & -1-\alpha & -\delta & 0 \\ \nu\alpha - \nu - \omega\xi\delta & -\alpha\xi - \xi + \nu\delta & -\alpha^2 + 1 + \omega\delta^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

On the other hand, if $\delta = 0$ then we have rank 4 unless $\alpha = 1$ and $\xi = 0$ or $\alpha = -1$ and $\nu = 0$, when we have rank 3. So again, this situation arises p times for every solution of $-\alpha^2 + 1 + \omega\delta^2 = 0$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & \xi & 0 & -1-\alpha & 0 & \nu \\ 0 & \nu & 0 & \xi & 0 & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & 0 & -1-\alpha & 0 & \nu \\ 0 & \xi & 0 & \alpha-1 & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Finally consider the case when $\alpha^2 - \omega\delta^2 = 1$, when $\gamma \neq \pm 1$.

$$\left(\begin{array}{ccccccc} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta & 0 & 0 & 0 \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & \gamma+\alpha\gamma & \gamma\delta & -\gamma\nu \\ 0 & \nu & 0 & \xi & -\omega\gamma\delta & \gamma-\alpha\gamma & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{array}\right)$$

$$\left(\begin{array}{cccccc} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta & 0 & 0 \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma & 0 & 0 \\ \nu & 0 & \xi & 0 & 1+\alpha & \delta \\ 0 & \nu & 0 & \xi & -\omega\delta & 1-\alpha \end{array}\right)$$

$$\left(\begin{array}{cccc} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma \end{array}\right)$$

has determinant: $\left(\alpha^2-\omega\delta^2\right)^2(\gamma-1)^2(\gamma+1)^2$, which is not zero. So the rank is 6.

Note that there are $p+1$ pairs α, β which give $\alpha^2-\omega\delta^2=1$.

So the contribution to Burnside's Lemma from matrices of the second type is
$((p^2-1)p^2-p(p+1))(p^2-1)+p(p+1)(p^3-1)$
$+((p^2-1)p^2-p(p+1))(p^3-1)+p(p+1)(p^4-1)$
$+(p-3)(p+1)p^2(p-1)+(p^2-1)(p-1)p^2$
$=p^2(p+1)\left(p^2+3p+6\right)(p-1)^2$.

So the contribution from both types is
$p^2(p+1)\left(p^3+2p^2+2p+5\right)(p-1)^2$
$+p^2(p+1)\left(p^2+3p+6\right)(p-1)^2$
$=p^2(p+1)\left(p^3+3p^2+5p+11\right)(p-1)^2$.

So the total number of orbits is $(p^3+3p^2+5p+11)/2$.

Now we need to count the orbits when pc is a linear combination of ba, ca. Let

$$\left(\begin{array}{c} pa \\ pd \\ pc \end{array}\right) = \left(\begin{array}{cc} u & v \\ w & x \\ y & z \end{array}\right)\left(\begin{array}{c} ba \\ ca \end{array}\right)$$

$$\left(\begin{array}{ccc} \alpha & \delta & 0 \\ \omega\delta & \alpha & 0 \\ \nu & \xi & \gamma \end{array}\right)\left(\begin{array}{cc} u & v \\ w & x \\ y & z \end{array}\right) - \left(\begin{array}{cc} u & v \\ w & x \\ y & z \end{array}\right)\left(\begin{array}{cc} \alpha\gamma & (-\omega\gamma\delta) \\ -\gamma\delta & \alpha\gamma \end{array}\right)$$

$$\begin{pmatrix} \alpha u+\delta w-u\alpha\gamma+v\gamma\delta & \alpha v+\delta x+u\omega\gamma\delta-v\alpha\gamma \\ \omega\delta u+\alpha w-w\alpha\gamma+x\gamma\delta & \omega\delta v+\alpha x+w\omega\gamma\delta-x\alpha\gamma \\ \nu u+\xi w+\gamma y-y\alpha\gamma+z\gamma\delta & \nu v+\xi x+\gamma z+y\omega\gamma\delta-z\alpha\gamma \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 \\ \omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 \\ \omega\delta & 0 & \alpha-\alpha\gamma & \gamma\delta & 0 & 0 \\ 0 & \omega\delta & \omega\gamma\delta & \alpha-\alpha\gamma & 0 & 0 \\ \nu & 0 & \xi & 0 & \gamma-\alpha\gamma & \gamma\delta \\ 0 & \nu & 0 & \xi & \omega\gamma\delta & \gamma-\alpha\gamma \end{pmatrix}$$

Determinant: $\gamma^2\left(-\omega\delta^2+\alpha^2\right)\left(-\omega\delta^2+(\alpha-1)^2\right)(\gamma-1)^2\left(-\omega\delta^2(\gamma+1)^2+\alpha^2(\gamma-1)^2\right)$

So the rank is 6 unless $\gamma=1$ or $\alpha=1$ and $\delta=0$ or $\gamma=-1$ and $\alpha=0$.

First consider the case when $\gamma=1$. We then have

$$\begin{pmatrix} 0 & \delta & \delta & 0 & 0 & 0 \\ \omega\delta & 0 & 0 & \delta & 0 & 0 \\ \omega\delta & 0 & 0 & \delta & 0 & 0 \\ 0 & \omega\delta & \omega\delta & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & 1-\alpha & \delta \\ 0 & \nu & 0 & \xi & \omega\delta & 1-\alpha \end{pmatrix}$$

If $\delta=0$ this has rank 2 unless $\alpha=1$ and $\nu=\xi=0$ in which case it has rank 0. If $\delta\neq 0$ then we have

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \nu & -\xi & \xi & 0 & 1-\alpha & \delta \\ -\omega\xi & \nu & 0 & \xi & \omega\delta & 1-\alpha \end{pmatrix}$$

which has rank 4.

Next consider the case when $\alpha=1$ and $\delta=0$ and $\gamma\neq 1$. Then we have

$$\begin{pmatrix} 1-\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\gamma & 0 & 0 \\ \nu & 0 & \xi & 0 & 0 & 0 \\ 0 & \nu & 0 & \xi & 0 & 0 \end{pmatrix}$$

which has rank 4.

And now consider the case when $\gamma=-1$ and $\alpha=0$. Then we have

$$\begin{pmatrix} 0 & -\delta & \delta & 0 & 0 & 0 \\ -\omega\delta & 0 & 0 & \delta & 0 & 0 \\ \omega\delta & 0 & 0 & -\delta & 0 & 0 \\ 0 & \omega\delta & -\omega\delta & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & -1 & -\delta \\ 0 & \nu & 0 & \xi & -\omega\delta & -1 \end{pmatrix}$$

which has rank 4 since $\delta\neq 0$.

The contribution to Burnside's Lemma for matrices of this sort is

$$
\begin{aligned}
&(p^3-p^2-1)(p^4-1)+(p^6-1)+p(p-1)p^2(p^2-1)\\
&+(p-2)p^2(p^2-1)+(p-1)p^2(p^2-1)+(p^2-1)(p-1)p^2\\
&=p^2\left(p+1\right)\left(p^2+2p+5\right)\left(p-1\right)^2
\end{aligned}
$$

$$
\begin{pmatrix} \alpha & \delta & 0 \\ -\omega\delta & -\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}
\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix}
-
\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix}
\begin{pmatrix} -\alpha\gamma & \omega\gamma\delta \\ -\gamma\delta & \alpha\gamma \end{pmatrix}
$$

$$
\begin{pmatrix}
\alpha u+\delta w+u\alpha\gamma+v\gamma\delta & \alpha v+\delta x-u\omega\gamma\delta-v\alpha\gamma \\
-\omega\delta u-\alpha w+w\alpha\gamma+x\gamma\delta & -\omega\delta v-\alpha x-w\omega\gamma\delta-x\alpha\gamma \\
\nu u+\xi w+\gamma y+y\alpha\gamma+z\gamma\delta & \nu v+\xi x+\gamma z-y\omega\gamma\delta-z\alpha\gamma
\end{pmatrix}
$$

$$
\begin{pmatrix}
\alpha+\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 \\
-\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 \\
-\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta & 0 & 0 \\
0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma & 0 & 0 \\
\nu & 0 & \xi & 0 & \gamma+\alpha\gamma & \gamma\delta \\
0 & \nu & 0 & \xi & -\omega\gamma\delta & \gamma-\alpha\gamma
\end{pmatrix}
$$

Determinant: $-\gamma^2\left(\alpha^2-1-\omega\delta^2\right)\left(-\omega\delta^2+\alpha^2\right)^2\left(\gamma-1\right)^2\left(\gamma+1\right)^2$.

So the rank is 6 unless $\gamma=\pm1$ or $\alpha^2-\omega\delta^2=1$.

Consider the case when $\gamma=1$. Then we have

$$
\begin{pmatrix}
2\alpha & \delta & \delta & 0 & 0 & 0 \\
-\omega\delta & 0 & 0 & \delta & 0 & 0 \\
-\omega\delta & 0 & 0 & \delta & 0 & 0 \\
0 & -\omega\delta & -\omega\delta & -2\alpha & 0 & 0 \\
\nu & 0 & \xi & 0 & \alpha+1 & \delta \\
0 & \nu & 0 & \xi & -\omega\delta & 1-\alpha
\end{pmatrix}
$$

$$
\begin{pmatrix}
2\alpha & 0 & \delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\nu & -\xi & \xi & \alpha+1 & \delta \\
\omega\xi & \nu & 0 & -\omega\delta & 1-\alpha
\end{pmatrix}
$$

If $\delta=0$ then this has rank 4 unless $\alpha=1$ and $\nu=0$ or $\alpha=-1$ and $\xi=0$. And if $\delta\neq0$ then we again have rank 4 unless

$$
\delta\nu\alpha-\delta\nu+\xi\omega\delta^2-2\alpha^2\xi+2\xi\alpha=-\xi\alpha+\xi+\delta\nu=\alpha^2-1-\omega\delta^2=0
$$

when we get rank 3. Now if $\alpha^2-1-\omega\delta^2=0$ then

$$
\begin{aligned}
&\delta\nu\alpha-\delta\nu+\xi\omega\delta^2-2\alpha^2\xi+2\xi\alpha\\
=\;&\delta\nu\alpha-\delta\nu+\xi\omega\delta^2-2\alpha^2\xi+2\xi\alpha-(-\xi\alpha+\xi+\delta\nu)\alpha+(\alpha^2-1-\omega\delta^2)\xi\\
=\;&-\delta\nu+\xi\alpha-\xi
\end{aligned}
$$

So (whether or not $\delta=0$) we get p situations where we get rank three for each of the $p+1$ solutions of $\alpha^2-1-\omega\delta^2=0$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \alpha-1 & \delta \end{pmatrix} \begin{pmatrix} 2\alpha & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & -\xi & \xi & \alpha+1 & \delta \\ \omega\xi & \nu & 0 & -\omega\delta & 1-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & -\xi & \xi & \alpha+1 & \delta \\ \nu\alpha-\nu+\xi\omega\delta & -\xi\alpha+\xi+\delta\nu & (\alpha-1)\,\xi & \alpha^2-1-\omega\delta^2 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2\alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \delta\nu-2\xi\alpha & -\xi & \xi & \alpha+1 & \delta \\ \delta\nu\alpha-\delta\nu+\xi\omega\delta^2-2\alpha^2\xi+2\xi\alpha & -\xi\alpha+\xi+\delta\nu & (\alpha-1)\,\xi & \alpha^2-1-\omega\delta^2 & 0 \end{pmatrix}$$

Now consider the case when $\gamma=-1$. Then we get

$$\begin{pmatrix} 0 & -\delta & \delta & 0 & 0 & 0 \\ \omega\delta & 2\alpha & 0 & \delta & 0 & 0 \\ -\omega\delta & 0 & -2\alpha & -\delta & 0 & 0 \\ 0 & -\omega\delta & \omega\delta & 0 & 0 & 0 \\ \nu & 0 & \xi & 0 & -1-\alpha & -\delta \\ 0 & \nu & 0 & \xi & \omega\delta & \alpha-1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2\alpha & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \nu & \xi & 0 & -1-\alpha & -\delta \\ -\omega\xi & \nu & \xi & \omega\delta & \alpha-1 \end{pmatrix}$$

This has rank 4 except in $p(p+1)$ situations when it has rank 3.

Finally consider the case when $\gamma\neq\pm1$ and $\alpha^2-\omega\delta^2=1$.

$$\begin{pmatrix} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 & 0 & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta & 0 & 0 \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta & 0 & 0 \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma & 0 & 0 \\ \nu & 0 & \xi & 0 & \gamma+\alpha\gamma & \gamma\delta \\ 0 & \nu & 0 & \xi & -\omega\gamma\delta & \gamma-\alpha\gamma \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha\gamma & \gamma\delta & \delta & 0 \\ -\omega\gamma\delta & \alpha-\alpha\gamma & 0 & \delta \\ -\omega\delta & 0 & -\alpha+\alpha\gamma & \gamma\delta \\ 0 & -\omega\delta & -\omega\gamma\delta & -\alpha-\alpha\gamma \end{pmatrix}$$

Determinant: $\left(-\omega\delta^2+\alpha^2\right)^2(\gamma-1)^2(\gamma+1)^2$. So the matrix has rank 6 unless $\alpha^2-\omega\delta^2=1$, when it has rank 5.

The contribution to Burnside's Lemma from matrices of this form is
$2((p^2-1)p^2-p(p+1))(p^2-1)+2p(p+1)(p^3-1)$

$+(p-3)(p+1)p^2(p-1)+(p^2-1)(p-1)p^2$
$= 2p^2\,(p+3)\,(p+1)\,(p-1)^2\,.$
So the contribution to Burnside's Lemma from both types of matrices is
$p^2\,(p+1)\left(p^2+2p+5\right)(p-1)^2 + 2p^2\,(p+3)\,(p+1)\,(p-1)^2$
$= p^2\,(p+1)\left(p^2+4p+11\right)(p-1)^2$
So the number of orbits is $(p^2+4p+11)/2$.

Next we need to count the orbits when pa, pd lie in the span of ba, ca and $pb=0$, pc does not lie in the span of ba, ca, but pa, pd are not linearly independant. If they are not both zero then we can assume that $pa=0$, but then we need $\delta=0$.

$$\begin{pmatrix} \alpha & \delta & 0 \\ \omega\delta & \alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} \alpha\gamma & (-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ (-\gamma\nu) & \omega\gamma\xi & \gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta & 0 \\ -\omega\delta & -\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} -\alpha\gamma & \omega\gamma\delta & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \gamma\nu & -\omega\gamma\xi & -\gamma^2 \end{pmatrix}$$

We want these two above with $u=v=0$ and $\delta=0$.

$$\begin{pmatrix} \alpha & 0 \\ \xi & \gamma \end{pmatrix}\begin{pmatrix} w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} \alpha\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ (-\gamma\nu) & \omega\gamma\xi & \gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha w - w\alpha\gamma & \alpha x - x\alpha\gamma & 0 \\ \xi w + \gamma y - y\alpha\gamma + t\nu\gamma & \xi x + \gamma z - z\alpha\gamma - t\omega\gamma\xi & \gamma t - t\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha\gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha-\alpha\gamma & 0 & 0 & 0 \\ \xi & 0 & \gamma-\alpha\gamma & 0 & \nu\gamma \\ 0 & \xi & 0 & \gamma-\alpha\gamma & -\omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma-\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha & 0 \\ \xi & \gamma \end{pmatrix}\begin{pmatrix} w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} -\alpha\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ \gamma\nu & -\omega\gamma\xi & -\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha w + w\alpha\gamma & -\alpha x - x\alpha\gamma & 0 \\ \xi w + \gamma y + y\alpha\gamma - t\nu\gamma & \xi x + \gamma z - z\alpha\gamma + t\omega\gamma\xi & \gamma t + t\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha+\alpha\gamma & 0 & 0 & 0 & 0 \\ 0 & -\alpha-\alpha\gamma & 0 & 0 & 0 \\ \xi & 0 & \gamma+\alpha\gamma & 0 & -\nu\gamma \\ 0 & \xi & 0 & \gamma-\alpha\gamma & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$$

Now try the øst of these.

$$\begin{pmatrix} \alpha-\alpha\gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha-\alpha\gamma & 0 & 0 & 0 \\ \xi & 0 & \gamma-\alpha\gamma & 0 & \nu\gamma \\ 0 & \xi & 0 & \gamma-\alpha\gamma & -\omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma-\gamma^2 \end{pmatrix}$$

Determinant: $-\alpha^2 (\gamma-1)^3 \gamma^3 (-1+\alpha)^2$
So rank is øve unless $\gamma = 1$ or $\alpha = 1$. (Note that $\alpha \neq 0$ since $\delta = 0$.)
Try $\gamma = 1$. We get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \xi & 0 & 1-\alpha & 0 & \nu \\ 0 & \xi & 0 & 1-\alpha & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This has rank 2 unless $\xi = 0$ and $\alpha = 1$. It then has rank 1 if $\nu \neq 0$ and rank 0 if $\nu = 0$.
Finally try $\alpha = 1$, $\gamma \neq 1$ We get

$$\begin{pmatrix} 1-\gamma & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 & \nu\gamma \\ 0 & \xi & 0 & 0 & -\omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma-\gamma^2 \end{pmatrix}$$

which has rank 3.
So the contribution to Burnside's Lemma from matrices of this sort is
$((p-1)p-1)p(p^3-1)+(p-1)(p^4-1)+(p^5-1)$
$+(p-2)p^2(p^2-1)+(p-1)^2p^2$
$= p^2 (p+2)^2 (p-1)^2$
Next try

$$\begin{pmatrix} -\alpha+\alpha\gamma & 0 & 0 & 0 & 0 \\ 0 & -\alpha-\alpha\gamma & 0 & 0 & 0 \\ \xi & 0 & \gamma+\alpha\gamma & 0 & -\nu\gamma \\ 0 & \xi & 0 & \gamma-\alpha\gamma & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$$

Determinant: $\alpha^2 (\gamma-1)(1+\gamma)^2 \gamma^3 (1+\alpha)(-1+\alpha)$
So we get rank 5 unless $\gamma = \pm 1$, or $\alpha = \pm 1$.
Try $\gamma = 1$. We get

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2\alpha & 0 & 0 & 0 \\ \xi & 0 & 1+\alpha & 0 & -\nu \\ 0 & \xi & 0 & 1-\alpha & \omega\xi \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \xi & 1+\alpha & 0 \\ 0 & 0 & 1-\alpha \\ 0 & 0 & 0 \end{pmatrix}$$

If $\alpha \neq \pm 1$ this has rank 4. If $\alpha = 1$ it has rank 3.

$$\begin{pmatrix} 0 & 0 & 0 \\ \xi & 1+\alpha & 0 \\ 0 & 0 & 1-\alpha \end{pmatrix}$$

If $\alpha = -1$ it has rank 4 if $\xi \neq 0$ and rank 3 if $\xi = 0$.

Now try $\gamma = -1$. We get

$$\begin{pmatrix} -2\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \xi & 0 & -1-\alpha & 0 & \nu \\ 0 & \xi & 0 & -1+\alpha & -\omega\xi \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\alpha \neq \pm 1$ this has rank 3.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1-\alpha & 0 & \nu \\ \xi & 0 & -1+\alpha & -\omega\xi \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\alpha = 1$ it has rank 3 if $\xi \neq 0$ and rank 2 if $\xi = 0$. And if $\alpha = -1$ it has rank 3 if $\nu \neq 0$ and rank 2 if $\nu = 0$.

Next $\alpha = 1$, $\gamma \neq \pm 1$. We get

$$\begin{pmatrix} \gamma-1 & 0 & 0 & 0 & 0 \\ 0 & -1-\gamma & 0 & 0 & 0 \\ \xi & 0 & 2\gamma & 0 & -\nu\gamma \\ 0 & \xi & 0 & 0 & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$$

which has rank 4.

Finally $\alpha = -1$, $\gamma \neq \pm 1$. We get

$$\begin{pmatrix} 1-\gamma & 0 & 0 & 0 & 0 \\ 0 & 1+\gamma & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 & -\nu\gamma \\ 0 & \xi & 0 & 2\gamma & \omega\gamma\xi \\ 0 & 0 & 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$$

which has rank 4.

So the contribution to Burnside's Lemma from matrices of this form is
$(p-3)p^2(p-1) + p^2(p^2-1) + (p-1)p(p-1) + p(p^2-1)$
$+(p-3)p^2(p^2-1) + 2(p-1)p(p^2-1) + 2p(p^3-1)$
$+2(p-3)p^2(p-1) + (p-1)^2p^2$
$= p^2\,(p+8)\,(p-1)^2$

So the contribution to Burnside's Lemma from both types of matrices is
$p^2\,(p+2)^2\,(p-1)^2 + p^2\,(p+8)\,(p-1)^2$
$= p^2\,\left(p^2+5p+12\right)(p-1)^2$
and the number of orbits is $(p^2+5p+12)/2$.

Next we need to count the orbits when pa, pd lie in the span of ba, ca and $pb = 0$, pc does lie in the span of ba, ca, but pa, pd are not linearly independant. If they are not both zero then we can assume that $pa = 0$, but then we need $\delta = 0$.

$$\begin{pmatrix} \alpha & \delta & 0 \\ \omega\delta & \alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix} \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} \begin{pmatrix} \alpha\gamma & (-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ (-\gamma\nu) & \omega\gamma\xi & \gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta & 0 \\ -\omega\delta & -\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix} \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} - \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} \begin{pmatrix} -\alpha\gamma & \omega\gamma\delta & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \gamma\nu & -\omega\gamma\xi & -\gamma^2 \end{pmatrix}$$

We want these two above with $u = v = 0$ and $t = 0$ and $\delta = 0$.

$$\begin{pmatrix} \alpha & 0 \\ \xi & \gamma \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} - \begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} \alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha w - w\alpha\gamma & \alpha x - x\alpha\gamma \\ \xi w + \gamma y - y\alpha\gamma & \xi x + \gamma z - z\alpha\gamma \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \alpha\gamma & 0 & 0 & 0 \\ 0 & \alpha - \alpha\gamma & 0 & 0 \\ \xi & 0 & \gamma - \alpha\gamma & 0 \\ 0 & \xi & 0 & \gamma - \alpha\gamma \end{pmatrix}$$

$$\begin{pmatrix} -\alpha & 0 \\ \xi & \gamma \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} - \begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} -\alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha w + w\alpha\gamma & -\alpha x - x\alpha\gamma \\ \xi w + \gamma y + y\alpha\gamma & \xi x + \gamma z - z\alpha\gamma \end{pmatrix}$$

$$\begin{pmatrix} -\alpha + \alpha\gamma & 0 & 0 & 0 \\ 0 & -\alpha - \alpha\gamma & 0 & 0 \\ \xi & 0 & \gamma + \alpha\gamma & 0 \\ 0 & \xi & 0 & \gamma - \alpha\gamma \end{pmatrix}$$

Try the ørst one

$$\begin{pmatrix} \alpha - \alpha\gamma & 0 & 0 & 0 \\ 0 & \alpha - \alpha\gamma & 0 & 0 \\ \xi & 0 & \gamma - \alpha\gamma & 0 \\ 0 & \xi & 0 & \gamma - \alpha\gamma \end{pmatrix}$$

Determinant: $\alpha^2 (\gamma - 1)^2 \gamma^2 (-1 + \alpha)^2$
So the rank is 4 unless $\gamma = 1$ or $\alpha = 1$.
If $\gamma = 1$ we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 1 - \alpha & 0 \\ 0 & \xi & 0 & 1 - \alpha \end{pmatrix}$$

which has rank 2 unless $\alpha = 1$ and $\xi = 0$, in which case it has rank 0.
If $\alpha = 1$ and $\gamma \neq 1$ we have

$$\begin{pmatrix} 1 - \gamma & 0 & 0 & 0 \\ 0 & 1 - \gamma & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 \end{pmatrix}$$

which has rank 2.
The contribution to Burnside's Lemma from this matrix is
$((p - 1)p - 1)(p^2 - 1) + (p^4 - 1) + (p - 2)p(p^2 - 1) + (p - 1)^2 p$
$= p(3p + 4)(p - 1)^2$

Now look at the second matrix.

$$\begin{pmatrix} -\alpha+\alpha\gamma & 0 & 0 & 0 \\ 0 & -\alpha-\alpha\gamma & 0 & 0 \\ \xi & 0 & \gamma+\alpha\gamma & 0 \\ 0 & \xi & 0 & \gamma-\alpha\gamma \end{pmatrix}$$

Determinant: $\alpha^2(\gamma-1)(1+\gamma)\gamma^2(1+\alpha)(-1+\alpha)$
So the matrix has rank 4 unless $\gamma=\pm 1$ or $\alpha=\pm 1$.
Try $\gamma=1$. We have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\alpha & 0 & 0 \\ \xi & 0 & 1+\alpha & 0 \\ 0 & \xi & 0 & 1-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ \xi & 1+\alpha & 0 \\ 0 & 0 & 1-\alpha \end{pmatrix}$$

This has rank 3 if $\alpha\neq\pm 1$. Rank 2 if $\alpha=1$ and rank 3 if $\alpha=-1$, $\xi\neq 0$ and rank 2 if $\alpha=-1$, $\xi=0$.

Try $\gamma=-1$. We have

$$\begin{pmatrix} -2\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & -1-\alpha & 0 \\ 0 & \xi & 0 & -1+\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1-\alpha & 0 \\ \xi & 0 & -1+\alpha \end{pmatrix}$$

This has rank 3 if $\alpha\neq\pm 1$. Rank 2 if $\alpha=-1$ and rank 3 if $\alpha=1$, $\xi\neq 0$ and rank 2 if $\alpha=1$, $\xi=0$.

Now $\alpha=1$, $\gamma\neq\pm 1$. This gives

$$\begin{pmatrix} \gamma-1 & 0 & 0 & 0 \\ 0 & -1-\gamma & 0 & 0 \\ \xi & 0 & 2\gamma & 0 \\ 0 & \xi & 0 & 0 \end{pmatrix}$$

which has rank 3.

And if $\alpha=-1$, $\gamma\neq\pm 1$ we have

$$\begin{pmatrix} 1-\gamma & 0 & 0 & 0 \\ 0 & 1+\gamma & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & \xi & 0 & 2\gamma \end{pmatrix}$$

which also has rank 3.

So the contribution to Burnside's Lemma from matrices of this sort is
$2((p-3)p(p-1)+p(p^2-1)+(p-1)^2+(p^2-1))$

$+(p-3)2p(p-1)+(p-1)^2p = 7\left(p-1\right)^2 p$
The contribution to Burnside's Lemma from both matrices is
$p\left(3p+4\right)\left(p-1\right)^2 + 7\left(p-1\right)^2 p = p\left(3p+11\right)\left(p-1\right)^2$
So the number of orbits is $(3p+11)/2$.
Now do the same counts when $pa = pd = 0$, but we have kept the restriction $\delta = 0$.

$$\gamma\left(\begin{array}{ccc} y & z & t \end{array}\right) - \left(\begin{array}{ccc} y & z & t \end{array}\right)\left(\begin{array}{ccc} \alpha\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ (-\gamma\nu) & \omega\gamma\xi & \gamma^2 \end{array}\right)$$

$$= \left(\begin{array}{ccc} \gamma y - y\alpha\gamma + t\nu\gamma & \gamma z - z\alpha\gamma - t\omega\gamma\xi & \gamma t - t\gamma^2 \end{array}\right)$$

$$\begin{array}{ccc} \gamma-\alpha\gamma & 0 & \nu\gamma \\ 0 & \gamma-\alpha\gamma & -\omega\gamma\xi \\ 0 & 0 & \gamma-\gamma^2 \end{array}$$

$$\gamma\left(\begin{array}{ccc} y & z & t \end{array}\right) - \left(\begin{array}{ccc} y & z & t \end{array}\right)\left(\begin{array}{ccc} -\alpha\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ \gamma\nu & -\omega\gamma\xi & -\gamma^2 \end{array}\right)$$

$$= \left(\begin{array}{ccc} \gamma y + y\alpha\gamma - t\nu\gamma & \gamma z - z\alpha\gamma + t\omega\gamma\xi & \gamma t + t\gamma^2 \end{array}\right)$$

$$\begin{array}{ccc} \gamma+\alpha\gamma & 0 & -\nu\gamma \\ 0 & \gamma-\alpha\gamma & +\omega\gamma\xi \\ 0 & 0 & \gamma+\gamma^2 \end{array}$$

Try the ørst matrix

$$\begin{array}{ccc} \gamma-\alpha\gamma & 0 & \nu\gamma \\ 0 & \gamma-\alpha\gamma & -\omega\gamma\xi \\ 0 & 0 & \gamma-\gamma^2 \end{array}$$

Determinant: $-\gamma^3\left(-1+\alpha\right)^2\left(-1+\gamma\right)$
So rank is three unless $\gamma = 1$ or $\alpha = 1$.
Try $\gamma = 1$. We get

$$\left(\begin{array}{ccc} 1-\alpha & 0 & \nu \\ 0 & 1-\alpha & -\omega\xi \\ 0 & 0 & 0 \end{array}\right)$$

so the rank is 2 if $\alpha \neq 1$ and if $\alpha = 1$ the rank is one unless ν, ξ are both zero and rank 0 if both are zero.
Try $\alpha = 1$, $\gamma \neq 1$. We have

$$\left(\begin{array}{ccc} 0 & 0 & \nu\gamma \\ 0 & 0 & -\omega\gamma\xi \\ 0 & 0 & \gamma-\gamma^2 \end{array}\right)$$

which has rank 1.
Contribution to Burnside's Lemma is
$(p-2)p^2(p-1)+(p^2-1)(p^2-1)+(p^3-1)$
$+(p-2)p^2(p^2-1)+(p-1)^2p^2$

$= p^2\,(p+3)\,(p-1)^2$

Now the second matrix

$$\begin{matrix} \gamma+\alpha\gamma & 0 & -\nu\gamma \\ 0 & \gamma-\alpha\gamma & +\omega\gamma\xi \\ 0 & 0 & \gamma+\gamma^2 \end{matrix}$$

Determinant: $-\gamma^3\,(1+\alpha)\,(-1+\alpha)\,(1+\gamma)$

So rank is 3 unless $\gamma=-1$ or $\alpha=\pm1$.

Try $\gamma=-1$. we have

$$\begin{pmatrix} -1-\alpha & 0 & \nu \\ 0 & -1+\alpha & -\omega\xi \\ 0 & 0 & 0 \end{pmatrix}$$

If $\alpha\neq\pm1$ this has rank 2. In fact it has rank 2 except when $\alpha=1$ and $\xi=0$ or $\alpha=-1$ and $\nu=0$ when it has rank 1.

Try $\alpha=1$, $\gamma\neq-1$. We have $\begin{pmatrix} 2\gamma & 0 & -\nu\gamma \\ 0 & 0 & \omega\gamma\xi \\ 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$ which has rank 2.

Try $\alpha=-1$, $\gamma\neq-1$. We have $\begin{pmatrix} 0 & 0 & -\nu\gamma \\ 0 & 2\gamma & \omega\gamma\xi \\ 0 & 0 & \gamma+\gamma^2 \end{pmatrix}$ which has rank 2.

Contribution to Burnside's Lemma is

$((p-1)p^2-2p)(p-1)+2p(p^2-1)+(p-2)2p^2(p-1)+(p-1)^2p^2$

$=4\,(p-1)^2\,p^2$

So the contribution to Burnsides Lemma from both matrices is

$p^2\,(p+3)\,(p-1)^2+4\,(p-1)^2\,p^2=p^2\,(p+7)\,(p-1)^2$,

so the number of orbits is $(p+7)/2$.

Now do the same counts when $pa=pd=0$, $t=0$ but we have kept the restriction $\delta=0$.

$$\gamma\begin{pmatrix} y & z \end{pmatrix}\begin{pmatrix} \alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} \frac{y}{\alpha} & \frac{z}{\alpha} \end{pmatrix}$$

$$\gamma\begin{pmatrix} y & z \end{pmatrix}\begin{pmatrix} -\alpha\gamma & 0 \\ 0 & \alpha\gamma \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} -\frac{y}{\alpha} & \frac{z}{\alpha} \end{pmatrix}$$

Number of orbits is $(p+5)/2$.

And ønally we need to count the number of orbits when $pa=pd=0$, but we no longer have the restriction $\delta=0$.

$$\begin{pmatrix} y & z & t \end{pmatrix}-\begin{pmatrix} y & z & t \end{pmatrix}\begin{pmatrix} \alpha & (-\omega\delta) & 0 \\ -\delta & \alpha & 0 \\ (-\nu) & \omega\xi & \gamma \end{pmatrix}$$

$$=\begin{pmatrix} y-y\alpha+z\delta+t\nu & z+y\omega\delta-z\alpha-t\omega\xi & t-\gamma t \end{pmatrix}$$

$$\begin{pmatrix} 1-\alpha & \delta & \nu \\ \omega\delta & 1-\alpha & -\omega\xi \\ 0 & 0 & 1-\gamma \end{pmatrix}$$

$$\begin{pmatrix} y & z & t \end{pmatrix} - \begin{pmatrix} y & z & t \end{pmatrix} \begin{pmatrix} -\alpha & \omega\delta & 0 \\ -\delta & \alpha & 0 \\ \nu & -\omega\xi & -\gamma \end{pmatrix}$$

$$= \begin{pmatrix} y + y\alpha + z\delta - t\nu & z - y\omega\delta - z\alpha + t\omega\xi & t + \gamma t \end{pmatrix}$$

$$\begin{pmatrix} 1+\alpha & \delta & -\nu \\ -\omega\delta & 1-\alpha & \omega\xi \\ 0 & 0 & 1+\gamma \end{pmatrix}$$

First look at the øerst matrix

$$\begin{pmatrix} 1-\alpha & \delta & \nu \\ \omega\delta & 1-\alpha & -\omega\xi \\ 0 & 0 & 1-\gamma \end{pmatrix}$$

Determinant: $(-1+\gamma)\left(-(\alpha-1)^2 + \omega\delta^2\right)$

So the rank is three unless $\gamma = 1$ or $\alpha = 1$, $\delta = 0$.

Try $\gamma = 1$. Get rank 2 unless $\alpha = 1$, $\delta = 0$. Then get rank 1 unless $\nu = \xi = 0$ when we get rank 0.

If $\gamma \neq 1$ and $\alpha = 1$, $\delta = 0$ we get rank 1 (otherwise rank 3).

Contribution to Burnside's Lemma is

$((p^2-1)p^2 - p^2)(p-1) + (p^2-1)(p^2-1) + (p^3-1)$

$+(p-2)p^2(p^2-1) + (p-1)(p^2-1)p^2$

$= 3p^2\,(p+1)\,(p-1)^2$

Now the second matrix.

$$\begin{pmatrix} 1+\alpha & \delta & -\nu \\ -\omega\delta & 1-\alpha & \omega\xi \\ 0 & 0 & 1+\gamma \end{pmatrix}$$

Determinant: $(1+\gamma)\left(1 - \alpha^2 + \omega\delta^2\right)$

So rank is three unless $\gamma = -1$ or $\alpha^2 - \omega\delta^2 = 1$.

Let $\gamma = -1$. Then we get rank 2 in all $(p^2-1)p^2$ situtations except for $p(p+1)$ situations when we get rank 1.

If $\gamma \neq -1$ then (for each of the $p-2$ values of γ) we get rank 3 except in $(p+1)p^2$ situations, when we get rank 2.

Contribution to Burnside's Lemma is

$((p^2-1)p^2 - p(p+1))(p-1) + p(p+1)(p^2-1) + (p-2)(p+1)p^2(p-1)$

$+(p-1)(p^2-1)p^2$

$= 3p^2\,(p+1)\,(p-1)^2$

So the total number of orbits is 3.

And last of all (I hope!!!!!) is the above calculation with $z = 0$.

$$\begin{pmatrix} y & z \end{pmatrix} - \begin{pmatrix} y & z \end{pmatrix} \begin{pmatrix} \alpha & (-\omega\delta) \\ -\delta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} y - y\alpha + z\delta & z + y\omega\delta - z\alpha \end{pmatrix}$$

$$\begin{pmatrix} 1-\alpha & \delta \\ \omega\delta & 1-\alpha \end{pmatrix}$$

Contribution to Burnside's Lemma is $(p^2-1)+(p^2-1)$

$$\begin{pmatrix} y & z \end{pmatrix} - \begin{pmatrix} y & z \end{pmatrix}\begin{pmatrix} -\alpha & \omega\delta \\ -\delta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} y+y\alpha+z\delta & z-y\omega\delta-z\alpha \end{pmatrix}$$

$$\begin{pmatrix} 1+\alpha & \delta \\ -\omega\delta & 1-\alpha \end{pmatrix}$$

Determinant: $1-\alpha^2+\omega\delta^2$
Contribution to Burnsides Lemma is $(p+1)(p-1)+(p^2-1)$
So the number of orbits is 2.

61.10 Summary

We consider the possibilities for pa, pd. They are $0,0$; $0,cb$; $0,ca$; cb,ca; and p possibilities when pa, pd span $\langle ba, ca\rangle$.

61.11 $pa = pd = 0$

There are $p+4$ algebras with $pa = pd = 0$.

61.12 $pa = 0, pd = cb$

So the number of algebras is $p^2 + (3p+5)/2$.

61.13 $pa = 0, pd = ca$

So the total number of algebras here is $p^2 + (3p+1)/2$. Dec4.16e computes these.

61.14 $pa = cb, pd = ca$

So we have $\frac{1}{2}p^2 + \frac{1}{2}p^4$ algebras here.

61.15 pa, pd span ba, ca

We can assume that $pb = 0$.

First consider the case when pa, pd, pc are linearly dependent. Then we can assume that $pc = 0$, though this restricts us to a', b', c', d' with λ, μ, ν, ξ all zero. The evidence is that there are p orbits. See dec4.16b.

Next consiser the case when pa, pd, pc are linearly independent, and $pb = 0$. This restricts us to a', b', c', d' with $\lambda = \beta = \mu = 0$.

Let

$$\begin{pmatrix} pa \\ pd \\ pc \end{pmatrix} = \begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} ba \\ ca \\ cb \end{pmatrix}.$$

Then

$$\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \delta & 0 \\ \pm\omega\delta & \pm\alpha & 0 \\ \nu & \xi & \gamma \end{pmatrix}\begin{pmatrix} u & v & 0 \\ w & x & 0 \\ y & z & t \end{pmatrix}\begin{pmatrix} \pm\alpha\gamma & \pm(-\omega\gamma\delta) & 0 \\ -\gamma\delta & \alpha\gamma & 0 \\ \pm(-\gamma\nu) & \pm\omega\gamma\xi & \pm\gamma^2 \end{pmatrix}^{-1}$$

Number of orbits is $(p^3+3p^2+5p+11)/2$.

Now we need to count the orbits when pc is a linear combination of ba, ca. Let

$$\begin{pmatrix} pa \\ pd \\ pc \end{pmatrix} = \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} ba \\ ca \end{pmatrix}$$

The number of orbits here is $(p^2+4p+11)/2$.

So under $\lambda = \beta = \mu = 0$ there are $((p^3+3p^2+5p+11)-(p^2+4p+11))/2 = (p^3+2p^2+p)/2$ orbits with pa, pd in the span of ba, ca, $pb = 0$, pc not in the span of ba, ca.

Next we need to count the orbits when pa, pd lie in the span of ba, ca and $pb = 0$, pc does not lie in the span of ba, ca, but pa, pd are not linearly independant. If they are not both zero then we can assume that $pa = 0$, but then we need $\delta = 0$.

The number of orbits is $(p^2+5p+12)/2$.

Next we need to count the orbits when pa, pd lie in the span of ba, ca and $pb = 0$, pc does lie in the span of ba, ca, but pa, pd are not linearly independnt. If they are not both zero then we can assume that $pa = 0$, but then we need $\delta = 0$.

The number of orbits is $(3p+11)/2$.

So under $\lambda = \beta = \mu = 0$, $\delta = 0$, there are $((p^2+5p+12)-(3p+11)) = (p^2+2p+1)/2$ orbits with pa, pd in the span of ba, ca, $pb = 0$, pc not in the span of ba, ca, ba, ca linearly dependent.

Now do the same counts when $pa = pd = 0$, but we have kept the restriction $\delta = 0$.

the number of orbits is $(p+7)/2$.

Now do the same counts when $pa = pd = 0$, $t = 0$ but we have kept the restriction $\delta = 0$. Number of orbits is $(p+5)/2$.

So under $\lambda = \beta = \mu = 0$, $\delta = 0$ there is one orbit with $pa = pb = pd = 0$, pc not in the span of ba, ca.

And ønally we need to count the number of orbits when $pa = pd = 0$, but we no longer have the restriction $\delta = 0$. The total number of orbits is 3.

And last of all (I hope!!!!!) is the above calculation with $z = 0$. The number of orbits is 2.

So under $\lambda = \beta = \mu = 0$, any δ there is one orbit with $pa = pb = pd = 0$, pc not in the span of ba, ca.

61.16 Grand totals when pa, pd span ba, ca and $pb = 0$, pa, pd, pc l.i.

There are $(p^3+2p^2+p)/2$ orbits under $\lambda = \beta = \mu = 0$ with pa, pd in the span of ba, ca, $pb = 0$, pc not in the span of ba, ca.

Of these, $(p^2+2p+1)/2 - 1$ have pa, pd spanning a space of dimension 1, and one orbit has $pa = pd = 0$.

So the total number of orbits here is $((p^3+2p^2+p)-(p^2+2p+1)) = (p^3-p+p^2-1)/2$.

61.17 Grand total of everything!

There are $p+4$ algebras with $pa = pd = 0$.

The number of algebras with $pa = 0$, $pd = cb$ is $p^2 + (3p+5)/2$.

So the total number of algebras with $pa = 0$, $pd = ca$ is $p^2 + (3p+1)/2$. Dec4.16e computes these.

There are $\frac{1}{2}p^2 + \frac{1}{2}p^4$ algebras with $pa = cb$, $pd = ca$.

There are p algebras with pa, pd spanning ba, ca, and pL spanned by ba, ca. See dec4.16b.

There are $(p^3-p+p^2-1)/2$ algebras when pa, pd span ba, ca and $pL = L^2$.
So the grand total is $p+4+p^2+(3p+5)/2+p^2+(3p+1)/2+\frac{1}{2}p^2$
$+\frac{1}{2}p^4+p+(p^3-p+p^2-1)/2$
$=\frac{9}{2}p+\frac{13}{2}+3p^2+\frac{1}{2}p^4+\frac{1}{2}p^3$
This gives 101 when $p=3$, 479 when $p=5$, and 1557 when $p=7$.

62 Appendix B

From Case 6 in the descendants of 5.3

$$\begin{aligned}
a' &= \alpha a - \beta b + \gamma c + \delta d, \\
b' &= \pm(\beta a + \alpha b + \lambda c + \mu d), \\
c' &= (\alpha^2-\beta^2)c - 4\alpha\beta d, \\
d' &= \pm(\alpha\beta c + (\alpha^2-\beta^2)d),
\end{aligned}$$

$$\begin{aligned}
b'a'a' &= \pm(\alpha^2+\beta^2)(\alpha baa - \beta bab), \\
b'a'b' &= (\alpha^2+\beta^2)(\beta baa + \alpha bab).
\end{aligned}$$

The group has order $(p^2-1)p^4$ if $p = 3 \bmod 4$ and order $(p-1)^2p^4$ if $p = 1 \bmod 4$.

$$\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} - (\alpha^2+\beta^2) \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u - \beta w + \gamma y + \delta t - \alpha^3 u - \alpha u\beta^2 - \beta v\alpha^2 - \beta^3 v & \alpha v - \beta x + \gamma z + \delta m + \beta u\alpha^2 + \beta^3 u - \alpha^3 v - \alpha v\beta^2 \\ \beta u + \alpha w + \lambda y + \mu t - \alpha^3 w - \alpha w\beta^2 - \beta x\alpha^2 - \beta^3 x & \beta v + \alpha x + \lambda z + \mu m + \beta w\alpha^2 + \beta^3 w - \alpha^3 x - \alpha x\beta^2 \\ y\alpha^2 - y\beta^2 - 4\alpha\beta t - y\alpha^3 - y\alpha\beta^2 - z\beta\alpha^2 - z\beta^3 & z\alpha^2 - z\beta^2 - 4\alpha\beta m + y\beta\alpha^2 + y\beta^3 - z\alpha^3 - z\alpha\beta^2 \\ \alpha\beta y + t\alpha^2 - t\beta^2 - t\alpha^3 - t\alpha\beta^2 - m\beta\alpha^2 - m\beta^3 & \alpha\beta z + m\alpha^2 - m\beta^2 + t\beta\alpha^2 + t\beta^3 - m\alpha^3 - m\alpha\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 & -\beta & 0 & \gamma & 0 \\ \beta\alpha^2+\beta^3 & \alpha-\alpha^3-\alpha\beta^2 & 0 & -\beta & 0 & \gamma \\ \beta & 0 & \alpha-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 & \lambda & 0 \\ 0 & \beta & \beta\alpha^2+\beta^3 & \alpha-\alpha^3-\alpha\beta^2 & 0 & \lambda \\ 0 & 0 & 0 & 0 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 \\ 0 & 0 & 0 & 0 & \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 \\ 0 & 0 & 0 & 0 & \alpha\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta \end{pmatrix}$$

runmatrixon
$$\begin{pmatrix} \delta & 0 \\ 0 & \delta \\ \mu & 0 \\ 0 & \mu \\ -4\alpha\beta & 0 \\ 0 & -4\alpha\beta \\ \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 \\ \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 \end{pmatrix}$$

Determinant:
$(\beta^2+(1+\alpha)^2)\left(\beta^4+6\beta^2\alpha+\beta^2+2\beta^2\alpha^2+\alpha^2-2\alpha^3+\alpha^4\right)\left(\beta^2+(1-\alpha)^2\right)^2\left(\beta^2-1+\alpha^2\right)^2\left(\beta^2+\alpha^2\right)^5$

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2+\beta^2)\begin{pmatrix} u & v \\ w & x \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u - \beta w - \alpha u\beta^2 - \alpha^3 u - \beta^3 v - \beta v\alpha^2 & \alpha v - \beta x + \beta^3 u + \beta u\alpha^2 - \alpha v\beta^2 - \alpha^3 v \\ \beta u + \alpha w - \alpha w\beta^2 - \alpha^3 w - \beta x\alpha^2 - \beta^3 x & \beta v + \alpha x + \beta^3 w + \beta w\alpha^2 - \alpha x\beta^2 - \alpha^3 x \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \alpha\beta^2 - \alpha^3 & -\beta^3 - \beta\alpha^2 & -\beta & 0 \\ \beta^3 + \beta\alpha^2 & \alpha - \alpha\beta^2 - \alpha^3 & 0 & -\beta \\ \beta & 0 & \alpha - \alpha\beta^2 - \alpha^3 & -\beta\alpha^2 - \beta^3 \\ 0 & \beta & \beta^3 + \beta\alpha^2 & \alpha - \alpha\beta^2 - \alpha^3 \end{pmatrix}$$

Determinant: $\left(\beta^2+1+2\alpha+\alpha^2\right)\left(\beta^2+1-2\alpha+\alpha^2\right)\left(\beta^2-1+\alpha^2\right)^2\left(\beta^2+\alpha^2\right)^2$

Suppose that $p = 3 \bmod 4$, so that $\alpha^2+\beta^2 = 0$ has no sulutions other than $\alpha = \beta = 0$. So the group has order p^2-1.

The rank is 4 unless $\alpha = \pm 1$ and $\beta = 0$, or $\alpha^2+\beta^2 = 1$, which happens for $p+1$ pairs α, β (including $\alpha = \pm 1$ and $\beta = 0$).

First suppose that $\alpha = 1$, $\beta = 0$: we get rank 0.

Next suppose $\alpha = -1$, $\beta = 0$: again we get rank 0.

Now suppose that $\alpha^2+\beta^2 = 1$, $\beta \neq 0$.

$$\begin{pmatrix} 0 & -\beta & -\beta & 0 \\ \beta & 0 & 0 & -\beta \\ \beta & 0 & 0 & -\beta \\ 0 & \beta & \beta & 0 \end{pmatrix}$$

We get rank 2.

So the contribution Burnsides Lemma is $2(p^4-1)+(p-1)(p^2-1)+(p^2-1) = (p-1)(p+1)\left(2p^2+p+2\right)$.

$$\begin{pmatrix} \alpha & -\beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2+\beta^2)\begin{pmatrix} u & v \\ w & x \end{pmatrix}\begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u - \beta w + \alpha u\beta^2 + \alpha^3 u - \beta^3 v - \beta v\alpha^2 & \alpha v - \beta x - \beta^3 u - \beta u\alpha^2 - \alpha v\beta^2 - \alpha^3 v \\ -\beta u - \alpha w + \alpha w\beta^2 + \alpha^3 w - \beta x\alpha^2 - \beta^3 x & -\beta v - \alpha x - \beta^3 w - \beta w\alpha^2 - \alpha x\beta^2 - \alpha^3 x \end{pmatrix}$$

$$\begin{pmatrix} \alpha + \alpha\beta^2 + \alpha^3 & -\beta^3 - \beta\alpha^2 & -\beta & 0 \\ -\beta^3 - \beta\alpha^2 & \alpha - \alpha\beta^2 - \alpha^3 & 0 & -\beta \\ -\beta & 0 & -\alpha + \alpha\beta^2 + \alpha^3 & -\beta\alpha^2 - \beta^3 \\ 0 & -\beta & -\beta^3 - \beta\alpha^2 & -\alpha - \alpha\beta^2 - \alpha^3 \end{pmatrix}$$

Determinant: $\left(\beta^2-1+\alpha^2\right)^2\left(\beta^2+\alpha^2\right)^2\left(1+\beta^2+\alpha^2\right)^2$.

Suppose $\alpha^2+\beta^2 = 1$.

$$\begin{pmatrix} 2\alpha & -\beta & 0 & 0 \\ -\beta & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & -2\alpha \end{pmatrix}$$

If $\beta = 0$ this has rank 2. So suppose $\beta \neq 0$.

$$\begin{pmatrix} 1 & -2\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\alpha & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & -2\alpha \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\alpha \end{pmatrix}$$

So the rank is always 2.

Suppose $\alpha^2 + \beta^2 = -1$.

$$\begin{pmatrix} 0 & \beta & -\beta & 0 \\ 0 & 2\alpha & 0 & -\beta \\ 0 & 0 & -2\alpha & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Again rank is always 2.

Contribution to Burnside's Lemma $2(p+1)(p^2-1) + (p^2-1) = (p-1)(2p+3)(p+1)$

So the number of orbits is $p^2 + (3p+5)/2$.

Now what if $p = 1 \bmod 4$. Then there are $2(p-1)$ non-zero pairs α, β with $\alpha^2 + \beta^2 = 0$, and the group has size $(p-1)^2$.

So $\beta^2 + 1 + 2\alpha + \alpha^2$ will equal 0 for $2(p-1)$ non-zero pairs α, β. But could $\alpha^2 + \beta^2 = 0$ for one of these pairs. Then $\alpha = -\frac{1}{2}$ and $(1+\alpha)^2 = \alpha^2$, so this only happens (twice) if $p = 1 \bmod 3$. Similarly $\beta^2 + 1 - 2\alpha + \alpha^2 = 0$ occurs $2(p-1)$ times.

Suppose $\beta^2 + 1 + 2\alpha + \alpha^2 = 0$

Then $\alpha^2 + \beta^2 = -1 - 2\alpha$

$$\begin{pmatrix} \alpha - \alpha(-1-2\alpha) & -\beta(-1-2\alpha) & -\beta & 0 \\ \beta(-1-2\alpha) & \alpha - \alpha(-1-2\alpha) & 0 & -\beta \\ \beta & 0 & \alpha - \alpha(-1-2\alpha) & -\beta(-1-2\alpha) \\ 0 & \beta & \beta(-1-2\alpha) & \alpha - \alpha(-1-2\alpha) \end{pmatrix}$$

$$\begin{pmatrix} \beta & 0 & -(2\alpha+2\alpha^2) & 0 \\ 0 & 1 & 1+2\alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\alpha+2\alpha^2 & \beta(1+2\alpha) & -\beta & 0 \\ -\beta(1+2\alpha) & 2\alpha+2\alpha^2 & 0 & -\beta \\ \beta & 0 & 2\alpha+2\alpha^2 & \beta(1+2\alpha) \\ 0 & \beta & -\beta(1+2\alpha) & 2\alpha+2\alpha^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\beta(1+2\alpha) \\ 0 & \beta & -(2\alpha+2\alpha^2) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta^2(1+2\alpha) & -\beta^2 - 4\alpha^2 - 8\alpha^3 - 4\alpha^4 & -2\alpha(1+\alpha)\beta(1+2\alpha) \\ 2\alpha+2\alpha^2 & 2(1+2\alpha)\alpha(1+\alpha) & 4\beta\alpha + 4\beta\alpha^2 \\ \beta & -\beta(1+2\alpha) & 2\alpha+2\alpha^2 \end{pmatrix}$$

$$\begin{pmatrix} -4\alpha^2 - 8\alpha^3 - 4\alpha^4 + 4\beta^2\alpha + 4\beta^2\alpha^2 & -4\beta\alpha - 12\beta\alpha^2 - 8\beta\alpha^3 \\ 4\beta\alpha + 12\beta\alpha^2 + 8\beta\alpha^3 & -4\alpha^2 - 8\alpha^3 - 4\alpha^4 + 4\beta^2\alpha + 4\beta^2\alpha^2 \end{pmatrix}$$

$4\beta\alpha + 12\beta\alpha^2 + 8\beta\alpha^3 = 4\alpha(1+\alpha)\beta(1+2\alpha)$

$-4\alpha^2 - 8\alpha^3 - 4\alpha^4 + 4\beta^2\alpha + 4\beta^2\alpha^2 = 4\alpha(1+\alpha)\left(-\alpha^2 - \alpha + \beta^2\right)$

So rank is 3 unless $\alpha = 0$ or $\alpha = -1$ or ($\alpha = -\frac{1}{2}$ and $-\alpha^2 - \alpha + \beta^2 = 0$).

$\beta^2 + 1 + 2\alpha + \alpha^2 = 0$ and $-\alpha^2 - \alpha + \beta^2 = 0$ implies $0 = 1 + 2\alpha + \alpha^2 + \alpha^2 + \alpha = (1+\alpha)(1+2\alpha)$. But $\alpha = -\frac{1}{2}$ has to be excluded.

$\alpha = 0$ occurs twice.

$\alpha=-1$ does not arise.

$\alpha=-\frac{1}{2}$ gives $\beta^2=-\frac{3}{4}$ which has a solution if $p=1 \bmod 3$.

So if $p=1 \bmod 3$ there are $2(p-1)-4$ occurences of rank 3, and 2 occurences of rank 2.

If $p=2 \bmod 3$ there are $2(p-1)-2$ occurrences of rank 3, and 2 occurences of rank 2.

Now consider the situation when $\beta^2+1-2\alpha+\alpha^2=0$. Then $\alpha^2+\beta^2=2\alpha-1$.

$$\begin{pmatrix} \alpha-\alpha(2\alpha-1) & -\beta(2\alpha-1) & -\beta & 0 \\ \beta(2\alpha-1) & \alpha-\alpha(2\alpha-1) & 0 & -\beta \\ \beta & 0 & \alpha-\alpha(2\alpha-1) & -\beta(2\alpha-1) \\ 0 & \beta & \beta(2\alpha-1) & \alpha-\alpha(2\alpha-1) \end{pmatrix}$$

$$\begin{pmatrix} \beta & 0 & -2\alpha+2\alpha^2) & 0 \\ 0 & 1 & 1-2\alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\alpha-2\alpha^2 & -\beta\,(2\alpha-1) & -\beta & 0 \\ \beta\,(2\alpha-1) & 2\alpha-2\alpha^2 & 0 & -\beta \\ \beta & 0 & 2\alpha-2\alpha^2 & -\beta\,(2\alpha-1) \\ 0 & \beta & \beta\,(2\alpha-1) & 2\alpha-2\alpha^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \beta\,(2\alpha-1) \\ 0 & \beta & -2\alpha+2\alpha^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta^2\,(2\alpha-1) & -\beta^2-4\alpha^2+8\alpha^3-4\alpha^4 & -2\alpha\,(\alpha-1)\,\beta\,(2\alpha-1) \\ 2\alpha-2\alpha^2 & 2\,(2\alpha-1)\,\alpha\,(\alpha-1) & -4\beta\alpha+4\beta\alpha^2 \\ \beta & \beta\,(2\alpha-1) & 2\alpha-2\alpha^2 \end{pmatrix}$$

$$\begin{pmatrix} -4\alpha^2+8\alpha^3-4\alpha^4+4\beta^2\alpha^2-4\beta^2\alpha & -8\beta\alpha^3+12\beta\alpha^2-4\beta\alpha \\ 8\beta\alpha^3-12\beta\alpha^2+4\beta\alpha & -4\alpha^2+8\alpha^3-4\alpha^4+4\beta^2\alpha^2-4\beta^2\alpha \end{pmatrix}$$

$8\beta\alpha^3-12\beta\alpha^2+4\beta\alpha=4\alpha\,(\alpha-1)\,\beta\,(2\alpha-1)$

$-4\alpha^2+8\alpha^3-4\alpha^4+4\beta^2\alpha^2-4\beta^2\alpha=4\alpha\,(\alpha-1)\left(-\alpha^2+\alpha+\beta^2\right)$

So rank is 3 unless $\alpha=0$, which occurs twice (but we have already counted these!), or $\alpha=\frac{1}{2}$ which occurs twice if $p=1 \bmod 3$.

So if $p=1 \bmod 3$ there are $2(p-1)-4$ occurences of rank 3.

If $p=2 \bmod 3$ there are $2(p-1)-2$ occurrences of rank 3.

Contribution to Burnsides Lemma is $2(2(p-1)-4)(p-1))+(p^2-1)=5p^2-16p+11$ if $p=1 \bmod 3$ and

$2((2(p-1)-2)(p-1)+2(p^2-1))=4\,(2p-1)\,(p-1)$ if $p=2 \bmod 3$.

$2(p^4-1)+(p-1)(p^2-1)+2(p^2-1)+2(2p-6)(p-1)+(p-1)^2$

The ørst matrix has rank 0 twice, rank 2 $p-1$ times, and rank 3 $4p-12$ times if $p=1 \bmod 4$, but no times if $p=3 \bmod 4$.

So the contribution from the ørst matrix is $2(p^4-1)+(p-1)(p^2-1)+(p^2-1)=(p-1)\,(p+1)\left(2p^2+p+2\right)$ if $p=3 \bmod 4$, and

$2(p^4-1)+(p-1)(p^2-1)+(4p-12)(p-1)+(p-1)^2=\left(2p^2+5p+12\right)(p-1)^2$ if $p=1 \bmod 4$.

The second matrix has rank 2 $2(p-1)$ times when $p=1 \bmod 4$ and $2(p+1)$ times when $p=3 \bmod 4$.

So the contribution from the secondmatrix is $2(p-1)(p^2-1)+(p-1)^2=(2p+3)\,(p-1)^2$ when $p=1 \bmod 4$, and

$2(p+1)(p^2-1)+(p^2-1)=(p-1)\,(2p+3)\,(p+1)$ when $p=3 \bmod 4$.

So the number of orbits is $p^2+(7p+15)/2$ when $p=1 \bmod 4$ and $p^2+(3p+5)/2$ when $p=3 \bmod 4$.

62.2 Possibilities for pc, pd

$$\begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2+\beta^2) \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} u\alpha^2 - u\beta^2 - 4\alpha\beta w - u\alpha^3 - u\alpha\beta^2 - v\beta\alpha^2 - v\beta^3 & v\alpha^2 - v\beta^2 - 4\alpha\beta x + u\beta\alpha^2 + u\beta^3 - v\alpha^3 - v\alpha\beta^2 \\ \alpha\beta u + w\alpha^2 - w\beta^2 - w\alpha^3 - w\alpha\beta^2 - x\beta\alpha^2 - x\beta^3 & \alpha\beta v + x\alpha^2 - x\beta^2 + w\beta\alpha^2 + w\beta^3 - x\alpha^3 - x\alpha\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2 - \beta^2 - \alpha^3 - \alpha\beta^2 & -\beta\alpha^2 - \beta^3 & -4\alpha\beta & 0 \\ \beta\alpha^2 + \beta^3 & \alpha^2 - \beta^2 - \alpha^3 - \alpha\beta^2 & 0 & -4\alpha\beta \\ \alpha\beta & 0 & \alpha^2 - \beta^2 - \alpha^3 - \alpha\beta^2 & -\beta\alpha^2 - \beta^3 \\ 0 & \alpha\beta & \beta\alpha^2 + \beta^3 & \alpha^2 - \beta^2 - \alpha^3 - \alpha\beta^2 \end{pmatrix}$$

Determinant: $\left(\beta^2 + 1 - 2\alpha + \alpha^2\right)\left(\beta^4 + \beta^2 + 2\alpha^2\beta^2 + 6\alpha\beta^2 + \alpha^2 + \alpha^4 - 2\alpha^3\right)\left(\alpha^2 + \beta^2\right)^3$

$$\begin{pmatrix} \alpha^2 - \beta^2 & -4\alpha\beta \\ -\alpha\beta & -(\alpha^2 - \beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2 + \beta^2)\begin{pmatrix} u & v \\ w & x \end{pmatrix}\begin{pmatrix} -\alpha & +\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} u\alpha^2 - u\beta^2 - 4\alpha\beta w + u\alpha^3 + u\alpha\beta^2 - v\beta\alpha^2 - v\beta^3 & v\alpha^2 - v\beta^2 - 4\alpha\beta x - u\beta\alpha^2 - u\beta^3 - v\alpha^3 - v\alpha\beta^2 \\ -\alpha\beta u - w\alpha^2 + w\beta^2 + w\alpha^3 + w\alpha\beta^2 - x\beta\alpha^2 - x\beta^3 & -\alpha\beta v - x\alpha^2 + x\beta^2 - w\beta\alpha^2 - w\beta^3 - x\alpha^3 - x\alpha\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2 - \beta^2 + \alpha^3 + \alpha\beta^2 & -\beta\alpha^2 - \beta^3 & -4\alpha\beta & 0 \\ -\beta\alpha^2 - \beta^3 & \alpha^2 - \beta^2 - \alpha^3 - \alpha\beta^2 & 0 & -4\alpha\beta \\ -\alpha\beta & 0 & -\alpha^2 + \beta^2 + \alpha^3 + \alpha\beta^2 & -\beta\alpha^2 - \beta^3 \\ 0 & -\alpha\beta & -\beta\alpha^2 - \beta^3 & -\alpha^2 + \beta^2 - \alpha^3 - \alpha\beta^2 \end{pmatrix}$$

Determinant: $\left(\beta^2 - 1 + \alpha^2\right)^2\left(\alpha^2 + \beta^2\right)^4$

It turns out that the ørst matrix has has 1 occurence of rank 0; 4,2,0,2 of rank 2 depending on whether $p = 1, 5, 7, 11 \bmod p$; and $4p-16, 4p-12, 0, 0$ of rank 3. This is proved below. So the contribution to Burnside's Lemma is

$(p^4-1)+4(p^2-1)+(4p-16)(p-1)+(p-1)^2 = \left(p^2+2p+12\right)(p-1)^2$ if $p = 1 \bmod 12$,
$(p^4-1)+2(p^2-1)+(4p-12)(p-1)+(p-1)^2 = \left(p^2+2p+10\right)(p-1)^2$ if $p = 5 \bmod 12$,
$(p^4-1)+(p^2-1) = (p-1)(p+1)\left(p^2+2\right)$ if $p = 7 \bmod 12$,
$(p^4-1)+2(p^2-1)+(p^2-1) = (p-1)(p+1)\left(p^2+4\right)$ if $p = 11 \bmod 12$.

The second matrix has $p-1$ occurrences of rank 2 if $p = 1 \bmod 4$ and $p+1$ occurences if $p = 3 \bmod 4$. So the contribution to Burnside's Lemma is

$(p-1)(p^2-1)+(p-1)^2 = (p+2)(p-1)^2$ if $p = 1 \bmod 12$,
$(p-1)(p^2-1)+(p-1)^2 = (p+2)(p-1)^2$ if $p = 5 \bmod 12$,
$(p+1)(p^2-1)+(p^2-1) = (p-1)(p+2)(p+1)$ if $p = 7 \bmod 12$,
$(p+1)(p^2-1)+(p^2-1) = (p-1)(p+2)(p+1)$ if $p = 11 \bmod 12$.

So the number of orbits
$(p^2+3p+14)/2$ if $p = 1 \bmod 12$,
$(p^2+3p+12)/2$ if $p = 5 \bmod 12$,
$(p^2+p+4)/2$ if $p = 7 \bmod 12$,
$(p^2+p+6)/2$ if $p = 11 \bmod 12$.

Try to øgure out possibilitiles for pc, pd spanning a one dimensional space.

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha^2 - \beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2 - \beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha^2 - \beta^2 & -4\alpha\beta \\ -\alpha\beta & -(\alpha^2 - \beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \end{pmatrix}\begin{pmatrix} -\alpha & +\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

If $u = v = 0$ and $(w,x) \neq (0,0)$ we need $\alpha = 0$ or $\beta = 0$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha^2 - \beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2 - \beta^2) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ w & x \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{w}{\alpha} & \frac{x}{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{x}{\beta} & -\frac{w}{\beta} \end{pmatrix}$$

$$(\alpha^2+\beta^2)^{-1} \begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ w & x \end{pmatrix} \begin{pmatrix} -\alpha & +\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{w}{\alpha} & -\frac{x}{\alpha} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{x}{\beta} & \frac{w}{\beta} \end{pmatrix}$$

So we get $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and if w, x are both non-zero we have $(1,x) \sim (1,-x) \sim (1,x^{-1}) \sim (1,-x^{-1})$. Space has size $p-1$. Identity øxes everything, $-$ øxes nothing, $x \mapsto x^{-1}$ øxes 2, $x \mapsto -x^{-1}$ øxes 2 if $p = 1 \bmod 4$, and øxes 0 if $p = 3 \bmod 4$. So the number of orbits is $(p+1)/4$ if $p = 3 \bmod 4$ and $(p+3)/4$ if $p = 1 \bmod 4$.

Now consider the case when both rows are non-zero. We would like to get the top row equal to $(0,1)$.

$$(\alpha^2+\beta^2) \begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ uw & vw \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} (-u\alpha+v\beta)\left(-\alpha^2+4\beta w\alpha+\beta^2\right) & -(u\beta+v\alpha)\left(-\alpha^2+4\beta w\alpha+\beta^2\right) \\ -(-u\alpha+v\beta)\left(w\alpha^2+\alpha\beta-w\beta^2\right) & (u\beta+v\alpha)\left(w\alpha^2+\alpha\beta-w\beta^2\right) \end{pmatrix}$$

$$(\alpha^2+\beta^2) \begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ uw & vw \end{pmatrix} \begin{pmatrix} -\alpha & +\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -(-u\alpha+v\beta)\left(-\alpha^2+4\beta w\alpha+\beta^2\right) & -(u\beta+v\alpha)\left(-\alpha^2+4\beta w\alpha+\beta^2\right) \\ -(-u\alpha+v\beta)\left(w\alpha^2+\alpha\beta-w\beta^2\right) & -(u\beta+v\alpha)\left(w\alpha^2+\alpha\beta-w\beta^2\right) \end{pmatrix}$$

So we can get $pc = 0$ if we can solve $-\alpha^2+4\beta w\alpha+\beta^2 = 0$. This has a solution if $(2w)^2+1$ is a square. But if $(2w)^2+1 = 0$ then the solution α, β satisøes $\alpha^2+\beta^2 = 0$. We can also get $u = 0$ unless $u^2+v^2 = 0$.Try $\beta = 0$, $\alpha = 0$.

$$(\alpha^2+\beta^2)^{-1} \begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ uw & vw \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

We get $\begin{pmatrix} \frac{u}{\alpha} & \frac{v}{\alpha} \\ u\frac{w}{\alpha} & v\frac{w}{\alpha} \end{pmatrix}, \begin{pmatrix} \frac{v}{\beta} & -\frac{u}{\beta} \\ v\frac{w}{\beta} & -u\frac{w}{\beta} \end{pmatrix}$ and from the second matrix we get $\begin{pmatrix} -\frac{u}{\alpha} & \frac{v}{\alpha} \\ u\frac{w}{\alpha} & -v\frac{w}{\alpha} \end{pmatrix}, \begin{pmatrix} -\frac{v}{\beta} & -\frac{u}{\beta} \\ v\frac{w}{\beta} & u\frac{w}{\beta} \end{pmatrix}$

So we are only stuck with $u \neq 0$ if $u^2+v^2 = 0$ and $(2w)^2+1$ is not a square, or $(2w)^2+1 = 0$. We can then take $u = 1$ and take v to be of the two possible solutions of $1+v^2 = 0$. To maintain that ratio we need to stick with matrices of the ørst kind.

The orbit of a given w are the diœerent values of

$$-\frac{w\alpha^2+\alpha\beta-w\beta^2}{-\alpha^2+4\beta w\alpha+\beta^2}.$$

We need to count the number of w such that $(2w)^2+1$ is not a square (the number is $(p-1)/2$ when $p = 1 \bmod 4$, and $(p+1)/2$ when $p = 3 \bmod 4$).

First let $p = 1 \bmod 4$. Consider the number of solutions of $k^2+l^2 = ns$. Since $p = 1 \bmod 4$ the number of solutions is $p-1$ for each ns. So the total number of solutions is $(p-1)^2/2$. Now for each ratio $k : l$ for which k^2+l^2 is not a square, there are $p-1$ pairs k', l' such that $k' : l' = k : l$. So the number of w is $(p-1)/2$.

Next let $p = 3 \bmod 4$. Consider the number of solutions of $k^2 + l^2 = ns$. Since $p = 3 \bmod 4$ the number of solutions is $p+1$ for each ns. So the total number of solutions is $(p^2-1)/2$. Now for each ratio $k : l$ for which $k^2 + l^2$ is not a square, there are $p - 1$ pairs k', l' such that $k' : l' = k : l$. So the number of w is $(p+1)/2$.

Now we want to count the number of orbits of these $w's$ under the transformation

$$w \mapsto -\frac{w\alpha^2 + \alpha\beta - w\beta^2}{-\alpha^2 + 4\beta w\alpha + \beta^2}.$$

So suppose that

$$w = -\frac{w\alpha^2 + \alpha\beta - w\beta^2}{-\alpha^2 + 4\beta w\alpha + \beta^2}.$$

Then

$$w(-\alpha^2 + 4\beta w\alpha + \beta^2) + (w\alpha^2 + \alpha\beta - w\beta^2) = 0.$$

$$w(-\alpha^2 + 4\beta w\alpha + \beta^2) + (w\alpha^2 + \alpha\beta - w\beta^2) = \alpha\beta\left(4w^2 + 1\right).$$

So $\alpha = 0$ øxes everything and $\beta = 0$ øxes everything. Contribution to Burnside's Lemma is $(p-1)^2$.

So there is only one orbit of $w's$ with $(2w)^2 + 1$ not a square (while preserving the ratio $u : v$).

On the other hand if $(4w^2 + 1) = 0$ then w is øxed by everything.

So there are three orbits of matrices with $u \neq 0$.

Now consider orbits of matrices with $u = 0$, $v \neq 0$.

$$(\alpha^2+\beta^2)\begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & vw \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \beta\left(-\alpha^2 + 4\beta w\alpha + \beta^2\right)v & -\alpha\left(-\alpha^2 + 4\beta w\alpha + \beta^2\right)v \\ -\beta\left(w\alpha^2 + \alpha\beta - w\beta^2\right)v & \alpha\left(w\alpha^2 + \alpha\beta - w\beta^2\right)v \end{pmatrix}$$

$$(\alpha^2+\beta^2)\begin{pmatrix} \alpha^2-\beta^2 & -4\alpha\beta \\ -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & vw \end{pmatrix}\begin{pmatrix} -\alpha & +\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\beta\left(-\alpha^2 + 4\beta w\alpha + \beta^2\right)v & -\alpha\left(-\alpha^2 + 4\beta w\alpha + \beta^2\right)v \\ -\beta\left(w\alpha^2 + \alpha\beta - w\beta^2\right)v & -\alpha\left(w\alpha^2 + \alpha\beta - w\beta^2\right)v \end{pmatrix}$$

Once again we can get back to $pc = 0$ unless $-\alpha^2 + 4\beta w\alpha + \beta^2 = 0$ has no solution. So if $p = 1 \bmod 4$ we can take $v = 1$ and w to be one of the $(p-1)/2$ elements such that $(2w)^2 + 1$ is not a square, or one of the two elements such that $(2w)^2 + 1 = 0$. But we need to stick to $\beta = 0$, which means that $w \sim -w$. so there are $(p+3)/4$ orbits. And if $p = 3 \bmod 4$ then we can take $v = 1$ and w to be one of the $(p+1)/2$ elements such that $(2w)^2 + 1$ is not a square. But we need to stick to $\beta = 0$, which means that $w \sim -w$. so there are $(p+1)/4$ orbits.

So the total number of orbits with pc, pd spanning a one dimensional space is

$1 + 3 + (p+3)/4 + (p+3)/4 = (p+11)/2$ if $p = 1 \bmod 4$.

$1 + (p+1)/4 + (p+1)/4 = (p+3)/2$ if $p = 3 \bmod 4$.

Now for each of these choices of pc, pd we need to work out the possibilities for pa, pb.

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ -\beta & -\alpha & -\lambda & -\mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix}\begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

Now if $pc=0$, $pd\neq 0$ we need $\alpha=0$ or $\beta=0$.

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} \frac{1}{\alpha^2}u & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^2}w & \frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ 0 & \frac{m}{\alpha} \end{pmatrix}$$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ -\beta & -\alpha & -\lambda & -\mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} -\frac{1}{\alpha^2}u & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^2}w & -\frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ 0 & -\frac{m}{\alpha} \end{pmatrix}$$

So there are $p(p+1)/2$ orbits here.

Now try $y=z=0$, $t=1$, $m\neq 0$. $\alpha=0$ or $\beta=0$, for $\beta=0$ need $\alpha=1$. If $\alpha=0$ we need $\beta=m$, $m^2=-1$, $\delta=x$, $\mu=-v$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & x \\ 0 & 0 \\ 1 & m \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & mx \\ 0 & -mv \\ 0 & 0 \\ 1 & m \end{pmatrix}$$

This case arises when $m^2=-1=\begin{pmatrix} \frac{x-\delta}{m^2} & \frac{1}{m^3}\delta \\ -\frac{v+\mu}{m^2} & \frac{1}{m^3}\mu \\ 0 & 0 \\ 1 & -\frac{1}{m} \end{pmatrix}=\begin{pmatrix} -\frac{-x\beta+\delta m}{\beta^3} & \frac{1}{\beta^3}\delta \\ -\frac{v\beta+\mu m}{\beta^3} & \frac{1}{\beta^3}\mu \\ 0 & 0 \\ \frac{m}{\beta} & -\frac{1}{\beta} \end{pmatrix}=\begin{pmatrix} \frac{1}{\alpha^3}\delta & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^3}\mu & \frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & \frac{m}{\alpha} \end{pmatrix}$

v,x unchanged

For the second matrix need $\beta=m$, $m=\pm 1$, $\delta=x$, $\mu=-v$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ -\beta & -\alpha & -\lambda & -\mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & x \\ 0 & 0 \\ 1 & m \end{pmatrix}\begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & \frac{1}{m^3}x \\ 0 & \frac{1}{m^3}v \\ 0 & 0 \\ 1 & \frac{1}{m} \end{pmatrix}.$$

We only need $m = 1$, so no change.$= \begin{pmatrix} \frac{-x\beta+\delta m}{\beta^3} & \frac{1}{\beta^3}\delta \\ -\frac{v\beta+\mu m}{\beta^3} & -\frac{1}{\beta^3}\mu \\ 0 & 0 \\ \frac{m}{\beta} & \frac{1}{\beta} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^3}\delta & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^3}\mu & -\frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & -\frac{m}{\alpha} \end{pmatrix}$
no good as it changes m.

So for $y = z = 0$, $t = 1$, $m \neq 0$ we have p^2 orbits, except for one case when $m^2 = \pm 1$ when the number of orbits is $(p^2 + p)/2$.

$$(\alpha^2+\beta^2)^{-1} \begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^3}\delta & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^3}\mu & \frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & \frac{m}{\alpha} \end{pmatrix}$$

no change$= \begin{pmatrix} \frac{x-\delta}{m^2} & mx \\ -\frac{v+\mu}{m^2} & -mv \\ 0 & 0 \\ 1 & -\frac{1}{m} \end{pmatrix} = \begin{pmatrix} -\frac{-x\beta+\delta m}{\beta^3} & \frac{1}{\beta^3}\delta \\ -\frac{v\beta+\mu m}{\beta^3} & \frac{1}{\beta^3}\mu \\ 0 & 0 \\ \frac{m}{\beta} & -\frac{1}{\beta} \end{pmatrix}$

Need $\beta = m$, $m^2 = -1$

$$(\alpha^2+\beta^2)^{-1} \begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ -\beta & -\alpha & -\lambda & -\mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & -\alpha\beta & -(\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{x-\delta}{m^2} & x \\ -\frac{v+\mu}{m^2} & v \\ 0 & 0 \\ 1 & \frac{1}{m} \end{pmatrix} = \begin{pmatrix} \frac{-x\beta+\delta m}{\beta^3} & \frac{1}{\beta^3}\delta \\ -\frac{v\beta+\mu m}{\beta^3} & -\frac{1}{\beta^3}\mu \\ 0 & 0 \\ \frac{m}{\beta} & \frac{1}{\beta} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^3}\delta & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{1}{\alpha^3}\mu & -\frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & -\frac{m}{\alpha} \end{pmatrix}$$

$\alpha = 0$ $\beta = m$, $m^2 = 1$.
Now consider

$$(\alpha^2+\beta^2)^{-1} \begin{pmatrix} \alpha & -\beta & \gamma & \delta \\ \beta & \alpha & \lambda & \mu \\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 1 \\ 0 & m \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2}u & \frac{\alpha v+\gamma+\delta m}{\alpha^3} \\ \frac{1}{\alpha^2}w & \frac{\alpha x+\lambda+\mu m}{\alpha^3} \\ 0 & \frac{1}{\alpha} \\ 0 & \frac{m}{\alpha} \end{pmatrix}$$

$$(\alpha^2+\beta^2)^{-1}\begin{pmatrix}\alpha & -\beta & \gamma & \delta\\ \beta & \alpha & \lambda & \mu\\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta\\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2)\end{pmatrix}\begin{pmatrix}u & v\\ w & x\\ 0 & 1\\ 0 & m\end{pmatrix}\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}\frac{1}{\alpha^2}u & \frac{\alpha v+\gamma+\delta m}{\alpha^3}\\ \frac{1}{\alpha^2}w & \frac{\alpha x+\lambda+\mu m}{\alpha^3}\\ 0 & \frac{1}{\alpha}\\ 0 & \frac{m}{\alpha}\end{pmatrix}2$$

Need $\beta=0$, $\alpha=1$. Can take $v=x=0$, but no change in u,w, so p^2 orbits.
Finally

$$(\alpha^2+\beta^2)\begin{pmatrix}\alpha & -\beta & \gamma & 0\\ \beta & \alpha & \lambda & 0\\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta\\ 0 & 0 & \alpha\beta & (\alpha^2-\beta^2)\end{pmatrix}\begin{pmatrix}0 & v\\ 0 & x\\ 1 & z\\ t & tz\end{pmatrix}\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}\beta^2x-\beta\alpha v-\beta\gamma z+\alpha\gamma & \beta\gamma-\alpha\beta x+\alpha^2v+\alpha\gamma z\\ -\beta^2v-\beta\lambda z-\alpha\beta x+\alpha\lambda & \beta\lambda+\beta\alpha v+\alpha^2x+\alpha\lambda z\\ (-\alpha+z\beta)\left(\beta^2+4\alpha\beta t-\alpha^2\right) & -(\beta+z\alpha)\left(\beta^2+4\alpha\beta t-\alpha^2\right)\\ (-\alpha+z\beta)\left(t\beta^2-\alpha\beta-t\alpha^2\right) & -(\beta+z\alpha)\left(t\beta^2-\alpha\beta-t\alpha^2\right)\end{pmatrix}$$

$$(\alpha^2+\beta^2)\begin{pmatrix}\alpha & -\beta & \gamma & 0\\ -\beta & -\alpha & -\lambda & 0\\ 0 & 0 & \alpha^2-\beta^2 & -4\alpha\beta\\ 0 & 0 & -\alpha\beta & -(\alpha^2-\beta^2)\end{pmatrix}\begin{pmatrix}0 & v\\ 0 & x\\ 1 & z\\ t & tz\end{pmatrix}\begin{pmatrix}-\alpha & \beta\\ \beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}-\beta^2x+\beta\alpha v+\beta\gamma z-\alpha\gamma & \beta\gamma-\alpha\beta x+\alpha^2v+\alpha\gamma z\\ -\beta^2v-\beta\lambda z-\alpha\beta x+\alpha\lambda & -\beta\lambda-\beta\alpha v-\alpha^2x-\alpha\lambda z\\ -(-\alpha+z\beta)\left(\beta^2+4\alpha\beta t-\alpha^2\right) & -(\beta+z\alpha)\left(\beta^2+4\alpha\beta t-\alpha^2\right)\\ (-\alpha+z\beta)\left(t\beta^2-\alpha\beta-t\alpha^2\right) & (\beta+z\alpha)\left(t\beta^2-\alpha\beta-t\alpha^2\right)\end{pmatrix}$$

Guess that number of orbits is $2p$ when $4t^2+1=0$ and $p(p+1)/2$ for the third case. We can take $\delta=\mu=0$. We can always take $u=w=0$.

To preserve the ratio $1:z$ we need to stick to the ørst type of transformation.

If $4t^2+1$ is not a square then (from above) we need $\alpha=0$ or $\beta=0$. If $\beta=0$ then we need $\alpha=1$, and there is no change. If $\alpha=0$ then we need $\beta=1$, and $(v,x)\mapsto(zx,-zv)$. So there are $p(p+1)/2$ orbits.

If $4t^2+1=0$, then (from above) the ratio of pc to pd is preserved for all α,β, but we still need

$$(-\alpha+z\beta)\left(\beta^2+4\alpha\beta t-\alpha^2\right)=(\alpha^2+\beta^2)^2.$$

Note that $2t=\pm z$. This restricts us to $p-1$ pairs α,β. We see that

$$(v,x)\mapsto(\alpha^2+\beta^2)^{-2}((\alpha+z\beta)(-x\beta+\alpha v),(\alpha+z\beta)(x\beta+\alpha v))$$

So

$$\begin{pmatrix}v\\ x\end{pmatrix}\mapsto\frac{\alpha+z\beta}{(\alpha^2+\beta^2)^2}\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}v\\ x\end{pmatrix}$$

Consider the case when $t=-z/2$. Then

$$\alpha^2-4\alpha\beta t-\beta^2=(\alpha+z\beta)^2$$

so

$$(\alpha - z\beta)(\alpha + z\beta)^2 = (\alpha^2 + \beta^2)^2 = (\alpha - z\beta)^2(\alpha + z\beta)^2.$$

It follows that $(\alpha - z\beta) = 1$ and $(\alpha^2 + \beta^2)^2 = (\alpha + z\beta)^2$. So (v, x) is øxed if

$$\begin{pmatrix} 1 + z\beta & -\beta \\ \beta & 1 + z\beta \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} = (1 + 2z\beta) \begin{pmatrix} v \\ x \end{pmatrix}.$$

This is equivalent to $z\beta v = -\beta x$, $\beta v = z\beta x$. So if $\beta = 0$ then all p^2 values of (v, x) are øxed, and if $\beta \neq 0$ then p values are øxed. The number of allowable values of β is $p - 1$ so the number of orbits is $(p^2 + (p-2)p)/(p-1) = 2p$.

Next let $t = z/2$.

$$(\alpha^2 + \beta^2) \left(\begin{pmatrix} \alpha & -\beta & \gamma & 0 \\ \beta & \alpha & \lambda & 0 \\ 0 & 0 & \alpha^2 - \beta^2 & -4\alpha\beta \\ 0 & 0 & \alpha\beta & (\alpha^2 - \beta^2) \end{pmatrix} \right) \left(\begin{pmatrix} 0 & v \\ 0 & x \\ 1 & z \\ z/2 & -1/2 \end{pmatrix} \right) \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \gamma\alpha - \alpha\beta v + \beta^2 x - \beta\gamma z & \alpha^2 v + \alpha\gamma z - \alpha\beta x + \gamma\beta \\ \lambda\alpha - \alpha\beta x - v\beta^2 - \beta\lambda z & x\alpha^2 + \alpha\beta v + \alpha\lambda z + \lambda\beta \\ \alpha^3 - 3\beta z\alpha^2 - 3\alpha\beta^2 + z\beta^3 & z\alpha^3 + 3\beta\alpha^2 - 3\alpha z\beta^2 - \beta^3 \\ \frac{1}{2}z\alpha^3 + \frac{3}{2}\beta\alpha^2 - \frac{3}{2}\alpha z\beta^2 - \frac{1}{2}\beta^3 & -\frac{1}{2}\alpha^3 + \frac{3}{2}\beta z\alpha^2 + \frac{3}{2}\alpha\beta^2 - \frac{1}{2}z\beta^3 \end{pmatrix}$$

We require $\alpha^3 - 3\beta z\alpha^2 - 3\alpha\beta^2 + z\beta^3 = (\alpha^2 + \beta^2)^2$.
$\alpha^3 - 3\beta z\alpha^2 + 3z^2\alpha\beta^2 - z^3\beta^3 = (\alpha - z\beta)^3$
$(\alpha^2 + \beta^2)^2 = (\alpha - z\beta)^2(\alpha + z\beta)^2$,
so we require $(\alpha - z\beta) = (\alpha + z\beta)^2$.
As above we have

$$\begin{pmatrix} v \\ x \end{pmatrix} \mapsto \frac{\alpha + z\beta}{(\alpha^2 + \beta^2)^2} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix}$$

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} - (\alpha - z\beta)(\alpha^2 + \beta^2) \begin{pmatrix} v \\ x \end{pmatrix}$$

$$\begin{pmatrix} -x\beta + \alpha v + (z\beta - \alpha)\left(\alpha^2 + \beta^2\right) v \\ \beta v + \alpha x + (z\beta - \alpha)\left(\alpha^2 + \beta^2\right) x \end{pmatrix}$$

$$\begin{pmatrix} \alpha + (z\beta - \alpha)\left(\alpha^2 + \beta^2\right) & -\beta \\ \beta & \alpha + (z\beta - \alpha)\left(\alpha^2 + \beta^2\right) \end{pmatrix}$$

Determinant: $\left(\alpha^2 + \beta^2\right)\left(\alpha^4 - 2z\alpha^3\beta - 2\alpha^2 + \alpha^2\beta^2z^2 + \alpha^2\beta^2 - 2z\alpha\beta^3 + 2z\alpha\beta + 1 + \beta^4z^2\right)$
$\left(\alpha^4 - 2z\alpha^3\beta - 2\alpha^2 - 2z\alpha\beta^3 + 2z\alpha\beta + 1 - \beta^4\right) = -\left(\alpha^2 - 1 + \beta^2\right)\left(-\alpha^2 + 2z\alpha\beta + 1 + \beta^2\right)$
$-\left(-\alpha^2 + 2z\alpha\beta + 1 + \beta^2\right) = \alpha^2 - 2z\alpha\beta - 1 - \beta^2 = (\alpha - z\beta)^2 - 1.$
So the rank is 2 unless $\alpha^2 + \beta^2 = 1$ or $(\alpha - zb)^2 = 1$.
Note that the rank is at least1 if $\beta \neq 0$,
and that if $\beta = 0$ then we require $\alpha = 1$, which gives rank 0. Also note that if $\alpha^2 + \beta^2 = 1$ then $(\alpha - z\beta)^3 = 1$.

If $(\alpha - z\beta) = 1$ then $(\alpha + z\beta) = \pm 1$, so that we either have $\alpha = 1$, $\beta = 0$ (which gives rank 0), or $\alpha = 0$, $\beta = z$.

If $(\alpha - z\beta) = -1$ then $(\alpha + z\beta) = \pm z$, so that we either have $\alpha = (z-1)/2$, $\beta = -\alpha$, or $\alpha = -(1+z)/2$, $\beta = \alpha$.

Next, suppose that $p = 1 \bmod 12$, so that there is an element $\varepsilon \neq 1$ such that $\varepsilon^3 = 1$. Then we consider the cases $(\alpha - z\beta) = \varepsilon, \varepsilon^2$ and $\alpha^2 + \beta^2 = 1$.

First consider the case when $(\alpha - z\beta) = \varepsilon$. Then $(\alpha + z\beta) = \varepsilon^2$.

Let $(\alpha - z\beta) = \varepsilon$, $(\alpha + z\beta) = \varepsilon^2$. Then $\alpha = -1/2$, $\beta = \frac{\varepsilon^2 - \varepsilon}{2z}$.

Next consider the case when $(\alpha - z\beta) = \varepsilon^2$. Then $(\alpha + z\beta) = \varepsilon$. This gives $\alpha = -1/2$, $\beta = \frac{-\varepsilon^2+\varepsilon}{2z}$.

So if $p \neq 1 \bmod 12$ then we have $((p^2-1) + 3(p-1) + (p-1))/(p-1) = p+5$ orbits.

And if $p = 1 \bmod 12$ then we have $((p^2-1) + 5(p-1) + (p-1))/(p-1) = p+7$ orbits.

62.3 Totals

If $pc = pd = 0$ the number of algebras is $p^2 + (7p+15)/2$ when $p = 1 \bmod 4$ and $p^2 + (3p+5)/2$ when $p = 3 \bmod 4$.

The possibilities when $\begin{pmatrix} pc \\ pd \end{pmatrix}$ has rank 1 are:

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ which gives $p(p+1)/2$ algebras.

$(p+1)/4$ if $p = 3 \bmod 4$ or $(p+3)/4$ if $p = 1 \bmod 4$ matrices of the form $\begin{pmatrix} 0 & 0 \\ 1 & k \end{pmatrix}$, each of which leads to p^2 algebras, except for $k = 1$ which gives $p(p+1)/2$ algebras, and (if $p = 1 \bmod 4$) one value of k such that $k^2 = -1$ which gives $p(p+1)/2$ algebras.

$(p+1)/4$ if $p = 3 \bmod 4$ or $(p+3)/4$ if $p = 1 \bmod 4$ matrices of the form $\begin{pmatrix} 0 & 1 \\ 0 & k \end{pmatrix}$, each of which leads to p^2 algebras.

If $p = 1 \bmod 4$ we also get 3 matrices of the form $\begin{pmatrix} 1 & z \\ t & tz \end{pmatrix}$ where $1 + z^2 = 0$. One gives $p(p+1)/2$ algebras, another $2p$ algebras and another $p+5$ algebras, or $p+7$ if $p = 1 \bmod 3$.

The total number of possibilities for pc, pd is
$(p^2+3p+14)/2$ if $p = 1 \bmod 12$,
$(p^2+3p+12)/2$ if $p = 5 \bmod 12$,
$(p^2+p+4)/2$ if $p = 7 \bmod 12$,
$(p^2+p+6)/2$ if $p = 11 \bmod 12$.

The total number of orbits with pc, pd spanning a one dimensional space is
$1 + 3 + (p+3)/4 + (p+3)/4 = (p+11)/2$ if $p = 1 \bmod 4$.
$1 + (p+1)/4 + (p+1)/4 = (p+3)/2$ if $p = 3 \bmod 4$.

If pc, pd span a space of dimension 2 then there is only one algebra with that pc, pd.

62.4 $p = 1 \bmod 12$

$(p^2+3p+14)/2 - (p+11)/2 - 1 = \frac{1}{2}p^2 + p + \frac{1}{2}$ with pc, pd rank 2.

$p^2 + (7p+15)/2$ with $pc = pd = 0$.

$p(p+1)/2 + ((p+3)/4 - 2)p^2 + p(p+1) + p^2(p+3)/4 + p(p+1)/2 + 2p + p + 7 = \frac{3}{2}p^2 + 5p + \frac{1}{2}p^3 + 7$ with pc, pd rank 1.

So the grand total is $\frac{1}{2}p^2+p+\frac{1}{2}+p^2+(7p+15)/2+\frac{3}{2}p^2+5p+\frac{1}{2}p^3+7 = 3p^2+\frac{19}{2}p+15+\frac{1}{2}p^3$

62.5 $p = 5 \bmod 12$

$(p^2+3p+12)/2 - (p+11)/2 - 1 = \frac{1}{2}p^2 + p - \frac{1}{2}$ with pc, pd rank 2.

$p^2 + (7p+15)/2$ with $pc = pd = 0$.

$p(p+1)/2 + ((p+3)/4 - 2)p^2 + p(p+1) + p^2(p+3)/4 + p(p+1)/2 + 2p + p + 5 = \frac{3}{2}p^2 + 5p + \frac{1}{2}p^3 + 5$ with pc, pd rank 1.

So the grand total is $\frac{1}{2}p^2+p-\frac{1}{2}+p^2+(7p+15)/2+\frac{3}{2}p^2+5p+\frac{1}{2}p^3+5 = 3p^2+\frac{19}{2}p+12+\frac{1}{2}p^3$

62.6 $\quad p = 7 \bmod 12$

$(p^2+p+4)/2-(p+3)/2-1 = \frac{1}{2}p^2 - \frac{1}{2}$ with pc, pd of rank 2.

$p^2 + (3p+5)/2$ with $pc = pd = 0$.

$p(p+1)/2 + p^2((p+1)/4-1) + p(p+1)/2 + p^2(p+1)/4 = \frac{1}{2}p^2 + p + \frac{1}{2}p^3$ with pc, pd of rank 1.

So the grand total is $\frac{1}{2}p^2 - \frac{1}{2} + p^2 + (3p+5)/2 + \frac{1}{2}p^2 + p + \frac{1}{2}p^3 = 2p^2 + 2 + \frac{5}{2}p + \frac{1}{2}p^3$.

62.7 $\quad p = 11 \bmod 12$

$(p^2+p+6)/2-(p+3)/2-1 = \frac{1}{2}p^2 + \frac{1}{2}$ with pc, pd of rank 2.

$p^2 + (3p+5)/2$ with $pc = pd = 0$.

$p(p+1)/2 + p^2((p+1)/4-1) + p(p+1)/2 + p^2(p+1)/4 = \frac{1}{2}p^2 + p + \frac{1}{2}p^3$ with pc, pd of rank 1.

So the grand total is $\frac{1}{2}p^2 + \frac{1}{2} + p^2 + (3p+5)/2 + \frac{1}{2}p^2 + p + \frac{1}{2}p^3 = 2p^2 + 3 + \frac{5}{2}p + \frac{1}{2}p^3$

62.8 Check for theory over experiment

$$\begin{pmatrix} \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 & -4\alpha\beta & 0 \\ \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & 0 & -4\alpha\beta \\ \alpha\beta & 0 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 \end{pmatrix}$$

Determinant: $\left(\beta^2+1-2\alpha+\alpha^2\right)\left(\beta^4+\beta^2+2\alpha^2\beta^2+6\alpha\beta^2+\alpha^2+\alpha^4-2\alpha^3\right)\left(\alpha^2+\beta^2\right)^3$

We have $\alpha^2+\beta^2 \neq 0$.

It turns out that this matrix has 1 occurence of rank 0; 4,2,0,2 of rank 2 depending on whether $p = 1, 5, 7, 11 \bmod 12$; and $4p-16, 4p-12, 0, 0$ of rank 3.

Consider the case when $\alpha = 0$. We have

$$\begin{pmatrix} -\beta^2 & -\beta^3 & 0 & 0 \\ \beta^3 & -\beta^2 & 0 & 0 \\ 0 & 0 & -\beta^2 & -\beta^3 \\ 0 & 0 & \beta^3 & -\beta^2 \end{pmatrix}$$

which has rank 4 if $\beta^2 \neq -1$, and rank 2 if $\beta^2 = -1$.

Next consider the case when $\beta = 0$. We have

$$\begin{pmatrix} \alpha^2-\alpha^3 & 0 & 0 & 0 \\ 0 & \alpha^2-\alpha^3 & 0 & 0 \\ 0 & 0 & \alpha^2-\alpha^3 & 0 \\ 0 & 0 & 0 & \alpha^2-\alpha^3 \end{pmatrix}$$

which has rank 4 if $\alpha \neq 1$ and rank 0 if $\alpha = 1$.

Now consider the case when $\alpha\beta \neq 0$.

$$\begin{pmatrix} -\alpha\beta & 0 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 \\ 0 & -\alpha\beta & \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 & -4\alpha\beta & 0 \\ \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & 0 & -4\alpha\beta \\ \alpha\beta & 0 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 & -\beta\alpha^2-\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3 & \alpha^2-\beta^2-\alpha^3-\alpha\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha^2\beta^2+\alpha^4-2\alpha^5+\beta^4+2\beta^4\alpha+\alpha^6+\alpha^4\beta^2-\alpha^2\beta^4-\beta^6 & 2\left(-\alpha^2+\beta^2+\alpha^3+\alpha\beta^2\right)\beta\left(\alpha^2+\beta^2\right) \\ -2\left(-\alpha^2+\beta^2+\alpha^3+\alpha\beta^2\right)\beta\left(\alpha^2+\beta^2\right) & 2\alpha^2\beta^2+\alpha^4-2\alpha^5+\beta^4+2\beta^4\alpha+\alpha^6+\alpha^4\beta^2-\alpha^2\beta^4-\beta^6 \end{pmatrix}$$

Determinant: $\left(\alpha^2-2\alpha+1+\beta^2\right)\left(\alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4\right)\left(\alpha^2+\beta^2\right)^3$
$2\alpha^2\beta^2+\alpha^4-2\alpha^5+\beta^4+2\beta^4\alpha+\alpha^6+\alpha^4\beta^2-\alpha^2\beta^4-\beta^6$
$=\left(\alpha^2+\beta^2\right)\left(\alpha^4-2\alpha^3+\alpha^2+2\alpha\beta^2-\beta^4+\beta^2\right)$

Note that the determinant is the sum of two squares, so if $p=3 \bmod 4$ then then we can only get determinant zero if all entries in the matrix are zero. We have already dealt with the cases $\alpha=0$ and $\beta=0$. So suppose that

$$-\alpha^2+\beta^2+\alpha^3+\alpha\beta^2=0.$$

We have $\beta^2=\alpha^2\frac{1-\alpha}{1+\alpha}$. This gives

$$2\alpha^2\beta^2+\alpha^4-2\alpha^5+\beta^4+2\beta^4\alpha+\alpha^6+\alpha^4\beta^2-\alpha^2\beta^4-\beta^6=-4\alpha^4\frac{-1+2\alpha^2-\alpha}{(\alpha+1)^3}$$

So we need $-1+2\alpha^2-\alpha=0$, Solution is : $\alpha=1,-\frac{1}{2}$. But $\alpha=1$ gives $\beta=0$, so we discount this case. If $\alpha=-\frac{1}{2}$ then $\beta^2=\frac{3}{4}$, and so we get two rank 2 matrices if 3 is a square modulo p. Now -3 is a square if $p=1 \bmod 3$ and not a square if $p=2 \bmod 3$. So 3 is a square if $p=1 \bmod 12$ or if $p=11 \bmod 12$, but not if $p=5$ or $7 \bmod 12$.

This gives a proof that numbers of rank 2 matrices are as given by the experimental evidence.

If $p=1 \bmod 4$ then we have the possibility of rank 3 matrices. We have $2(p-1)$ non-zero solutions to the equation

$$i^2+j^2=0,$$

and we get $2(p-1)$ rank 3 matrices taking $\alpha=1+i$, $\beta=j$. (Note that we only have the zero matrix if $\alpha=-\frac{1}{2}$ and $\beta^2=\frac{3}{4}$, which gives $\alpha^2-2\alpha+1+\beta^2=3$, so there is no overlap with rank 2 matrices.) But if $\alpha=\frac{1}{2}$ we have $\alpha^2+\beta^2=0$, and there are two solutions to this. So the actual contribution to the number of rank 3 matrices with $\alpha^2+\beta^2\neq 0$ is $2p-4$.

What about $\alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4=0$, Solution is :
$\left\{\beta=\frac{1}{2}\sqrt{(-8\alpha-1)}+\frac{1}{2}i\left(2\alpha+1\right)\right\}$,
$\left\{\beta=\frac{1}{2}\sqrt{(-8\alpha-1)}-\frac{1}{2}i\left(2\alpha+1\right)\right\}$,
$\left\{\beta=-\frac{1}{2}\sqrt{(-8\alpha-1)}+\frac{1}{2}i\left(2\alpha+1\right)\right\}$,
$\left\{\beta=-\frac{1}{2}\sqrt{(-8\alpha-1)}-\frac{1}{2}i\left(2\alpha+1\right)\right\}$?

So we have four solutions for each value of α such that $(-8\alpha-1)$ is a square. This gives $\alpha=-\frac{1}{8}$ and $(p-1)/2$ other values. However we want to discount $\alpha=0$, and $\alpha=1$ gives $\beta=0$ or $\sqrt{-9}$, so we need to discount that as well. So we have a total of $(p-3)/2$ possible values of α (including $\alpha=-\frac{1}{8}$).

Can any of these solutions overlap with $\alpha^2-2\alpha+1+\beta^2=0$? Let $\beta^2=-(\alpha-1)^2$. Then

$$\begin{aligned} & \alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4 \\ = & \alpha^4-2\alpha^3+\alpha^2-2\alpha^2(\alpha-1)^2-6\alpha(\alpha-1)^2-(\alpha-1)^2+(\alpha-1)^4 \\ = & -8\alpha\left(\alpha-1\right)^2 \end{aligned}$$

so the answer is ınoȷ.

Finally, can any of these solutions overlap with $\alpha^2+\beta^2=0$. In this case

$$\begin{aligned} & \alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4 \\ = & \alpha^4-2\alpha^3+\alpha^2-2\alpha^4-6\alpha^3-\alpha^2+\alpha^4 \\ = & -8\alpha^3 \end{aligned}$$

so again the answer is ınoȷ.

Can any of these solutions overlap with rank 2 matrices? Yes in the case when $\alpha=-\frac{1}{2}$ and when 3 is a square (which only happens when $p=1 \bmod 12$), because we then get $\beta=\pm\frac{1}{2}\sqrt{3}$, which gives rank 2 to the original 4×4 matrix. So we need to discount $\alpha=-\frac{1}{2}$ when $p=1 \bmod 12$.

Can there be any repeats among these solutions? If $\alpha=-\frac{1}{2}$ or $-\frac{1}{8}$ then we only get two values for β. Now $\alpha=-\frac{1}{2}$ gives $-8\alpha-1=3$, so this situation only arises when $p=1 \bmod 12$. And $\alpha=-\frac{1}{8}$ gives only two values for β

So when $p=5 \bmod 12$ we have $2(p-4)$ rank 3 matrices arising from

$$\alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4=0$$

and the total number of rank 3 matrices is $4p-12$, conørming the experimental evidence.

When $p=1 \bmod 12$ we have $2(p-6)$ rank 3 matrices arising from

$$\alpha^4-2\alpha^3+\alpha^2+2\alpha^2\beta^2+6\alpha\beta^2+\beta^2+\beta^4=0$$

so it seems that the total number of rank 3 matrices is $4p-16$, conørming the experimental evidence!

63 Appendix C

Case 7 in the descendants of 5.3

Now let L satisfy $da=dc=0$, $ca=db=bab$, $cb=\omega baa$. If a',b',c',d' satisfy the same commutator relations then

$$\begin{aligned} a' &= \alpha a+\beta b+\gamma c+\delta d, \\ b' &= \pm(-\omega\beta a+\alpha b+\lambda c+\mu d), \\ c' &= (\alpha^2-\omega\beta^2)c+4\omega\alpha\beta d, \\ d' &= \pm(-\alpha\beta c+(\alpha^2-\omega\beta^2)d) \end{aligned}$$

modulo L^2 which gives

$$\begin{aligned} b'a'a' &= \pm(\alpha^2+\omega\beta^2)(\alpha baa+\beta bab), \\ b'a'b' &= (\alpha^2+\omega\beta^2)(-\omega\beta baa+\alpha bab). \end{aligned}$$

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} - (\alpha^2+\omega\beta^2) \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u+\beta w+\gamma y+\delta t-\alpha^3u-\alpha u\omega\beta^2+\omega\beta v\alpha^2+\omega^2\beta^3v & \alpha v+\beta x+\gamma z+\delta m-u\beta\alpha^2-u\beta^3\omega-\alpha^3v-\alpha v\omega\beta^2 \\ -\omega\beta u+\alpha w+\lambda y+\mu t-\alpha^3w-\alpha w\omega\beta^2+x\omega\beta\alpha^2+x\omega^2\beta^3 & -\omega\beta v+\alpha x+\lambda z+\mu m-\beta w\alpha^2-\beta^3w\omega-\alpha^3x-\alpha x\omega\beta^2 \\ y\alpha^2-y\omega\beta^2+4\omega\alpha\beta t-y\alpha^3-y\alpha\omega\beta^2+z\omega\beta\alpha^2+z\omega^2\beta^3 & z\alpha^2-z\omega\beta^2+4\omega\alpha\beta m-y\beta\alpha^2-y\beta^3\omega-z\alpha^3-z\alpha\omega\beta^2 \\ -\alpha\beta y+t\alpha^2-t\omega\beta^2-t\alpha^3-t\alpha\omega\beta^2+m\omega\beta\alpha^2+m\omega^2\beta^3 & -\alpha\beta z+m\alpha^2-m\omega\beta^2-t\beta\alpha^2-t\beta^3\omega-m\alpha^3-m\alpha\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha u - \alpha^3 u - \alpha u\omega\beta^2 & \omega\beta v\alpha^2 + \omega^2\beta^3 v & \beta w & 0 & \gamma y \\ -u\beta\alpha^2 - u\beta^3\omega & \alpha v - \alpha^3 v - \alpha v\omega\beta^2 & 0 & \beta x & 0 \\ -\omega\beta u & 0 & \alpha w - \alpha^3 w - \alpha w\omega\beta^2 & x\omega\beta\alpha^2 + x\omega^2\beta^3 & \lambda y \\ 0 & -\omega\beta v & -\beta w\alpha^2 - \beta^3 w\omega & \alpha x - \alpha^3 x - \alpha x\omega\beta^2 & 0 \\ 0 & 0 & 0 & 0 & y\alpha^2 - y\omega\beta^2 - y\alpha^3 - y\alpha\omega\beta^2 \\ 0 & 0 & 0 & 0 & -y\beta\alpha^2 - y\beta^3\omega \\ 0 & 0 & 0 & 0 & -\alpha\beta y \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

runmatrixon $$\begin{pmatrix} 0 & \delta t & 0 \\ \gamma z & 0 & \delta m \\ 0 & \mu t & 0 \\ \lambda z & 0 & \mu m \\ z\omega\beta\alpha^2 + z\omega^2\beta^3 & 4\omega\alpha\beta t & 0 \\ z\alpha^2 - z\omega\beta^2 - z\alpha^3 - z\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta m \\ 0 & t\alpha^2 - t\omega\beta^2 - t\alpha^3 - t\alpha\omega\beta^2 & m\omega\beta\alpha^2 + m\omega^2\beta^3 \\ -\alpha\beta z & -t\beta\alpha^2 - t\beta^3\omega & m\alpha^2 - m\omega\beta^2 - m\alpha^3 - m\alpha\omega\beta^2 \end{pmatrix}$$

Determinant:
$\left(\beta^2\omega + 1 + 2\alpha + \alpha^2\right)\left(\omega^2\beta^4 + 6\beta^2\omega\alpha + \beta^2\omega + 2\omega\alpha^2\beta^2 + \alpha^2 - 2\alpha^3 + \alpha^4\right)$
$\times \left(\beta^2\omega - 1 + \alpha^2\right)^2 \left(\beta^2\omega + 1 - 2\alpha + \alpha^2\right)^2 \left(\beta^2\omega + \alpha^2\right)^5$
Not very promising!

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} - (\alpha^2 + \omega\beta^2) \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}$$

63.1 $pc = pd = 0$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2 + \omega\beta^2) \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u + \beta w - \alpha u\omega\beta^2 - \alpha^3 u + \beta^3\omega^2 v + \beta\omega v\alpha^2 & \alpha v + \beta x - u\beta\alpha^2 - u\beta^3\omega - \alpha v\beta^2\omega - \alpha^3 v \\ -\omega\beta u + \alpha w - \alpha w\omega\beta^2 - \alpha^3 w + x\omega^2\beta^3 + x\omega\beta\alpha^2 & -\beta\omega v + \alpha x - \beta^3 w\omega - \beta w\alpha^2 - \alpha x\beta^2\omega - \alpha^3 x \end{pmatrix}$$

$$\begin{pmatrix} \alpha u - \alpha^3 u - \alpha u\omega\beta^2 & \omega\beta v\alpha^2 + \omega^2\beta^3 v & \beta w & 0 \\ -u\beta\alpha^2 - u\beta^3\omega & \alpha v - \alpha^3 v - \alpha v\omega\beta^2 & 0 & \beta x \\ -\omega\beta u & 0 & \alpha w - \alpha^3 w - \alpha w\omega\beta^2 & x\omega\beta\alpha^2 + x\omega^2\beta^3 \\ 0 & -\omega\beta v & -\beta w\alpha^2 - \beta^3 w\omega & \alpha x - \alpha^3 x - \alpha x\omega\beta^2 \end{pmatrix}$$

Determinant: $\left(\beta^2\omega + 1 - 2\alpha + \alpha^2\right)\left(\beta^2\omega + 1 + 2\alpha + \alpha^2\right)\left(\beta^2\omega + \alpha^2\right)^2\left(\beta^2\omega - 1 + \alpha^2\right)^2$

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & -\alpha \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} - (\alpha^2 + \omega\beta^2) \begin{pmatrix} u & v \\ w & x \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha u + \beta w + \alpha u\omega\beta^2 + \alpha^3 u + \beta^3\omega^2 v + \beta\omega v\alpha^2 & \alpha v + \beta x + u\beta\alpha^2 + u\beta^3\omega - \alpha v\beta^2\omega - \alpha^3 v \\ \omega\beta u - \alpha w + \alpha w\omega\beta^2 + \alpha^3 w + x\omega^2\beta^3 + x\omega\beta\alpha^2 & \beta\omega v - \alpha x + \beta^3 w\omega + \beta w\alpha^2 - \alpha x\beta^2\omega - \alpha^3 x \end{pmatrix}$$

$$\begin{pmatrix} \alpha u + \alpha^3 u + \alpha u\omega\beta^2 & \omega\beta v\alpha^2 + \omega^2\beta^3 v & \beta w & 0 \\ u\beta\alpha^2 + u\beta^3\omega & \alpha v - \alpha^3 v - \alpha v\omega\beta^2 & 0 & \beta x \\ \omega\beta u & 0 & -\alpha w + \alpha^3 w + \alpha w\omega\beta^2 & x\omega\beta\alpha^2 + x\omega^2\beta^3 \\ 0 & \omega\beta v & \beta w\alpha^2 + \beta^3 w\omega & -\alpha x - \alpha^3 x - \alpha x\omega\beta^2 \end{pmatrix}$$

Determinant: $\left(\beta^2\omega - 1 + \alpha^2\right)^2\left(\beta^2\omega + \alpha^2\right)^2\left(\beta^2\omega + 1 + \alpha^2\right)^2$

First matrix for $pc = pd = 0$. Determinant is only zero if $\alpha = \pm 1$ and $\beta = 0$, or $\alpha^2 + \beta^2\omega = 1$, which happens $p+1$ times (including $\alpha = \pm 1$ and $\beta = 0$). Arghhhhhhh! There are lots of other zeros!

$$\begin{pmatrix} \alpha - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 & \beta & 0 \\ -\beta\alpha^2 - \beta^3\omega & \alpha - \alpha^3 - \alpha\omega\beta^2 & 0 & \beta \\ -\omega\beta & 0 & \alpha - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 \\ 0 & -\omega\beta & -\beta\alpha^2 - \beta^3\omega & \alpha - \alpha^3 - \alpha\omega\beta^2 \end{pmatrix}$$

If $\beta = 0$ we have

$$\begin{pmatrix} -\alpha^3 + \alpha & 0 & 0 & 0 \\ 0 & -\alpha^3 + \alpha & 0 & 0 \\ 0 & 0 & -\alpha^3 + \alpha & 0 \\ 0 & 0 & 0 & -\alpha^3 + \alpha \end{pmatrix}$$

which has rank 0 for $\alpha = \pm 1$.

So suppose that $\alpha^2 + \beta^2\omega = 1$, with $\beta \neq 0$.

$$\begin{pmatrix} \alpha - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 & \beta & 0 \\ -\beta\alpha^2 - \beta^3\omega & \alpha - \alpha^3 - \alpha\omega\beta^2 & 0 & \beta \\ -\omega\beta & 0 & \alpha - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 \\ 0 & -\omega\beta & -\beta\alpha^2 - \beta^3\omega & \alpha - \alpha^3 - \alpha\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \omega\beta & \beta & 0 \\ -\beta & 0 & 0 & \beta \\ -\omega\beta & 0 & 0 & \omega\beta \\ 0 & -\omega\beta & -\beta & 0 \end{pmatrix}$$

So the rank is 2.

The evidence is that the number of times the rank is 0,1,2,3 is $2, 0, p-1, 0$ if $p = 1 \bmod 4$ and $2, 0, p-1, 4p-12$ if $p = 3 \bmod 4$. This is proved in Case 1 below. The group has order $p^2 - 1$ if $p = 1 \bmod 4$, and order $(p-1)^2$ if $p = 3 \bmod 4$.

Contribution to Burnside's Lemma is $2(p^4-1)+(p-1)(p^2-1)+(p^2-1) = (p-1)(p+1)\left(2p^2+p+2\right)$ if $p = 1 \bmod 4$, and

$2(p^4-1) + (p-1)(p^2-1) + (4p-12)(p-1) + (p-1)^2 = \left(2p^2+5p+12\right)(p-1)^2$ if $p = 3 \bmod 4$.

Second matrix for $pc = pd = 0$. Determinant is only zero if $\alpha^2 + \beta^2\omega = \pm 1$, which happens $p+1$ times each if $p = 1 \bmod 4$ and $p-1$ times each if $p = 3 \bmod 4$.

$$\begin{pmatrix} \alpha + \alpha^3 + \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 & \beta & 0 \\ \beta\alpha^2 + \beta^3\omega & \alpha - \alpha^3 - \alpha\omega\beta^2 & 0 & \beta \\ \omega\beta & 0 & -\alpha + \alpha^3 + \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 \\ 0 & \omega\beta & \beta\alpha^2 + \beta^3\omega & -\alpha - \alpha^3 - \alpha\omega\beta^2 \end{pmatrix}$$

Try $\alpha^2 + \beta^2\omega = 1$.

$$\begin{pmatrix} 2\alpha & \omega\beta & \beta & 0 \\ \beta & 0 & 0 & \beta \\ \omega\beta & 0 & 0 & \omega\beta \\ 0 & \omega\beta & \beta & -2\alpha \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha & \beta & 0 \\ \beta & 0 & \beta \\ 0 & \beta & -2\alpha \end{pmatrix}$$

Determinant: 0, so the rank is 2.

Try $\alpha^2 + \beta^2\omega = -1$.

$$\begin{pmatrix} 0 & -\omega\beta & \beta & 0 \\ -\beta & 2\alpha & 0 & \beta \\ \omega\beta & 0 & -2\alpha & -\omega\beta \\ 0 & \omega\beta & -\beta & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\omega\beta & \beta \\ -\beta & 2\alpha & 0 \\ \omega\beta & 0 & -2\alpha \end{pmatrix}$$

Determinant: 0, so the rank is 2.

This proves that the rank is 0,1,2,3 the following number of times: 0, 0, $2p+2$ if $p = 1 \bmod 4$ and $2p-2$ if $p = 3 \bmod 4$, 0.

Contribution to Burnside's Lemma is $(2p+2)(p^2-1)+(p^2-1) = (p-1)(2p+3)(p+1)$ if $p = 1 \bmod 4$,

and $(2p-2)(p^2-1) + (p-1)^2 = (2p+3)(p-1)^2$ if $p = 3 \bmod 4$.

So the number of algebras with $pc = pd = 0$ is $(2p^2+p+2+2p+3)/2 = p^2 + \frac{3}{2}p + \frac{5}{2}$ if $p = 1 \bmod 4$,

and $(2p^2+5p+12+2p+3)/2 = p^2 + \frac{7}{2}p + \frac{15}{2}$ if $p = 3 \bmod 4$.

63.2 How many pc, pd?

$$\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} y & z \\ t & m \end{pmatrix} - (\alpha^2+\omega\beta^2) \begin{pmatrix} y & z \\ t & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} y\alpha^2 - y\omega\beta^2 + 4\alpha\omega\beta t - y\alpha\omega\beta^2 - y\alpha^3 + z\omega^2\beta^3 + z\omega\beta\alpha^2 & z\alpha^2 - z\omega\beta^2 + 4\alpha\omega\beta m - y\beta^3\omega - y\beta\alpha^2 - z\alpha\omega\beta^2 - z\alpha^3 \\ -\alpha\beta y + t\alpha^2 - t\omega\beta^2 - t\alpha\omega\beta^2 - t\alpha^3 + m\omega^2\beta^3 + m\omega\beta\alpha^2 & -\alpha\beta z + m\alpha^2 - m\omega\beta^2 - t\beta^3\omega - t\beta\alpha^2 - m\alpha\omega\beta^2 - m\alpha^3 \end{pmatrix}$$

$$\begin{pmatrix} y\alpha^2 - y\omega\beta^2 - y\alpha^3 - y\alpha\omega\beta^2 & z\omega\beta\alpha^2 + z\omega^2\beta^3 & 4\omega\alpha\beta t & 0 \\ -y\beta\alpha^2 - y\beta^3\omega & z\alpha^2 - z\omega\beta^2 - z\alpha^3 - z\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta m \\ -\alpha\beta y & 0 & t\alpha^2 - t\omega\beta^2 - t\alpha^3 - t\alpha\omega\beta^2 & m\omega\beta\alpha^2 + m\omega^2\beta^3 \\ 0 & -\alpha\beta z & -t\beta\alpha^2 - t\beta^3\omega & m\alpha^2 - m\omega\beta^2 - m\alpha^3 - m\alpha\omega\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2 - \omega\beta^2 - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ -\beta\alpha^2 - \beta^3\omega & \alpha^2 - \omega\beta^2 - \alpha^3 - \alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ -\alpha\beta & 0 & \alpha^2 - \omega\beta^2 - \alpha^3 - \alpha\omega\beta^2 & \omega\beta\alpha^2 + \omega^2\beta^3 \\ 0 & -\alpha\beta & -\beta\alpha^2 - \beta^3\omega & \alpha^2 - \omega\beta^2 - \alpha^3 - \alpha\omega\beta^2 \end{pmatrix}$$

Determinant: $\left(\omega\beta^2 + 1 - 2\alpha + \alpha^2\right)\left(\omega^2\beta^4 + 6\alpha\omega\beta^2 + 2\alpha^2\omega\beta^2 + \omega\beta^2 + \alpha^2 + \alpha^4 - 2\alpha^3\right)\left(\omega\beta^2 + \alpha^2\right)^3$

$$\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} y & z \\ t & m \end{pmatrix} - (\alpha^2+\omega\beta^2) \begin{pmatrix} y & z \\ t & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} y\alpha^2 - y\omega\beta^2 + 4\alpha\omega\beta t + y\alpha\omega\beta^2 + y\alpha^3 + z\omega^2\beta^3 + z\omega\beta\alpha^2 & z\alpha^2 - z\omega\beta^2 + 4\alpha\omega\beta m + y\beta^3\omega + y\beta\alpha^2 - z\alpha\omega\beta^2 - z\alpha^3 \\ \alpha\beta y - t\alpha^2 + t\omega\beta^2 + t\alpha\omega\beta^2 + t\alpha^3 + m\omega^2\beta^3 + m\omega\beta\alpha^2 & \alpha\beta z - m\alpha^2 + m\omega\beta^2 + t\beta^3\omega + t\beta\alpha^2 - m\alpha\omega\beta^2 - m\alpha^3 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2-\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ \beta\alpha^2+\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ \alpha\beta & 0 & -\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3\omega & -\alpha^2+\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

Determinant: $\left(\omega\beta^2-1+\alpha^2\right)^2\left(\omega\beta^2+\alpha^2\right)^4$

Consider the ørst matrix: it has rank 4 unless $\alpha=1$ and $\beta=0$, or that horrible quartic is zero.

$$\begin{pmatrix} \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ -\alpha\beta & 0 & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\alpha\beta & -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

If $\alpha=1$ and $\beta=0$ then the matrix has rank 0.

The evidence is that it has rank 0,1,2,3 the following number of times, depending on the value of $p \bmod 12$:

$1,0,0,0$ if $p=1 \bmod 12$,

$1,0,2,0$ if $p=5 \bmod 12$,

$1,0,4,4p-16$ if $p=7 \bmod 12$

$1,0,2,4p-12$ if $p=11 \bmod 12$.

This is proved in Case 2 below.

So the contribution to Burnside's Lemma is

$(p^4-1)+(p^2-1)=(p-1)(p+1)\left(p^2+2\right)$ if $p=1 \bmod 12$,

$(p^4-1)+2(p^2-1)+(p^2-1)=(p-1)(p+1)\left(p^2+4\right)$ if $p=5 \bmod 12$,

$(p^4-1)+4(p^2-1)+(4p-16)(p-1)+(p-1)^2=\left(p^2+2p+12\right)(p-1)^2$ if $p=7 \bmod 12$

$(p^4-1)+2(p^2-1)+(4p-12)(p-1)+(p-1)^2=\left(p^2+2p+10\right)(p-1)^2$ if $p=11 \bmod 12$.

Now consider the second matrix.

$$\begin{pmatrix} \alpha^2-\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ \beta\alpha^2+\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ \alpha\beta & 0 & -\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3\omega & -\alpha^2+\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

The evidence is that it has rank 0,1,2,3 the following number of times: 0, 0, $p+1$ if $p=1 \bmod 4$ and $p-1$ if $p=3 \bmod 4$, 0.

The contribution to Burnside's Lemma is $(p+1)(p^2-1)+(p^2-1)=(p-1)(p+2)(p+1)$ if $p=1 \bmod 4$, and

$(p-1)(p^2-1)+(p-1)^2=(p+2)(p-1)^2$ if $p=3 \bmod 4$.

So the number of possibilities for pc, pd is

$(p^2+2+p+2)/2=\frac{1}{2}p^2+2+\frac{1}{2}p$ if $p=1 \bmod 12$,

$(p^2+4+p+2)/2=\frac{1}{2}p^2+3+\frac{1}{2}p$ if $p=5 \bmod 12$,

$(p^2+2p+12+p+2)/2=\frac{1}{2}p^2+\frac{3}{2}p+7$ if $p=7 \bmod 12$,

$(p^2+2p+10+p+2)/2=\frac{1}{2}p^2+\frac{3}{2}p+6$ if $p=11 \bmod 12$.

Now investigate the possibilities for pc, pd spanning a space of dimension $1.\beta=0$

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} y & z \\ t & m \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} y & z \\ t & m \end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

First consider the case when $y = z = 0$. Then we require $\alpha = 0$ or $\beta = 0$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & m\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha} & \frac{m}{\alpha}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ -\frac{m}{\beta} & \frac{1}{\beta\omega}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ -\frac{m}{\beta} & \frac{t}{\beta\omega}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \frac{t}{\alpha} & \frac{m}{\alpha}\end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & m\end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & 0 \\ \frac{1}{\alpha} & -\frac{m}{\alpha}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ -\frac{m}{\beta} & -\frac{1}{\beta\omega}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ -\frac{m}{\beta} & -\frac{t}{\beta\omega}\end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \frac{t}{\alpha} & -\frac{m}{\alpha}\end{pmatrix}.$$

So we get $\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}$ and matrices $\begin{pmatrix} 0 & 0 \\ 1 & m\end{pmatrix}$ with $m \neq 0$, where $m \sim \pm m, \pm\frac{1}{\omega m}$. So there are $(p-1)/4$ orbits of matrices $\begin{pmatrix} 0 & 0 \\ 1 & m\end{pmatrix}$ if $p = 1 \bmod 4$ and $(p+1)/4$ if $p = 3 \bmod 4$.

Next, consider pc non-zero. Note that we have to consider the possibility that $pc \neq 0$, $pd = 0$.

$$(\alpha^2+\omega\beta^2)\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} y & z \\ wy & wz\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$\begin{pmatrix} (y\alpha+z\omega\beta)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) & -(y\beta-z\alpha)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) \\ -(y\alpha+z\omega\beta)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) & (y\beta-z\alpha)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)\end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} y & z \\ wy & wz\end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$\begin{pmatrix} -(y\alpha+z\omega\beta)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) & -(y\beta-z\alpha)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) \\ -(y\alpha+z\omega\beta)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) & -(y\beta-z\alpha)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)\end{pmatrix}$$

So we can always get $y = 0$ unless $y^2 + \omega z^2 = 0$. And we can also get $y = z = 0$ if we can solve $-\omega\beta^2 + 4\alpha w\omega\beta + \alpha^2 = 0$. We can solve this provided $16\omega^2 w^2 + 4\omega$ is a square, which occurs whenever $4\omega w^2 + 1$ is not a square, or when $4\omega w^2 + 1 = 0$. The case $4\omega w^2 + 1 = 0$ occurs twice when $p = 3 \bmod 4$ and not at all when $p = 1 \bmod 4$. But in the case when $4\omega w^2 + 1 = 0$ the solutions for α, β are disallowed, since $\alpha^2 + \omega\beta^2 = 0$.

So we can always get $y = 0$ if $p = 1 \bmod 4$.

Consider the case when $y = 0$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} 0 & z \\ 0 & wz\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$=\begin{pmatrix} 0 & \frac{z}{\alpha} \\ 0 & w\frac{z}{\alpha}\end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-2}\begin{pmatrix} \beta\omega\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)z & \alpha\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)z \\ -\beta\omega\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)z & -\alpha\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)z\end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2)\end{pmatrix}\begin{pmatrix} 0 & z \\ 0 & wz\end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha\end{pmatrix}^{-1}$$

$$\begin{pmatrix} 0 & \frac{z}{\alpha} \\ 0 & -w\frac{z}{\alpha} \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-2}\begin{pmatrix} -\beta\omega\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)z & \alpha\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)z \\ -\beta\omega\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)z & \alpha\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)z \end{pmatrix}$$

So we are only concerned with this situation when $4\omega w^2+1$ is a square, or we are back in the situation $y=z=0$. To preserve $y=0$ we need $\beta=0$. Every orbits has a pair $1,w$ with $4\omega w^2+1$ a square, and with $w\sim -w$. So the number of orbits is $(k+1)/2$, where k is the number of w such that $4\omega w^2+1$ is a square. Now $k=(p+1)/2$ if $p=1 \bmod 4$, and $k=(p+3)/2$ if $p=3 \bmod 4$, so the number of orbits is $(p+3)/4$ if $p=1 \bmod 4$ and $(p+5)/4$ if $p=3 \bmod 4$.

Finally consider the case when $p=3 \bmod 4$ and $y^2+\omega z^2=0$. Then we can take $y=1$ and z to be one of the two values such that $1+\omega z^2=0$.

$$(\alpha^2+\omega\beta^2)\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} 1 & z \\ w & wz \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$\begin{pmatrix} (\alpha+z\omega\beta)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) & (-\beta+z\alpha)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) \\ -(\alpha+z\omega\beta)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) & -(-\beta+z\alpha)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)\begin{pmatrix} (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} 1 & z \\ w & wz \end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$\begin{pmatrix} -(\alpha+z\omega\beta)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) & (-\beta+z\alpha)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right) \\ -(\alpha+z\omega\beta)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) & (-\beta+z\alpha)\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right) \end{pmatrix}$$

To preserve the ratio we need to restrict our attention to matrices of the ørst type. And we also require

$$(\alpha+z\omega\beta)\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)=(\alpha^2+\omega\beta^2)^2.$$

Clearly we can scale α,β to ensure that this is satisøed, and so w can get transformed to

$$\frac{-\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)}{\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)}$$

Suppose that w is øxed. Then

$$\left(w\omega\beta^2+\alpha\beta-w\alpha^2\right)+w\left(-\omega\beta^2+4\alpha w\omega\beta+\alpha^2\right)=\beta\alpha\left(1+4w^2\omega\right)=0.$$

So the two solutions of $1+4w^2\omega=0$ are øxed by everything, and transformations with $\alpha=0$ or $\beta=0$ øx all w. So the number of orbits of the $(p+3)/\ w's$ is

$$\frac{(p-1)(p+3)+2((p-1)^2-2(p-1))}{(p-1)^2}=3.$$

So we have three matrices of the form $\begin{pmatrix} 1 & z \\ w & wz \end{pmatrix}$.

So the total list of rank 1 possibilities for pc, pd is:

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

$(p-1)/4$ if $p=1 \bmod 4$ and $(p+1)/4$ if $p=3 \bmod 4$ matrices $\begin{pmatrix} 0 & 0 \\ 1 & m \end{pmatrix}$,

$(p+3)/4$ if $p=1 \bmod 4$ and $(p+5)/4$ if $p=3 \bmod 4$ matrices $\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$, 3 matrices $\begin{pmatrix} 1 & z \\ w & wz \end{pmatrix}$ when $p = 3 \bmod 4$, where $1 + \omega z^2 = 0$, and $4\omega w^2 + 1$ is a square. (In two of the cases $4\omega w^2 + 1 = 0$, and in the third we can take $w = 0$.)

Now, for each rank one matrix representing pc, pd we need to compute the number of associated algebras.

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & \alpha & -\lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \\ t & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

First consider $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2 + \omega\beta^2)^{-2} \begin{pmatrix} x\omega\beta^2 + \beta\omega\alpha v + \beta\omega\delta + \beta\alpha w + \alpha^2 u & -w\beta^2 + \beta\alpha x - \beta\alpha u + \alpha^2 v + \alpha\delta \\ -v\omega^2\beta^2 - \beta\alpha u\omega + \beta\omega\mu + \beta\alpha x\omega + w\alpha^2 & u\omega\beta^2 - \beta\omega\alpha v - \beta\alpha w + \alpha\mu + x\alpha^2 \\ 4\omega^2\alpha\beta^2 & 4\omega\beta\alpha^2 \\ -\left(-\alpha^2 + \omega\beta^2\right)\omega\beta & -\left(-\alpha^2 + \omega\beta^2\right)\alpha \end{pmatrix}$$

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2 + \omega\beta^2)^{-2} \begin{pmatrix} -x\omega\beta^2 - \beta\omega\alpha v - \beta\omega\delta - \beta\alpha w - \alpha^2 u & -w\beta^2 + \beta\alpha x - \beta\alpha u + \alpha^2 v + \alpha\delta \\ -\beta\alpha u\omega + w\alpha^2 - v\omega^2\beta^2 + \beta\alpha x\omega - \beta\omega\mu & -u\omega\beta^2 + \beta\alpha w + \beta\omega\alpha v - x\alpha^2 + \alpha\mu \\ -4\omega^2\alpha\beta^2 & 4\omega\beta\alpha^2 \\ -\left(-\alpha^2 + \omega\beta^2\right)\omega\beta & \left(-\alpha^2 + \omega\beta^2\right)\alpha \end{pmatrix}$$

So we need $\alpha = 0$ or $\beta = 0$.

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2}u & \frac{v\alpha+\delta}{\alpha^3} \\ \frac{1}{\alpha^2}w & \frac{x\alpha+\mu}{\alpha^3} \\ 0 & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\beta x+\delta}{\omega\beta^3} & -\frac{1}{\beta^2\omega^2}w \\ -\frac{\omega\beta v-\mu}{\omega\beta^3} & \frac{1}{\beta^2}\frac{u}{\omega} \\ 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix}$$

$$(\alpha^2 + \omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2 - \omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2 - \omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{1}{\alpha^2}u & \frac{v\alpha+\delta}{\alpha^3} \\ \frac{1}{\alpha^2}w & -\frac{x\alpha-\mu}{\alpha^3} \\ 0 & 0 \\ 0 & -\frac{1}{\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{\beta x+\delta}{\omega\beta^3} & -\frac{1}{\beta^2\omega^2}w \\ -\frac{\omega\beta v+\mu}{\omega\beta^3} & -\frac{1}{\beta^2}\frac{u}{\omega} \\ 0 & 0 \\ -\frac{1}{\beta} & 0 \end{pmatrix}$$

We need to restrict ourselves to $\beta = 0$. Clearly we can restrict ourselves to $v = x = 0$, and we see that $(u, w) \sim (-u, w)$. So there are $p(p+1)/2$ algebras.

Next consider $\begin{pmatrix} 0 & 0 \\ 1 & m \end{pmatrix}$.

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2+\omega\beta^2)^{-2} \begin{pmatrix} x\omega\beta^2+\beta\omega v\alpha+\beta\omega\delta m+\beta\alpha w+\alpha^2u+\alpha\delta & -w\beta^2-\beta\alpha u-\beta\delta+\beta x\alpha+v\alpha^2+\alpha\delta m \\ -v\omega^2\beta^2-\beta\alpha u\omega+\beta\omega\mu m+\beta\alpha x\omega+\alpha\mu+w\alpha^2 & u\omega\beta^2-\beta\omega v\alpha-\beta\mu-\beta\alpha w+x\alpha^2+\alpha\mu m \\ 4\omega\alpha^2\beta+4\omega^2\alpha\beta^2m & -4\alpha\omega\beta^2+4\omega\beta m\alpha^2 \\ \alpha^3-\alpha\omega\beta^2+\omega\beta m\alpha^2-\omega^2\beta^3m & -\beta\alpha^2+\omega\beta^3+m\alpha^3-\alpha m\omega\beta^2 \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2+\omega\beta^2)^{-2} \begin{pmatrix} -x\omega\beta^2-\beta\omega v\alpha-\beta\omega\delta m-\beta\alpha w-\alpha^2u-\alpha\delta & -w\beta^2-\beta\alpha u-\beta\delta+\beta x\alpha+v\alpha^2+\alpha\delta m \\ -v\omega^2\beta^2-\beta\alpha u\omega-\beta\omega\mu m+\beta\alpha x\omega-\alpha\mu+w\alpha^2 & -u\omega\beta^2+\beta\omega v\alpha-\beta\mu+\beta\alpha w-x\alpha^2+\alpha\mu m \\ -4\omega\alpha^2\beta-4\omega^2\alpha\beta^2m & -4\alpha\omega\beta^2+4\omega\beta m\alpha^2 \\ \alpha^3-\alpha\omega\beta^2+\omega\beta m\alpha^2-\omega^2\beta^3m & \beta\alpha^2-\omega\beta^3-m\alpha^3+\alpha m\omega\beta^2 \end{pmatrix}$$

So we need $\alpha = 0$ or $\beta = 0$ or $m^2 = -\frac{1}{\omega}$ and $\beta = m\alpha$. However this last possibility leads to $\alpha^2 + \omega\beta^2 = 0$, and so we can rule it out.

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{\alpha u+\delta}{\alpha^3} & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{\alpha w+\mu}{\alpha^3} & \frac{x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & \frac{m}{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\beta x+\delta m}{\omega\beta^3} & -\frac{\beta w+\delta}{\omega^2\beta^3} \\ \frac{-\omega\beta v+\mu m}{\omega\beta^3} & \frac{\omega\beta u-\mu}{\omega^2\beta^3} \\ 0 & 0 \\ -\frac{m}{\beta} & \frac{1}{\beta\omega} \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 0 \\ 1 & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{\alpha u+\delta}{\alpha^3} & \frac{v\alpha+\delta m}{\alpha^3} \\ \frac{-\mu+\alpha w}{\alpha^3} & \frac{-x\alpha+\mu m}{\alpha^3} \\ 0 & 0 \\ \frac{1}{\alpha} & -\frac{m}{\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{\beta x+\delta m}{\omega\beta^3} & -\frac{\beta w+\delta}{\omega^2\beta^3} \\ -\frac{\omega\beta v+\mu m}{\omega\beta^3} & -\frac{\omega\beta u+\mu}{\omega^2\beta^3} \\ 0 & 0 \\ -\frac{m}{\beta} & -\frac{1}{\beta\omega} \end{pmatrix}$$

If $\alpha = 0$ we need $\beta = -m$, and so we have nothing unless $m = \pm\frac{1}{m\omega}$. If $\beta = 0$ we need $\alpha = 1$.

So ørst consider the case when $\alpha = 1$ and $\beta = 0$. Clearly we can take $u = w = 0$, which requires $\delta = \mu = 0$. So there is no change.

So we get p^2 algebras for any given m, unless $m^2\omega = \pm 1$. Now $m^2\omega = 1$ is impossible, but if $p = 3 \bmod 4$ then there is one orbit of $m's$ with a solution. So assume that $m = -\frac{1}{m\omega}$. Then only the ørst matrix helps, with $\alpha = 0$, $\beta = -m$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & x \\ 0 & 0 \\ 1 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{x-\delta}{\omega m^2} & \frac{1}{\omega^2 m^3}\delta \\ -\frac{v\omega+\mu}{\omega m^2} & \frac{1}{\omega^2 m^3}\mu \\ 0 & 0 \\ 1 & -\frac{1}{m\omega} \end{pmatrix}$$

So we need $\delta = x$ and $\mu = -v\omega$, giving $\begin{pmatrix} 0 & \frac{1}{\omega^2 m^3}x \\ 0 & -\frac{1}{\omega m^3}v \\ 0 & 0 \\ 1 & -\frac{1}{m\omega} \end{pmatrix}$. So

$$(v,x) \sim (\frac{1}{\omega^2 m^3}x, -\frac{1}{\omega m^3}v) = (\frac{-1}{\omega m}x, \frac{1}{m}v).$$

So the pairs (v,x) are in orbits of size two, unless $v = \frac{-1}{\omega m}x$. So there are p orbits of size 1, and $(p^2-p)/2$ orbits of size 2. So the number of algebras here is $(p^2+p)/2$.

Next consider $\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ 0 & 1 \\ 0 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2+\omega\beta^2)^{-2}\begin{pmatrix} x\omega\beta^2+\beta\omega v\alpha+\beta\omega\delta m+\beta\omega\gamma+\beta\alpha w+\alpha^2 u & -w\beta^2+\beta x\alpha-\beta\alpha u+v\alpha^2+\alpha\delta m+\alpha\gamma \\ -v\omega^2\beta^2-\beta\alpha u\omega+\beta\omega\lambda+\beta\alpha x\omega+\beta\omega\mu m+w\alpha^2 & u\omega\beta^2-\beta\omega v\alpha-\beta\alpha w+\alpha\lambda+x\alpha^2+\alpha\mu m \\ \beta\omega\left(-\omega\beta^2+4\omega\alpha\beta m+\alpha^2\right) & \alpha\left(-\omega\beta^2+4\omega\alpha\beta m+\alpha^2\right) \\ -\beta\omega\left(\omega m\beta^2+\alpha\beta-m\alpha^2\right) & -\alpha\left(\omega m\beta^2+\alpha\beta-m\alpha^2\right) \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ 0 & 1 \\ 0 & m \end{pmatrix}\begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2+\omega\beta^2)^{-2}\begin{pmatrix} -x\omega\beta^2-\beta\omega v\alpha-\beta\omega\delta m-\beta\omega\gamma-\beta\alpha w-\alpha^2 u & -w\beta^2+\beta x\alpha-\beta\alpha u+v\alpha^2+\alpha\delta m+\alpha\gamma \\ -v\omega^2\beta^2-\beta\alpha u\omega-\beta\omega\lambda+\beta\alpha x\omega-\beta\omega\mu m+w\alpha^2 & -u\omega\beta^2+\beta\omega v\alpha+\beta\alpha w+\alpha\lambda-x\alpha^2+\alpha\mu m \\ -\beta\omega\left(-\omega\beta^2+4\omega\alpha\beta m+\alpha^2\right) & \alpha\left(-\omega\beta^2+4\omega\alpha\beta m+\alpha^2\right) \\ -\beta\omega\left(\omega m\beta^2+\alpha\beta-m\alpha^2\right) & \alpha\left(\omega m\beta^2+\alpha\beta-m\alpha^2\right) \end{pmatrix}$$

Now $-\omega\beta^2+4\omega\alpha\beta m+\alpha^2 = 0$ has no solution for these values of m, and so we need $\beta = 0$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} u & v \\ w & x \\ 0 & 1 \\ 0 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\alpha^2}u & \frac{v\alpha+\delta m+\gamma}{\alpha^3} \\ \frac{1}{\alpha^2}w & \frac{x\alpha+\lambda+\mu m}{\alpha^3} \\ 0 & \frac{1}{\alpha} \\ 0 & \frac{m}{\alpha} \end{pmatrix}$$

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \omega\beta & -\alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & \alpha\beta & -(\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 0 & 1 \\ 0 & m \end{pmatrix} \begin{pmatrix} -\alpha & -\beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{1}{\alpha^2}u & \frac{v\alpha+\delta m+\gamma}{\alpha^3} \\ \frac{1}{\alpha^2}w & \frac{-x\alpha+\lambda+\mu m}{\alpha^3} \\ 0 & \frac{1}{\alpha} \\ 0 & -\frac{m}{\alpha} \end{pmatrix}$$

So only the ørst matrix will do for $m \neq 0$, with $\alpha = 1$. Clearly we can take $v = x = 0$, and there are p^2 algebras for each $m \neq 0$. For $m = 0$ we have $p(p+1)/2$ algebras.

Finally consider $\begin{pmatrix} 1 & z \\ m & mz \end{pmatrix}$ when $p = 3 \bmod 4$, where $1 + \omega z^2 = 0$, and $4\omega m^2 + 1$ is a square. We can take $\omega = -1$, $z = 1$ and $m = 0, \pm 1/2$. In fact there are two with $4\omega m^2 + 1$, and one with $m = 0$. As we showed above, to preserve this matrix we are restricted to transformations of the ørst type with

$$(\alpha + z\omega\beta)\left(-\omega\beta^2 + 4\alpha m\omega\beta + \alpha^2\right) = (\alpha^2+\omega\beta^2)^2.$$

Take $\omega = -1$, $z = 1$, $m = 0$.

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 1 & z \\ m & mz \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= (\alpha^2+\omega\beta^2)^{-2} \begin{pmatrix} -x\beta^2+\beta\alpha w-\beta\alpha v-\beta\gamma+\alpha^2 u+\alpha\gamma & -w\beta^2-\beta\alpha u-\beta\gamma+\beta\alpha x+\alpha^2 v+\alpha\gamma \\ -v\beta^2+\beta\alpha u-\beta\lambda-\beta\alpha x+\alpha\lambda+w\alpha^2 & -u\beta^2-\beta\lambda-\beta\alpha w+\beta\alpha v+x\alpha^2+\alpha\lambda \\ \alpha^3+\alpha\beta^2-\beta\alpha^2-\beta^3 & \alpha^3+\alpha\beta^2-\beta\alpha^2-\beta^3 \\ -\beta\alpha^2+\alpha\beta^2 & -\beta\alpha^2+\alpha\beta^2 \end{pmatrix}$$

We must have $\alpha = 0$ or $\beta = 0$, since $\alpha = \beta$ is not possible with $\omega = -1$, and these give

$$\begin{pmatrix} \frac{\alpha u+\gamma}{\alpha^3} & \frac{\alpha v+\gamma}{\alpha^3} \\ \frac{\alpha w+\lambda}{\alpha^3} & \frac{\alpha x+\lambda}{\alpha^3} \\ \frac{1}{\alpha} & \frac{1}{\alpha} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{\beta x+\gamma}{\beta^3} & -\frac{\beta w+\gamma}{\beta^3} \\ -\frac{\beta v+\lambda}{\beta^3} & -\frac{\beta u+\lambda}{\beta^3} \\ -\frac{1}{\beta} & -\frac{1}{\beta} \\ 0 & 0 \end{pmatrix}$$

We can take $u = w = 0$ which in the ørst case above forces us to take $\alpha = 1$ and $\gamma = \lambda = 0$, giving no change, and in the second case forces us to take $\beta = -1$, $\gamma = -\beta x$, $\lambda = -\beta v$, changing (v, x) to (x, v). So there are $p(p+1)/2$ algebras here.

Now take $\omega = -1$, $z = 1$, $m = 1/2$.

$$(\alpha^2+\omega\beta^2)^{-1} \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\omega\beta & \alpha & \lambda & \mu \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ 1 & z \\ m & mz \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$= \quad (\alpha^2-\beta^2)^{-2}\times$$
$$\begin{pmatrix} -x\beta^2+\beta\alpha w-\beta\alpha v-\frac{1}{2}\beta\delta-\beta\gamma+\frac{1}{2}\alpha\delta+\alpha\gamma+\alpha^2 u & -w\beta^2-\frac{1}{2}\beta\delta+\beta\alpha x-\beta\gamma-\beta\alpha u+\frac{1}{2}\alpha\delta+\alpha^2 v+\alpha\gamma \\ -v\beta^2-\beta\lambda-\beta\alpha x+\beta\alpha u-\frac{1}{2}\beta\mu+w\alpha^2+\alpha\lambda+\frac{1}{2}\alpha\mu & -u\beta^2-\beta\lambda-\beta\alpha w+\beta\alpha v-\frac{1}{2}\beta\mu+x\alpha^2+\alpha\lambda+\frac{1}{2}\alpha\mu \\ (\alpha-\beta)^3 & (\alpha-\beta)^3 \\ \frac{1}{2}(\alpha-\beta)^3 & \frac{1}{2}(\alpha-\beta)^3 \end{pmatrix}$$

We can take $u=w=0$ and $\delta=\mu=0$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & 0 \\ -\omega\beta & \alpha & \lambda & 0 \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & x \\ 1 & z \\ m & mz \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$=(\alpha^2-\beta^2)^{-1}\begin{pmatrix} -x\beta^2-\beta\alpha v-\beta\gamma+\alpha\gamma & -\beta\gamma+\beta\alpha x+\alpha^2 v+\alpha\gamma \\ -v\beta^2-\beta\alpha x-\beta\lambda+\alpha\lambda & -\beta\lambda+\beta\alpha v+x\alpha^2+\alpha\lambda \\ (\alpha-\beta)^3 & (\alpha-\beta)^3 \\ \frac{1}{2}(\alpha-\beta)^3 & \frac{1}{2}(\alpha-\beta)^3 \end{pmatrix}$$

So we need to take $(\alpha-\beta)\gamma=x\beta^2+\beta\alpha v$ and $(\alpha-\beta)\lambda=v\beta^2+\beta\alpha x$ which gives

$$=(\alpha^2-\beta^2)^{-2}\begin{pmatrix} 0 & (\alpha+\beta)\,(\beta x+\alpha v) \\ 0 & (\alpha+\beta)\,(\beta v+\alpha x) \\ (\alpha-\beta)^3 & (\alpha-\beta)^3 \\ \frac{1}{2}(\alpha-\beta)^3 & \frac{1}{2}(\alpha-\beta)^3 \end{pmatrix}$$

Provided $v\neq\pm x$ we can take $v=0$. This then requires $\alpha=1$ and $\beta=0$ which gives no change to x. So we have p algebras with $v=0$.

Consider the case when $v=x$. Then

$$(x,x)\mapsto(\frac{x}{(\alpha-\beta)^2},\frac{x}{(\alpha-\beta)^2})$$

where $(\alpha+\beta)^2=\alpha-\beta$. And when $v=-x$ then

$$(-x,x)\mapsto(\frac{-x}{\alpha^2-\beta^2},\frac{x}{\alpha^2-\beta^2}).$$

There are $p-1$ pairs (α,β) satisfying $(\alpha^2-\beta^2)^2=(\alpha-\beta)^3$ with $\alpha^2\neq\beta^2$, and (x,x) is only øxed by $(1,0)$ and $(0,-1)$. So there are 2 orbits of pairs (x,x).

On the other hand, $(-x,x)$ is øxed if $\alpha^2-\beta^2=1$, which gives $(\alpha-\beta)^3=1$. So $(-x,x)$ is only øxed by the pair $(1,0)$ (giving one orbit), unless $p=1 \bmod 3$, so that there is an element $\varepsilon\neq1$ such that $\varepsilon^3=1$. Then $(-x,x)$ is øxed by two more pairs, so that there are 3 orbits $(-x,x)$.

So the total number of algebras with $\omega=-1$, $z=1$, $m=1/2$ is $p+3$ if $p\neq1 \bmod 3$ and $p+5$ if $p=1 \bmod 3$.

Finally consider the case when $\omega=-1$, $z=1$, $m=-1/2$.

$$(\alpha^2+\omega\beta^2)^{-1}\begin{pmatrix} \alpha & \beta & \gamma & 0 \\ -\omega\beta & \alpha & \lambda & 0 \\ 0 & 0 & (\alpha^2-\omega\beta^2) & 4\omega\alpha\beta \\ 0 & 0 & -\alpha\beta & (\alpha^2-\omega\beta^2) \end{pmatrix}\begin{pmatrix} 0 & v \\ 0 & x \\ 1 & z \\ m & mz \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & \alpha \end{pmatrix}^{-1}$$

$$\left(\begin{array}{cc} -x\beta^2-\beta\alpha v-\beta\gamma+\alpha\gamma & -\beta\gamma+\beta\alpha x+\alpha^2 v+\alpha\gamma \\ -v\beta^2-\beta\alpha x-\beta\lambda+\alpha\lambda & -\beta\lambda+\beta\alpha v+x\alpha^2+\alpha\lambda \\ (\alpha-\beta)(\alpha+\beta)^2 & (\alpha-\beta)(\alpha+\beta)^2 \\ -\frac{1}{2}(\alpha-\beta)(\alpha+\beta)^2 & -\frac{1}{2}(\alpha-\beta)(\alpha+\beta)^2 \end{array}\right)$$

So we require $\alpha-\beta=1$, which gives

$$(1+2\beta)^{-2}\left(\begin{array}{cc} -x\beta^2-\beta v-v\beta^2+\gamma & \beta x+x\beta^2+v+2\beta v+v\beta^2+\gamma \\ -v\beta^2-\beta x-x\beta^2+\lambda & \beta v+v\beta^2+x+2\beta x+x\beta^2+\lambda \\ (1+2\beta)^2 & (1+2\beta)^2 \\ -\frac{1}{2}(1+2\beta)^2 & -\frac{1}{2}(1+2\beta)^2 \end{array}\right)$$

So we need $\gamma=x\beta^2+\beta v+v\beta^2$ and $\lambda=v\beta^2+\beta x+x\beta^2$ which gives

$$(1+2\beta)^{-2}\left(\begin{array}{cc} 0 & (1+2\beta)(\beta x+\beta v+v) \\ 0 & (1+2\beta)(\beta x+\beta v+x) \\ (1+2\beta)^2 & (1+2\beta)^2 \\ -\frac{1}{2}(1+2\beta)^2 & -\frac{1}{2}(1+2\beta)^2 \end{array}\right)$$

So $(v,x)\mapsto(\frac{\beta x+\beta v+v}{1+2\beta},\frac{\beta x+\beta v+x}{1+2\beta})$. We see that (v,x) is øxed if

$$\begin{aligned} \beta x+\beta v+v &= (1+2\beta)v, \\ \beta x+\beta v+x &= (1+2\beta)x, \end{aligned}$$

that is if $\beta v=\beta x$. So $\beta=0$ øxes all p^2 possible (v,x), and if $\beta\neq 0$ then p pairs (v,x) are øxed. So the number of orbits is

$$\frac{p^2+(p-2)p}{p-1}=2p.$$

63.3 Totals

The number of algebras with $pc=pd=0$ is $p^2+\frac{3}{2}p+\frac{5}{2}$ if $p=1 \bmod 4$, and $p^2+\frac{7}{2}p+\frac{15}{2}$ if $p=3 \bmod 4$.

The number of possibilities for pc,pd is
$\frac{1}{2}p^2+2+\frac{1}{2}p$ if $p=1 \bmod 12$,
$\frac{1}{2}p^2+3+\frac{1}{2}p$ if $p=5 \bmod 12$,
$\frac{1}{2}p^2+\frac{3}{2}p+7$ if $p=7 \bmod 12$
$\frac{1}{2}p^2+\frac{3}{2}p+6$ if $p=11 \bmod 12$.
So the total list of rank 1 possibilities for pc,pd is:
$\left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right)$ giving $p(p+1)/2$ algebras.

$(p-1)/4$ if $p=1 \bmod 4$ all giving p^2 algebras and $(p+1)/4$ if $p=3 \bmod 4$ matrices $\left(\begin{array}{cc} 0 & 0 \\ 1 & m \end{array}\right)$ where all but one give p^2 algebras and one gives $(p^2+p)/2$ algebras,

$(p+3)/4$ if $p=1 \bmod 4$ and $(p+5)/4$ if $p=3 \bmod 4$ matrices $\left(\begin{array}{cc} 0 & 1 \\ 0 & m \end{array}\right)$ each giving p^2 algebras, except for $m=0$ which gives $p(p+1)/2$ algebras.

3 matrices $\left(\begin{array}{cc} 1 & z \\ w & wz \end{array}\right)$ when $p=3 \bmod 4$, where $1+\omega z^2=0$, and $4\omega w^2+1$ is a square. (In two of the cases $4\omega w^2+1=0$, and in the third we can take $w=0$.) These three possibilities give a total of

$(p^2+p)/2+p+3+2p=\frac{1}{2}p^2+\frac{7}{2}p+3$ algebras if $p\neq 1 \bmod 3$ and

$(p^2+p)/2+p+5+2p=\frac{1}{2}p^2+\frac{7}{2}p+5$ algebras if $p=1 \bmod 3$.

So the number of rank 1 possibilities for pc, pd is $(p+3)/2$ if $p=1 \bmod 4$ and $(p+11)/2$ when $p=3 \bmod 4$.

Thus the number of rank 2 possibilities for pc, pd is
$\frac{1}{2}p^2+2+\frac{1}{2}p-(p+3)/2-1=\frac{1}{2}p^2-\frac{1}{2}$ if $p=1 \bmod 12$,
$\frac{1}{2}p^2+3+\frac{1}{2}p-(p+3)/2-1=\frac{1}{2}p^2+\frac{1}{2}$ if $p=5 \bmod 12$,
$\frac{1}{2}p^2+\frac{3}{2}p+7-(p+11)/2-1=\frac{1}{2}p^2+p+\frac{1}{2}$ if $p=7 \bmod 12$,
$\frac{1}{2}p^2+\frac{3}{2}p+6-(p+11)/2-1=\frac{1}{2}p^2+p-\frac{1}{2}$ if $p=11 \bmod 12$.

63.4 Grand total

The total number of algebras is
$p^2+\frac{3}{2}p+\frac{5}{2}+\frac{1}{2}p^2-\frac{1}{2}+p(p+1)/2+p^2(p-1)/4+p^2(p-1)/4+p(p+1)/2$
$=2p^2+\frac{5}{2}p+2+\frac{1}{2}p^3$ if $p=1 \bmod 12$,
$p^2+\frac{3}{2}p+\frac{5}{2}+\frac{1}{2}p^2+\frac{1}{2}+p(p+1)/2+p^2(p-1)/4+p^2(p-1)/4+p(p+1)/2$
$=2p^2+\frac{5}{2}p+3+\frac{1}{2}p^3$ if $p=5 \bmod 12$,
$p^2+\frac{7}{2}p+\frac{15}{2}+\frac{1}{2}p^2+p+\frac{1}{2}+p(p+1)/2+p^2(p-3)/4+(p^2+p)/2+p^2(p+1)/4+p(p+1)/2+\frac{1}{2}p^2+\frac{7}{2}p+5$
$=3p^2+\frac{19}{2}p+13+\frac{1}{2}p^3$ if $p=7 \bmod 12$,
$p^2+\frac{7}{2}p+\frac{15}{2}+\frac{1}{2}p^2+p-\frac{1}{2}+p(p+1)/2+p^2(p-3)/4+(p^2+p)/2+p^2(p+1)/4+p(p+1)/2+\frac{1}{2}p^2+\frac{7}{2}p+3$
$=3p^2+\frac{19}{2}p+10+\frac{1}{2}p^3$ if $p=11 \bmod 12$.

63.5 Proofs conørming experiments

63.6 Case 1

$$\begin{pmatrix} \alpha-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & \beta & 0 \\ -\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3-\alpha\omega\beta^2 & 0 & \beta \\ -\omega\beta & 0 & \alpha-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\omega\beta & -\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

with $\left(\alpha^2+\omega\beta^2\right) \neq 0$.

If $\beta=0$ we have

$$\begin{pmatrix} -\alpha^3+\alpha & 0 & 0 & 0 \\ 0 & -\alpha^3+\alpha & 0 & 0 \\ 0 & 0 & -\alpha^3+\alpha & 0 \\ 0 & 0 & 0 & -\alpha^3+\alpha \end{pmatrix}$$

which has rank 0 for $\alpha=\pm 1$.

If $\beta \neq 0$ we have

The evidence is that the number of times the rank is 0,1,2,3 is $2,0,p-1,0$ if $p=1 \bmod 4$ and $2,0,p-1,4p-12$ if $p=3 \bmod 4$. The group has order p^2-1 if $p=1 \bmod 4$, and order $(p-1)^2$ if $p=3 \bmod 4$.

$$\begin{pmatrix} \omega\beta & 0 & \alpha-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \omega\beta & -\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3-\alpha\omega\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} \alpha-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & \beta & 0 \\ -\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3-\alpha\omega\beta^2 & 0 & \beta \\ -\omega\beta & 0 & \alpha-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\omega\beta & -\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

$$\left(\alpha^2+\omega\beta^2\right)\left(-1+\alpha^2+\omega\beta^2\right)\begin{pmatrix} \left(-\omega\beta^2-1+\alpha^2\right) & -2\alpha\omega\beta \\ 2\beta\alpha & \left(-\omega\beta^2-1+\alpha^2\right) \end{pmatrix}$$

Determinant: $\left(\omega\beta^2+1+2\alpha+\alpha^2\right)\left(\omega\beta^2+1-2\alpha+\alpha^2\right)\left(\alpha^2+\omega\beta^2\right)^2\left(-1+\alpha^2+\omega\beta^2\right)^2$

If $p=1 \bmod 4$ then $\left(\alpha^2+\omega\beta^2\right)=1$ has $p+1$ solutions, and the only solutions to

$$\left(\omega\beta^2+1+2\alpha+\alpha^2\right)\left(\omega\beta^2+1-2\alpha+\alpha^2\right)=0$$

are $\alpha=\pm 1$ and $\beta=0$. But the case $\beta=0$ was dealt with above. So proof conørms experiment!

If $p=3 \bmod 4$ then the group has order $(p-1)^2$, and there are $p-1$ solutions to $\left(\alpha^2+\omega\beta^2\right)=1$ (including 2 with $\beta=0$) And there are $2(p-2)$ solutions with $\beta\neq 0$ to each of $\left(\omega\beta^2+1\pm 2\alpha+\alpha^2\right)=0$. However $\alpha=0$ gives the same two solutions to both equations, and these solutions give the 2×2 matrix equal to zero. So we have $p-1$ rank two matrices, and $4p-12$ rank 3 matrices, conørming experiment.

63.7 Case 2

$$\begin{pmatrix} \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ -\alpha\beta & 0 & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\alpha\beta & -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

If $\alpha=0$ we have

$$\begin{pmatrix} -\omega\beta^2 & \omega^2\beta^3 & 0 & 0 \\ -\beta^3\omega & -\omega\beta^2 & 0 & 0 \\ 0 & 0 & -\omega\beta^2 & \omega^2\beta^3 \\ 0 & 0 & -\beta^3\omega & -\omega\beta^2 \end{pmatrix}$$

Determinant: $\omega^4\beta^8\left(\omega\beta^2+1\right)^2$, so we get two rank 2 matrices if $p=3 \bmod 4$, and none if $p=1 \bmod 4$.

If $\beta=0$ we have

$$\begin{pmatrix} \alpha^2-\alpha^3 & 0 & 0 & 0 \\ 0 & \alpha^2-\alpha^3 & 0 & 0 \\ 0 & 0 & \alpha^2-\alpha^3 & 0 \\ 0 & 0 & 0 & \alpha^2-\alpha^3 \end{pmatrix}$$

and so we get rank 0 if $\alpha=1$.

If $\alpha\beta\neq 0$ we have

$$\begin{pmatrix} \alpha\beta & 0 & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \alpha\beta & -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\times$$
$$\begin{pmatrix} \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ -\alpha\beta & 0 & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\alpha\beta & -\beta\alpha^2-\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{pmatrix}$$

$$\left(\alpha^2+\omega\beta^2\right)\begin{pmatrix} \left(\alpha^4-2\alpha^3+\alpha^2+2\alpha\omega\beta^2-\omega^2\beta^4+\omega\beta^2\right) & -2\omega\beta\left(-\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2\right) \\ 2\beta\left(-\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2\right) & \left(\alpha^4-2\alpha^3+\alpha^2+2\alpha\omega\beta^2-\omega^2\beta^4+\omega\beta^2\right) \end{pmatrix}$$

Determinant: $\left(\omega\beta^2-2\alpha+\alpha^2+1\right)\left(\alpha^4-2\alpha^3+2\alpha^2\omega\beta^2+\alpha^2+6\alpha\omega\beta^2+\omega\beta^2+\omega^2\beta^4\right)\left(\alpha^2+\omega\beta^2\right)^3$

First calculate when this 2×2 matrix is zero (giving rank 2 for the original 4×4 matrix). Recall that α,β are non-zero.

$-\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2=0$, Solution is : $\beta^2=(-\frac{-\alpha^2+\alpha^3}{\alpha\omega+\omega})$. Substitute this value for β^2 into

$$\alpha^4-2\alpha^3+\alpha^2+2\alpha\omega\beta^2-\omega^2\beta^4+\omega\beta^2$$

and we get

$$\alpha^4-2\alpha^3+\alpha^2+2\alpha\omega(-\frac{-\alpha^2+\alpha^3}{\alpha\omega+\omega})-\omega^2(-\frac{-\alpha^2+\alpha^3}{\alpha\omega+\omega})^2+\omega(-\frac{-\alpha^2+\alpha^3}{\alpha\omega+\omega})=-2\alpha^2\frac{2\alpha^2-\alpha-1}{(\alpha+1)^2}$$

$2\alpha^2-\alpha-1=0$, Solution is : $\alpha=1,-\frac{1}{2}$. But $\alpha=1$ gives $\beta=0$ so we discount this case. If $\alpha=-\frac{1}{2}$ then

$$\beta^2=\frac{3}{4\omega}$$

which has 2 solutions if $p=5$ or $7 \bmod 12$, but none if $p=1$ or $11 \bmod 12$.

The 2×2 matrix has form $\begin{pmatrix} u & -\omega v \\ v & u \end{pmatrix}$, so it cannot have rank 1 if $p=1 \bmod 4$. But if $p=3 \bmod 4$ then rank 1 is a possibility.

Now when $p=3 \bmod 4$, $\omega\beta^2-2\alpha+\alpha^2+1=0$ has $2(p-1)$ solutions with $\beta\neq 0$. But we need to discount $\alpha=\frac{1}{2}$ since then $\alpha^2+\omega\beta^2=0$, and we need to discount $\alpha=0$. If $\alpha=-\frac{1}{2}$ then $\omega\beta^2=-\frac{9}{4}$, so there is no overlap with the case when the 2×2 matrix is zero. So we have $2(p-3)$ rank 1 matrices arising from $\omega\beta^2-2\alpha+\alpha^2+1=0$.

Next consider $\alpha^4-2\alpha^3+2\alpha^2\omega\beta^2+\alpha^2+6\alpha\omega\beta^2+\omega\beta^2+\omega^2\beta^4=0$. If we let $\omega\beta^2=x$ then we have

$$\alpha^4-2\alpha^3+2\alpha^2x+\alpha^2+6\alpha x+x+x^2=0$$

Solution is : $\omega\beta^2=-\alpha^2-3\alpha-\frac{1}{2}\pm\frac{1}{2}(1+2\alpha)\sqrt{(8\alpha+1)}$. Now $p=3 \bmod 4$ and so $-\omega$ is a square, and so if we set $y^2=-\omega\beta^2$, then we need to solve

$$y^2=\alpha^2+3\alpha+\frac{1}{2}\pm\frac{1}{2}(1+2\alpha)\sqrt{(8\alpha+1)}$$

Solution is :$y=\pm(\alpha+\frac{1}{2})\pm\frac{1}{2}\sqrt{(8\alpha+1)}$. For a solution we need $8\alpha+1$ to be a square, and there are $(p+1)/2$ values of α which give a square. We need to discount $\alpha=0$. Also $\alpha=1$ gives $\beta=0$, $\pm\sqrt{\frac{-9}{\omega}}$, $\alpha=-\frac{1}{2}$ (which only occurs if $p=7$ mod 12) gives $\beta^2=\frac{3}{4\omega}$ which gives the zero matrix, and $\alpha=-\frac{1}{8}$ gives $\beta=\pm\frac{3}{8\omega}\sqrt{(-\omega)}$.

Can there be any overlap between these cases? If $\omega\beta^2-2\alpha+\alpha^2+1=0$ then $\omega\beta^2=-(\alpha-1)^2$ and so

$$\alpha^4-2\alpha^3+2\alpha^2\omega\beta^2+\alpha^2+6\alpha\omega\beta^2+\omega\beta^2+\omega^2\beta^4=-8\alpha(\alpha-1)^2,$$

but when $\alpha=1$ and $\omega\beta^2-2\alpha+\alpha^2+1=0$ we have $\beta=0$, and we have already discounted that case.

This proves that the 4×4 matrix has rank 0,1,2,3 the following number of times, depending on the value of $p \bmod 12$:

$1,0,0,0$ if $p=1 \bmod 12$,
$1,0,2,0$ if $p=5 \bmod 12$,
$1,0,4,4p-16$ if $p=7 \bmod 12$
$1,0,2,4p-12$ if $p=11 \bmod 12$.

$$\left(\begin{array}{cccc} \alpha^2-\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ \beta\alpha^2+\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ \alpha\beta & 0 & -\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3\omega & -\alpha^2+\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{array}\right)$$

Determinant: $\left(-1+\alpha^2+\omega\beta^2\right)^2\left(\alpha^2+\omega\beta^2\right)^4$. So the determinant is only zero if $\alpha^2+\omega\beta^2=1$, which happens $p+1$ times if $p=1 \bmod 4$ and happens $p-1$ times if $p=3 \bmod 4$.

Now we cannot have $\alpha^2+\omega\beta^2=1$ with $\alpha=0$, but we can with $\beta=0$. This gives

$$\left(\begin{array}{cccc} \alpha^2+\alpha^3 & 0 & 0 & 0 \\ 0 & \alpha^2-\alpha^3 & 0 & 0 \\ 0 & 0 & -\alpha^2+\alpha^3 & 0 \\ 0 & 0 & 0 & -\alpha^2-\alpha^3 \end{array}\right),$$

and so we get rank 2 if $\alpha=\pm 1$.

If $\beta\neq 0$ then we have

$$\left(\begin{array}{llll} -\alpha\beta & 0 & \alpha^2-\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & -\alpha\beta & \beta\alpha^2+\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\times$$

$$\left(\begin{array}{cccc} \alpha^2-\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 & 4\omega\alpha\beta & 0 \\ \beta\alpha^2+\beta^3\omega & \alpha^2-\omega\beta^2-\alpha^3-\alpha\omega\beta^2 & 0 & 4\omega\alpha\beta \\ \alpha\beta & 0 & -\alpha^2+\omega\beta^2+\alpha^3+\alpha\omega\beta^2 & \omega\beta\alpha^2+\omega^2\beta^3 \\ 0 & \alpha\beta & \beta\alpha^2+\beta^3\omega & -\alpha^2+\omega\beta^2-\alpha^3-\alpha\omega\beta^2 \end{array}\right)$$

$$\left(\begin{array}{cc} \left(-1+\alpha^2+\omega\beta^2\right)\left(\alpha^2+\omega\beta^2\right)^2 & 0 \\ 0 & \left(-1+\alpha^2+\omega\beta^2\right)\left(\alpha^2+\omega\beta^2\right)^2 \end{array}\right)$$

So (as the evidence showed) we have rank 0,1,2,3 the following number of times: 0, 0, $p+1$ if $p=1 \bmod 4$ and $p-1$ if $p=3 \bmod 4$, 0.

64 Appendix D

Descendants of 6.178

$A=\left(\begin{array}{cc} \lambda & \mu \\ \nu & \xi \end{array}\right)$, and A ranges over a set of representatives for the orbits of non-singular 2×2 matrices under the action

$$A\to\frac{1}{\det P}PAP^{-1}$$

as P ranges over non-singular matrices

$$P=\left(\begin{array}{cc} \alpha & \beta \\ \pm\omega\beta & \pm\alpha \end{array}\right).$$

These algebras are terminal unless $\xi=-\lambda$.

$$\left(\begin{array}{cc} \alpha & \beta \\ \omega\beta & \alpha \end{array}\right)\left(\begin{array}{cc} x & y \\ z & -x \end{array}\right)-(\alpha^2-\beta^2\omega)\left(\begin{array}{cc} x & y \\ z & -x \end{array}\right)\left(\begin{array}{cc} \alpha & \beta \\ \omega\beta & \alpha \end{array}\right)$$

$$= \begin{pmatrix} \alpha x+\beta z-\alpha^3x+\alpha x\beta^2\omega-\omega\beta y\alpha^2+\omega^2\beta^3y & \alpha y-\beta x-\beta x\alpha^2+\beta^3x\omega-\alpha^3y+\alpha y\beta^2\omega \\ \omega\beta x+\alpha z-\alpha^3z+\alpha z\beta^2\omega+\omega\beta x\alpha^2-\omega^2\beta^3x & \omega\beta y-\alpha x-\beta z\alpha^2+\beta^3z\omega+\alpha^3x-\alpha x\beta^2\omega \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha^3+\alpha\beta^2\omega & -\omega\beta\alpha^2+\omega^2\beta^3 & \beta \\ -\beta-\beta\alpha^2+\beta^3\omega & \alpha-\alpha^3+\alpha\beta^2\omega & 0 \\ \omega\beta+\omega\beta\alpha^2-\omega^2\beta^3 & 0 & \alpha-\alpha^3+\alpha\beta^2\omega \\ -\alpha+\alpha^3-\alpha\beta^2\omega & \omega\beta & -\beta\alpha^2+\beta^3\omega \end{pmatrix}$$

$$\begin{pmatrix} \alpha-\alpha^3+\alpha\beta^2\omega & -\omega\beta\alpha^2+\omega^2\beta^3 & \beta \\ -\beta-\beta\alpha^2+\beta^3\omega & \alpha-\alpha^3+\alpha\beta^2\omega & 0 \\ \omega\beta+\omega\beta\alpha^2-\omega^2\beta^3 & 0 & \alpha-\alpha^3+\alpha\beta^2\omega \end{pmatrix}$$

Determinant: $-\alpha\left(\beta^2\omega-(1+\alpha)^2\right)\left(\beta^2\omega+1-\alpha^2\right)\left(\beta^2\omega-(1-\alpha)^2\right)\left(-\alpha^2+\beta^2\omega\right)$. So the rank is 3 unless $\alpha=0$, or $\beta=0$ and $\alpha=\pm1$, or $(\alpha^2-\beta^2\omega)=1$.

Let $\alpha=0$. Then we have

$$\begin{pmatrix} 0 & \omega^2\beta^3 & \beta \\ -\beta+\beta^3\omega & 0 & 0 \\ \omega\beta-\omega^2\beta^3 & 0 & 0 \\ 0 & \omega\beta & \beta^3\omega \end{pmatrix}$$

$$\begin{pmatrix} \omega^2\beta^3 & \beta \\ \omega\beta & \beta^3\omega \end{pmatrix}$$

Determinant: $\beta^2\omega\left(\beta^2\omega-1\right)\left(\beta^2\omega+1\right)$ so we have rank 3 when $\alpha=0$ except when $(\alpha^2-\beta^2\omega)=1$.

Let $\beta=0$ and $\alpha=\pm1$. Then we have the zero matrix. Contribution $2(p^3-1)$.

Let $(\alpha^2-\beta^2\omega)=1$. Then we have

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}-\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}=\begin{pmatrix} \beta z-\omega\beta y & -2\beta x \\ 2\omega\beta x & \omega\beta y-\beta z \end{pmatrix}$$

The case $\beta=0$ is covered above. If $\beta\neq0$ we have $x=0$, $z=\omega y$ so we have a contribution of $(p-1)$ for each non-zero value of β with $(\alpha^2-\beta^2\omega)=1$.

Now $(\alpha^2-\beta^2\omega)=1$ has $p+1$ solutions, and so the contribution to Burnside's Lemma is

$$2(p^3-1)+(p-1)(p-1)+(p^2-1)=2\,(p-1)\,(p+1)^2\,.$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}+(\alpha^2-\beta^2\omega)\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x+\beta z+\alpha^3x-\alpha x\beta^2\omega-\omega\beta y\alpha^2+\omega^2\beta^3y & \alpha y-\beta x+\beta x\alpha^2-\beta^3x\omega-\alpha^3y+\alpha y\beta^2\omega \\ -\omega\beta x-\alpha z+\alpha^3z-\alpha z\beta^2\omega+\omega\beta x\alpha^2-\omega^2\beta^3x & -\omega\beta y+\alpha x+\beta z\alpha^2-\beta^3z\omega+\alpha^3x-\alpha x\beta^2\omega \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha^3-\alpha\beta^2\omega & -\omega\beta\alpha^2+\omega^2\beta^3 & \beta \\ -\beta+\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3+\alpha\beta^2\omega & 0 \\ -\omega\beta+\omega\beta\alpha^2-\omega^2\beta^3 & 0 & -\alpha+\alpha^3-\alpha\beta^2\omega \\ \alpha+\alpha^3-\alpha\beta^2\omega & -\omega\beta & \beta\alpha^2-\beta^3\omega \end{pmatrix}$$

$$\begin{pmatrix} \alpha+\alpha^3-\alpha\beta^2\omega & -\omega\beta\alpha^2+\omega^2\beta^3 & \beta \\ -\beta+\beta\alpha^2-\beta^3\omega & \alpha-\alpha^3+\alpha\beta^2\omega & 0 \\ -\omega\beta+\omega\beta\alpha^2-\omega^2\beta^3 & 0 & -\alpha+\alpha^3-\alpha\beta^2\omega \end{pmatrix}$$

Determinant: $-\alpha\left(\beta^2\omega-\alpha^2\right)\left(-1-\alpha^2+\beta^2\omega\right)\left(\beta^2\omega+1-\alpha^2\right)^2$
If $\alpha=0$ we have

$$\begin{pmatrix} 0 & \omega^2\beta^3 & \beta \\ -\beta-\beta^3\omega & 0 & 0 \\ -\omega\beta-\omega^2\beta^3 & 0 & 0 \\ 0 & -\omega\beta & -\beta^3\omega \end{pmatrix}$$

$$\begin{pmatrix} \omega^2\beta^3 & \beta \\ -\omega\beta & -\beta^3\omega \end{pmatrix}$$

Determinant: $-\beta^6\omega^3+\beta^2\omega=-\beta^2\omega\left(\beta^2\omega-1\right)\left(\beta^2\omega+1\right)$
So we have rank 3 unless $(\alpha^2-\beta^2\omega)=\pm1$.
If $(\alpha^2-\beta^2\omega)=-1$ we have

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}-\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}=\begin{pmatrix} \beta z+\omega\beta y & 2\alpha y-2\beta x \\ -2\omega\beta x-2\alpha z & -\omega\beta y-\beta z \end{pmatrix}$$

So we have a contribution $(p+1)(p-1)$
If $(\alpha^2-\beta^2\omega)=1$ we have

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}+\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}=\begin{pmatrix} 2\alpha x+\beta z-\omega\beta y & 0 \\ 0 & 2\alpha x+\beta z-\omega\beta y \end{pmatrix}$$

So we have a contribution $(p+1)(p^2-1)$.
So the contribution to Burnside's Lemma from matrices of the second kind is

$$(p+1)(p-1)+(p+1)(p^2-1)+(p^2-1)=(p-1)\left(p+3\right)\left(p+1\right)$$

and the number of orbits is $(3p+5)/2$.

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1}=\begin{pmatrix} \beta\alpha\frac{-z+y\omega}{-\alpha^2+\beta^2\omega} & -\frac{-\beta^2z+\alpha^2y}{-\alpha^2+\beta^2\omega} \\ \frac{-\alpha^2z+\omega^2\beta^2y}{-\alpha^2+\beta^2\omega} & -\beta\alpha\frac{-z+y\omega}{-\alpha^2+\beta^2\omega} \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1}=\begin{pmatrix} \beta\alpha\frac{-z+y\omega}{-\alpha^2+\beta^2\omega} & \frac{-\beta^2z+\alpha^2y}{-\alpha^2+\beta^2\omega} \\ -\frac{-\alpha^2z+\omega^2\beta^2y}{-\alpha^2+\beta^2\omega} & -\beta\alpha\frac{-z+y\omega}{-\alpha^2+\beta^2\omega} \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1}=(\alpha^2-\beta^2\omega)^{-1}\begin{pmatrix} 0 & y \\ y\omega & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1}=(\alpha^2-\beta^2\omega)^{-1}\begin{pmatrix} 0 & y \\ y\omega & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}^{-1}=\alpha^{-2}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}^{-1}=\alpha^{-2}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} 0 & \beta \\ -\omega\beta & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\omega\beta & 0 \end{pmatrix}^{-1} = (-\beta^2\omega)^{-1}\begin{pmatrix} 0 & \frac{z}{\omega} \\ \omega y & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} 0 & \beta \\ \omega\beta & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ \omega\beta & 0 \end{pmatrix}^{-1} = (-\beta^2\omega)^{-1}\begin{pmatrix} 0 & \frac{z}{\omega} \\ \omega y & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ z & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1}$$
$$= (\alpha^2 - \beta^2\omega)^{-2}\begin{pmatrix} \beta^2\omega + \beta z\alpha - \alpha y\omega\beta + \alpha^2 & -2\beta\alpha - \beta^2 z + \alpha^2 y \\ 2\alpha\omega\beta + \alpha^2 z - \omega^2\beta^2 y & -\alpha^2 - \beta z\alpha + \alpha y\omega\beta - \beta^2\omega \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ z & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1}$$
$$= (\alpha^2 - \beta^2\omega)^{-2}\begin{pmatrix} -\alpha^2 - \beta z\alpha + \alpha y\omega\beta - \beta^2\omega & -2\beta\alpha - \beta^2 z + \alpha^2 y \\ 2\alpha\omega\beta + \alpha^2 z - \omega^2\beta^2 y & \beta^2\omega + \beta z\alpha - \alpha y\omega\beta + \alpha^2 \end{pmatrix}$$

64.1 Orbits

64.1.1 Orbit 1

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1} = (\alpha^2 - \beta^2\omega)^{-1}\begin{pmatrix} 0 & y \\ y\omega & 0 \end{pmatrix}$$

$$\frac{1}{\det P}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1} = (\alpha^2 - \beta^2\omega)^{-1}\begin{pmatrix} 0 & y \\ y\omega & 0 \end{pmatrix}$$

so all the elements $\begin{pmatrix} 0 & y \\ y\omega & 0 \end{pmatrix}$ are in the same orbit.

64.1.2 Orbit 2

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1}$$
$$= \quad (\alpha^2 - \beta^2\omega)^{-2}\begin{pmatrix} \omega x\beta^2 - 2\alpha y\omega\beta + \alpha^2 x & \omega\beta^2 y - 2\alpha x\beta + \alpha^2 y \\ \left(-\omega\beta^2 y + 2\alpha x\beta - \alpha^2 y\right)\omega & -\omega x\beta^2 + 2\alpha y\omega\beta - \alpha^2 x \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1}$$
$$= \quad (\alpha^2 - \beta^2\omega)^{-2}\begin{pmatrix} -\omega x\beta^2 + 2\alpha y\omega\beta - \alpha^2 x & \omega\beta^2 y - 2\alpha x\beta + \alpha^2 y \\ \left(-\omega\beta^2 y + 2\alpha x\beta - \alpha^2 y\right)\omega & \omega x\beta^2 - 2\alpha y\omega\beta + \alpha^2 x \end{pmatrix}$$

So elements of the form $\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}$ are mapped to other elements of the same form. Note that this matrix is non-singular unless $x = y = 0$.

As above, the plus matrix only øxes the element if $(\alpha^2 - \beta^2\omega) = 1$, and the minus matrix only øxes it if $(\alpha^2 - \beta^2\omega) = \pm 1$.

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix} - \begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} -2\omega\beta y & -2\beta x \\ 2\omega\beta x & 2\omega\beta y \end{pmatrix}$$

so contribution to Burnside's Lemma is $2p^2 + p^2 - 3 = 3\,(p-1)\,(p+1)$.

Consider the minus matrix with $(\alpha^2 - \beta^2\omega) = 1$.

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix} - \det P\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix} + \begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} 2\alpha x - 2\omega\beta y & 0 \\ 0 & 2\alpha x - 2\omega\beta y \end{pmatrix}$$

so the contribution to Burnside's Lemma is $(p+1)(p-1)$.

Finally consider the minus matrix with $(\alpha^2 - \beta^2\omega) = -1$.

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix} - \det P\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix} - \begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 2\alpha y - 2\beta x \\ -2\omega\beta x + 2\alpha\omega y & 0 \end{pmatrix}$$

so again the contribution is $(p+1)(p-1)$.

The Burnside's Lemma count from matrices of the second type is $3(p^2-1)$, and since the contribution from matrices of the plus type is also $3(p^2-1)$, there are three orbits (including the zero matrix). Experimental evidence gives the two non-zero orbit representatives as $\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ when $p = 1 \bmod 4$. We prove this immediately below! But when $p = 3 \bmod 4$ these two are in the same orbit, and the other orbit is harder to ønd.

When are $\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the same orbit?

$$0 = \begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix} - (\alpha^2 - \beta^2\omega)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\omega\beta + \left(-\alpha^2 + \beta^2\omega\right)\alpha & \alpha + \left(-\alpha^2 + \beta^2\omega\right)\beta \\ -\alpha\omega - \left(-\alpha^2 + \beta^2\omega\right)\omega\beta & \omega\beta - \left(-\alpha^2 + \beta^2\omega\right)\alpha \end{pmatrix}$$

So

$$\omega\alpha = \omega\left(\alpha^2 - \beta^2\omega\right)\beta = -\left(\alpha^2 - \beta^2\omega\right)^2\alpha$$

(with $\alpha \neq 0$), which is impossible if $p = 1 \bmod 4$ (but possible if $p = 3 \bmod 4$).

$$0 = \begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix} + (\alpha^2 - \beta^2\omega)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\omega\beta - \left(-\alpha^2 + \beta^2\omega\right)\alpha & \alpha - \left(-\alpha^2 + \beta^2\omega\right)\beta \\ \alpha\omega - \left(-\alpha^2 + \beta^2\omega\right)\omega\beta & -\omega\beta - \left(-\alpha^2 + \beta^2\omega\right)\alpha \end{pmatrix}$$

So

$$\omega\alpha = -\omega\left(\alpha^2 - \beta^2\omega\right)\beta = -\left(\alpha^2 - \beta^2\omega\right)^2\alpha$$

(with $\alpha \neq 0$), which is impossible if $p = 1 \bmod 4$.

64.1.3 Orbit 3

There are p^2-1 rank 1 matrices of the form $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$. These correspond to the situation when pa, pb span a one dimensional space in the class 3 quotient of the algebra. We can choose α, β so that $pa = 0$, and then $pb = zbaa$. We can take $z = 1$ or ω, so there are two orbits of rank 1 matrices.

Plus matrices stabilize $\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$ if

$$0 = \begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} \beta z & 0 \\ 0 & -\beta z \end{pmatrix}$$

so we need $\alpha = \pm 1$, $\beta = 0$.

Minus matrices with determinant 1 cannot stabilize $\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$ since

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} \beta z & 0 \\ -2\alpha z & -\beta z \end{pmatrix}$$

And minus matrices with determinant -1 stabilize $\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$ if $\alpha = \pm 1$, $\beta = 0$.

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} \beta z & 0 \\ 0 & \beta z \end{pmatrix}$$

So there are two orbits of rank 1 matrices, both of size $(p^2-1)/2$.

64.1.4 Remaining orbits

Non-singular matrices which are not in any of the three orbits already covered are not øxed by a minus matrix with $(\alpha^2 - \beta^2\omega) = -1$. Minus matrices with $(\alpha^2 - \beta^2\omega) = 1$ øx $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ if $2\alpha x + \beta z - \omega\beta y = 0$. So minus matrices with $\alpha = \pm 1$, $\beta = 0$ øx anything with $x = 0$. And if $\beta \neq 0$ then the element is øxed if $z = \omega y - 2\alpha\beta^{-1}x$. So $\alpha\beta^{-1} = \frac{\omega y - z}{2x}$, and we need $\left(\frac{\omega y - z}{2x}\right)^2 - \omega$ to be a square.

The number of orbits of matrices in this situation is $(3p-7)/2$, and they split up into k orbits of size p^2-1 (with stabilzers of size 2 consisting of plus matrices with $\alpha = \pm 1$, $\beta = 0$) and l orbits of size $(p^2-1)/2$ (with stabilizers of size 4). So we have

$$k(p^2-1) + l(p^2-1)/2 = p^3 - (p-1) - p^2 - (p^2-1) = (p^2-1)(p-2).$$

It follows that $k = (p-1)/2$ and $l = p-3$.

64.2 Summary

The rank 2 matrices split up into one orbit of size $p-1$ (matrices $\begin{pmatrix} 0 & y \\ \omega y & 0 \end{pmatrix}$), $p-1$ orbits of size $(p^2-1)/2$, and $(p-1)/2$ orbits of size p^2-1. This calculation is correct, though the calculations above are slightly Æawed since whether matrices of the form $\begin{pmatrix} x & y \\ -\omega y & -x \end{pmatrix}$ have stabilizers of size two or four depends on the value of $p \bmod 4$.

$$(\alpha^2 - \beta^2\omega)\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} \beta y\alpha - \alpha x\omega\beta & -\beta^2 y + \alpha^2 x \\ \alpha^2 y - \omega^2\beta^2 x & -\beta y\alpha + \alpha x\omega\beta \end{pmatrix}$$

$$(-\alpha^2+\beta^2\omega)\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1} = \begin{pmatrix} -\beta y\alpha + \alpha x\omega\beta & -\beta^2 y + \alpha^2 x \\ \alpha^2 y - \omega^2\beta^2 x & \beta y\alpha - \alpha x\omega\beta \end{pmatrix}$$

So (except in the case $y = \omega x$) we need $\alpha\beta = 0$ to get the $(1,1)$ position eqal to 0.

Consider case $\alpha = 0$.

$$(\alpha^2 - \beta^2\omega)^{-1}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{\beta^2\omega^2}y \\ -\frac{1}{\beta^2}x & 0 \end{pmatrix}$$

$$(-\alpha^2 + \beta^2\omega)^{-1}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{\beta^2\omega^2}y \\ -\frac{1}{\beta^2}x & 0 \end{pmatrix}$$

And now consider the case $\beta = 0$.

$$(\alpha^2 - \beta^2\omega)^{-1}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\alpha^2}x \\ \frac{1}{\alpha^2}y & 0 \end{pmatrix}$$

$$(-\alpha^2 + \beta^2\omega)^{-1}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\alpha^2}x \\ \frac{1}{\alpha^2}y & 0 \end{pmatrix}$$

So $(x,y) \sim \alpha^{-2}(x,y) \sim -\beta^{-2}(\omega^{-2}y, x)$.

We can take the representatives to chosen from $(1,k^2)$, $(1,\omega k^2)$, (ω,k^2), $(\omega,\omega k^2)$ with

$$(1,k^2) \sim -\beta^{-2}(\omega^{-2}k^2,1) \sim -(1,\omega^2 k^{-2}),$$

$$(1,\omega k^2) \sim -\beta^{-2}(\omega^{-2}\omega k^2,1) \sim -(\omega,\omega^2 k^{-2}),$$

$$(\omega,k^2) \sim -\beta^{-2}(\omega^{-2}k^2,\omega) \sim -(1,\omega\omega^2 k^{-2}),$$

$$(\omega,\omega k^2) \sim -\beta^{-2}(\omega^{-2}\omega k^2,\omega) \sim -(\omega,\omega\omega^2 k^{-2}),$$

If $p = 1 \bmod 4$ we can take the representatives under the full equivalence relation to be $(1,k^2)$ where $k^2 \sim (\frac{\omega}{k})^2$, $(1,\omega k^2)$, $(\omega,\omega k^2)$ where $k^2 \sim (\frac{\omega}{k})^2$, giving $(p-1)$ eqivalence classes, including the class of $(1,\omega)$. If $p = 3 \bmod 4$ then we can take the representatives under the full equivalence relation to be $(1,k^2)$, $(1,\omega k^2)$ with $k^2 \sim k^{-2}$, (ω,k^2) with $k^2 \sim \omega^4 k^{-2}$ giving p equivalence classes, including the class of $(1,\omega)$.

Note that if $p = 1 \bmod 4$ then $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not equivalent to a matrix with $x = 0$ though it does have stabilizer of order 4, since if $\alpha^2 = -1$ and $\beta = 0$

$$(-\alpha^2 + \beta^2\omega)^{-1}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{\alpha^2} & 0 \\ 0 & \frac{1}{\alpha^2} \end{pmatrix}$$

So we have identiøed all p orbits with stabilizers of order 4.

64.3 Orbits with stabilizers of size two

These are orbits of matrices of form $\begin{pmatrix} 1 & x \\ y & -1 \end{pmatrix}$ which are not in the same orbit as a matrix with $(1,1)$ entry equal to zero, except for $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the case $p = 1 \bmod 4$.

Suppose $\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$ is not in the same orbit as an element with a zero in the $(1,1)$ position.

$$
(\alpha^2-\beta^2\omega)\begin{pmatrix}\alpha & \beta\\ \omega\beta & \alpha\end{pmatrix}\begin{pmatrix}x & y\\ z & -x\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \omega\beta & \alpha\end{pmatrix}^{-1}
$$
$$
=\begin{pmatrix}\omega x\beta^2-\omega\alpha y\beta+\alpha z\beta+\alpha^2x & -z\beta^2-2\alpha x\beta+\alpha^2 y\\ -y\beta^2\omega^2+2\alpha x\beta\omega+z\alpha^2 & -\omega x\beta^2+\omega\alpha y\beta-\alpha z\beta-\alpha^2 x\end{pmatrix}
$$
$$
(\alpha^2-\beta^2\omega)\begin{pmatrix}\alpha & \beta\\ -\omega\beta & -\alpha\end{pmatrix}\begin{pmatrix}x & y\\ z & -x\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ -\omega\beta & -\alpha\end{pmatrix}^{-1}
$$
$$
=\begin{pmatrix}\omega x\beta^2-\omega\alpha y\beta+\alpha z\beta+\alpha^2x & z\beta^2+2\alpha x\beta-\alpha^2 y\\ y\beta^2\omega^2-2\alpha x\beta\omega-z\alpha^2 & -\omega x\beta^2+\omega\alpha y\beta-\alpha z\beta-\alpha^2 x\end{pmatrix}
$$

So

$$
\omega x\beta^2-\omega\alpha y\beta+\alpha z\beta+\alpha^2x=0
$$

has no solutions, which implies that $(z-\omega y)^2-4\omega x^2$ is not a square.

64.4 Desperate Measures!

Try to compute the number of orbits of groups under the subgroup of transformations with $\alpha=\pm1$, $\beta=0$.

We have

$$
xbaaa+ybaab=zbaab-xbabb=zbaaa+ybabb=0.
$$

For ease of notation denote tail on $pa-xbaa-ybab$ by pa, etc.

$$
\begin{aligned}
pa' &= \pm pa+\gamma pc+(\omega x\gamma+\omega y\varepsilon)baaa-x\varepsilon baab-(x\gamma+2y\varepsilon)babb,\\
pb' &= \pm pb+\varepsilon pc+(2\omega z\gamma-\omega x\varepsilon)baaa-\omega x\gamma baab+(-z\gamma+x\varepsilon)babb,\\
pc' &= pc.
\end{aligned}
$$

If $x=0$ we have $baab=0$ and $babb=-y^{-1}zbaaa$. So

$$
\begin{aligned}
pa' &= \pm pa+\gamma pc+(\omega y+2z)\varepsilon baaa,\\
pb' &= \pm pb+\varepsilon pc+(2\omega z+y^{-1}z^2)\gamma baaa,\\
pc' &= pc.
\end{aligned}
$$

If $x\neq 0$ then we have $babb=x^{-1}zbaab$, $baaa=-x^{-1}ybaab$. So

$$
\begin{aligned}
pa' &= \pm pa+\gamma pc-x^{-1}(\omega yx\gamma+\omega y^2\varepsilon+x^2\varepsilon+zx\gamma+2zy\varepsilon)baab,\\
pb' &= \pm pb+\varepsilon pc-x^{-1}(2\omega yz\gamma-\omega yx\varepsilon+\omega x^2\gamma+z^2\gamma-zx\varepsilon)baab,\\
pc' &= pc.
\end{aligned}
$$

$(\omega+2k^2)(2\omega k^2+k^4)=\left(\omega+2k^2\right)k^2\left(2\omega+k^2\right)$ with $\sim$
$(\omega+2\omega k^2)(2\omega\omega k^2+\omega^2k^4)=\omega^3\left(1+2k^2\right)k^2\left(k^2+2\right)$
$(\omega\omega+2\omega k^2)(2\omega\omega k^2+\omega k^4)=\omega^2\left(\omega+2k^2\right)k^2\left(2\omega+k^2\right)$ with $\sim$

If $p=1 \bmod 4$ we can take the representatives under the full equivalence relation to be $(1,k^2)$ where $k^2\sim(\frac{\omega}{k})^2$, $(1,\omega k^2)$, $(\omega,\omega k^2)$ where $k^2\sim(\frac{\omega}{k})^2$, giving $(p-1)$ eqivalence classes, including the class of $(1,\omega)$. If $p=3 \bmod 4$ then we can take the representatives under the full equivalence relation to be $(1,k^2)$, $(1,\omega k^2)$ with $k^2\sim k^{-2}$, (ω,k^2) with $k^2\sim\omega^4k^{-2}$ giving p equivalence classes, including the class of $(1,\omega)$.

$x=1$
$z=-\omega y$
$\gamma t-x^{-1}(\omega yx\gamma+\omega y^2\varepsilon+x^2\varepsilon+zx\gamma+2zy\varepsilon)=\gamma t+\omega y^2\varepsilon-\varepsilon$
$\varepsilon t-x^{-1}(2\omega yz\gamma-\omega yx\varepsilon+\omega x^2\gamma+z^2\gamma-zx\varepsilon)=\varepsilon t+\omega^2y^2\gamma-\omega\gamma$
$(\omega y^2-1)(\omega^2y^2-\omega)$

64.5 Another orbit

Let $A = \begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix}$.

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix} - (\alpha^2-\beta^2\omega)\begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$$
$$= \begin{pmatrix} -\omega\beta y + \left(-\alpha^2+\beta^2\omega\right) y\omega\beta & \alpha y + \left(-\alpha^2+\beta^2\omega\right) y\alpha \\ -\alpha\omega y - \left(-\alpha^2+\beta^2\omega\right)\omega y\alpha & \omega\beta y - \left(-\alpha^2+\beta^2\omega\right) y\omega\beta \end{pmatrix}$$

To have zero here we need $\alpha = \pm 1$ and $\beta = 0$.

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix} + (\alpha^2-\beta^2\omega)\begin{pmatrix} 0 & y \\ -\omega y & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$
$$= \begin{pmatrix} \omega y\beta\left(\beta^2\omega - 1 - \alpha^2\right) & y\alpha\left(-\alpha^2+1+\beta^2\omega\right) \\ \omega y\alpha\left(-\alpha^2+1+\beta^2\omega\right) & \omega y\beta\left(\beta^2\omega-1-\alpha^2\right) \end{pmatrix}$$

To have zero here we need $\alpha = \pm 1$ and $\beta = 0$.

64.6 Another orbit

Let $A = \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}$ with $y \neq 0$.

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} - (\alpha^2-\beta^2\omega)\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} + (\alpha^2-\beta^2\omega)\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}$$

Considerations of determinants imply that we need $(\alpha^2-\beta^2\omega) = \pm 1$ to get zero in either case.

Plus matrix with $(\alpha^2-\beta^2\omega) = 1$ gives

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} - \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} -2\omega\beta y & -2\beta \\ 2\omega\beta & 2\omega\beta y \end{pmatrix}$$

so only possibility is $\alpha = \pm 1$, $\beta = 0$.

Plus matrix with $(\alpha^2-\beta^2\omega) = -1$ gives

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} + \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} 2\alpha & 2\alpha y \\ -2\alpha\omega y & -2\alpha \end{pmatrix}$$

which is impossible.

Minus matrix with $(\alpha^2-\beta^2\omega) = 1$ gives

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} + \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} 2\alpha - 2\omega\beta y & 0 \\ 0 & 2\alpha - 2\omega\beta y \end{pmatrix}$$

so we need $\alpha = \omega\beta y$ which gives $\alpha^2 - \beta^2\omega = \omega^2\beta^2 y^2 - \beta^2\omega = \omega\beta^2\left(\omega y^2 - 1\right)$, so we can ønd 2 matrices of this form if $\omega y^2 - 1$ is not a square.

Minus matrix with $(\alpha^2-\beta^2\omega) = -1$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix} - \begin{pmatrix} 1 & y \\ -\omega y & -1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 2\alpha y - 2\beta \\ -2\omega\beta + 2\alpha\omega y & 0 \end{pmatrix}$$

so we need $\beta = \alpha y$ and $\alpha^2 - \beta^2\omega = \alpha^2 - \alpha^2 y^2\omega = -\alpha^2\left(\omega y^2 - 1\right)$, so we can ønd two matrices of this form if $\omega y^2 - 1$ is a square.

64.7 Another orbit

Let $A = \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}$ with $z \neq \omega y$.

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} \beta z - \omega\beta y & 0 \\ 0 & \omega\beta y - \beta z \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \omega\beta & \alpha \end{pmatrix} = \begin{pmatrix} \beta z + \omega\beta y & 2\alpha y \\ 2\alpha z & \beta z + \omega\beta y \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} \beta z - \omega\beta y & 0 \\ 0 & \beta z - \omega\beta y \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix}\begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\omega\beta & -\alpha \end{pmatrix} = \begin{pmatrix} \beta z + \omega\beta y & 2\alpha y \\ -2\alpha z & -\omega\beta y - \beta z \end{pmatrix}$$

So the stabilizer is plus and minus matrices with $\alpha = \pm 1$ and $\beta = 0$.
The plus matrix has

$$\begin{aligned} b'a'a'a' &= baaa, \\ pa' &= \pm pa + \gamma pc + (\omega y + 2z)\varepsilon baaa, \\ pb' &= \pm pb + \varepsilon pc + (2\omega z + y^{-1}z^2)\gamma baaa, \\ pc' &= pc \end{aligned}$$

and the minus matrix has

$$\begin{aligned} b'a'a'a' &= -baaa, \\ pa' &= \pm pa + \gamma pc - (\omega y + 2z)\varepsilon baaa, \\ pb' &= \mp pb + \varepsilon pc - (2\omega z + y^{-1}z^2)\gamma baaa, \\ pc' &= pc \end{aligned}$$

So we can take $pc = \lambda baaa$ with $0 \leq \lambda \leq (p-1)/2$. If $(\omega y + 2z)(2\omega z + y^{-1}z^2)$ is not a square then we can take tails on pa and pb to be zero. If $(\omega y + 2z)(2\omega z + y^{-1}z^2) = \lambda^2$ then $pc = \pm\lambda baaa$ gives problems, though for all other values of pc we can take tails on pa and pb to be zero.

If $\omega y + 2z = 0$ or $2\omega z + y^{-1}z^2 = 0$ then $pc = 0$ gives problems. If $\omega y + 2z = 0$ and $pc = 0$ then we can take $pb = 0$ and take the tail on pa to be $\mu baaa$ where $0 \leq \mu \leq (p-1)/2$. And if $2\omega z + y^{-1}z^2 = 0$ we can take $pa = 0$ and take the tail on pb to be $\mu baaa$ where $0 \leq \mu \leq (p-1)/2$.

If $(\omega y + 2z)(2\omega z + y^{-1}z^2) = \lambda^2$ where $\lambda^2 \neq 0$, then only one of $\pm\lambda baaa$ occurs as the value of pc, and for that value we can take tail on pa to be zero, and tail on pb to be $\mu baaa$ where $0 \leq \mu \leq (p-1)/2$.

So we have $(p+1)/2$ orbits, with an extra $(p-1)/2$ orbits if $(\omega y + 2z)(2\omega z + y^{-1}z^2)$ is a square.

64.8 The last orbit

Finally, consider the case when $A = \begin{pmatrix} 1 & y \\ z & -1 \end{pmatrix}$ with $z \neq \omega y$, and with A not in an orbit with $(1,1)$-entry equal to 0. (There are $(p-1)/2$ orbits of this kind.) They have stabilizers of order 2 consisting of plus matrices with $\alpha = \pm 1$ and $\beta = 0$. We can take $pc = \lambda baab$ with $0 \leq \lambda < p$.

We have $b'a'a'a' = baaa$, $pc' = pc$, and

$$\begin{aligned} pa' &= \pm pa + \gamma pc - ((\omega y + z)\gamma + (\omega y^2 + 2yz + 1)\varepsilon)baaa, \\ pb' &= \pm pb + \varepsilon pc - ((\omega + 2\omega yz + z^2)\gamma - (\omega y + z)\varepsilon)baaa. \end{aligned}$$

$$\begin{pmatrix} \lambda - (\omega y + z) & (\omega y^2 + 2yz + 1) \\ (\omega + 2\omega yz + z^2) & \lambda + (\omega y + z) \end{pmatrix}$$

Determinant: $\lambda^2 - (1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right)$

So if $(1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right)$ is not a square we can take the tails on pa and pb to be zero, giving p algebras.

If $(1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right) = 0$ then $pc = 0$ gives problems, and when $pc = 0$ we can take the tail on pa to be zer0, and the tail on pb to be $\mu baab$ where $0 \leq \mu \leq (p-1)/2$.

If $(1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right)$ is a non-zero square then there are two values of pc which give problems, and for each value we similarly get an extra $(p - 1)/2$ algebras.

So if $(1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right) = 0$ we get an extra $(p - 1)/2$ algebras, and if $(1 + yz)\left(2(\omega y + z)^2 + \omega(1 + yz)\right)$ is a non-zero square we get an extra $p - 1$ algebras. Note that since A is not in the same orbit as a matrix with $(1, 1)$ position zero, $(z - \omega y)^2 - 4\omega$ is not a square.

It turns out that we need to add $(p - 1)/2$ to the total exactly $p + 1$ times!!!!!!!!!! But why?????????

In general, when $x \neq 0$, we have

$$\begin{pmatrix} \lambda - (\omega y + z) & (\omega x^{-1}y^2 + 2x^{-1}yz + x) \\ (\omega x + 2\omega x^{-1}yz + x^{-1}z^2) & \lambda + (\omega y + z) \end{pmatrix}$$

Determinant: $\lambda^2 - x^{-2}(2x^2\omega^2y^2 + 6x^2\omega yz + 2x^2z^2 + 2\omega^2y^3z + 5\omega y^2z^2 + 2yz^3 + x^4\omega)$So we get extra algebras if $(2x^2\omega^2y^2 + 6x^2\omega yz + 2x^2z^2 + 2\omega^2y^3z + 5\omega y^2z^2 + 2yz^3 + x^4\omega)$ is a square.

$$(2x^2\omega^2y^2 + 6x^2\omega yz + 2x^2z^2 + 2\omega^2y^3z + 5\omega y^2z^2 + 2yz^3 + x^4\omega) = \left(x^2 + yz\right)\left(2(\omega y + z)^2 + \omega(x^2 + yz)\right)$$

Note that when $x = 0$ this expression equals $(\omega y + 2z)(2\omega z + y^{-1}z^2)y^2$, and so the general test for extra algebras is that $\left(x^2 + yz\right)\left(2(\omega y + z)^2 + \omega(x^2 + yz)\right)$ is a square (even when $x = 0$).

In the orbits which do not have any matrices with a zero in the $(1, 1)$ position we have $(z - \omega y)^2 - 4\omega x^2$ not a square. The experimental evidence implies that if this is the case then $\left(x^2 + yz\right)\left(2(\omega y + z)^2 + \omega(x^2 + yz)\right)$ cannot be zero.

$2(\omega y + z)^2 + \omega(x^2 + yz) = 0$, Solution is : $z = -\frac{5}{4}\omega y \pm \frac{1}{4}\sqrt{(9\omega^2y^2 - 8\omega x^2)}$

Set $z = -\frac{5}{4}\omega y - \frac{1}{4}t$, where $t^2 = 9\omega^2y^2 - 8\omega x^2$.

$(z - \omega y)^2 - 4\omega x^2 = \frac{81}{16}\omega^2y^2 + \frac{9}{8}\omega yt + \frac{1}{16}t^2 - 4\omega x^2 =$

$\frac{81}{16}\omega^2y^2 + \frac{9}{8}\omega yt + \frac{1}{16}\left(9\omega^2y^2 - 8\omega x^2\right) - 4\omega x^2 = \frac{9}{8}\omega\left(5y^2\omega + yt - 4x^2\right)$

Now $\omega 2\left(5y^2\omega + yt - 4x^2\right) = 10\omega^2y^2 + 2\omega yt - 8\omega x^2 = \omega^2y^2 + 2\omega yt + t^2 = (\omega y + t)^2$.

So this proves that the experimental evidence is a correct reÆection of reality!!!

So each time $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is a square and $(z-\omega y)^2-4\omega x^2$ is a square we get an extra $(p-1)/2$ algebras, and each time $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is a square and $(z-\omega y)^2-4\omega x^2$ is not a square we get an extra $(p-1)$ algebras. Now in the situation when $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is a square, the orbits with $(z-\omega y)^2-4\omega x^2$ not a square have size p^2-1, and the orbits with $(z-\omega y)^2-4\omega x^2$ a square have size $(p^2-1)/2$. It follows that if we let k be the total number of times that $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is a square, with $\left(x^2+yz\right)\neq 0$, then the number of extra algebras is

$$\frac{p-1}{2}\cdot\frac{2k}{p^2-1}$$

The experimental evidence is that $\frac{2k}{p^2-1}=p+1$, i.e. that $k=(p+1)(p^2-1)/2$. (Checked for $p<100$.)

Total number of x,y,z with $\left(x^2+yz\right)\neq 0$ is

$$(p-1)^2+(p-1)(2p-1+(p-1)(p-2))=p^2\left(p-1\right)$$

Note that $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is homogeneous of degree 4 in x,y,z and so the number of square is $p-1$ times the number of squares with $x=0$ and $y=1$ plus $p-1$ times the number of squares with $x=1$.

Let $x=0$, $y=1$. Then we have $z\left(z+2\omega\right)\left(2z+\omega\right)$. (Note that $z\neq 0$.) We have two values of z which give 0. It follows that the number of x,y,z with $x^2+yz\neq 0$ and $2\left(\omega y+z\right)^2+\omega(x^2+yz)=0$ is p^2-1. So we need to show that we get a non-zero square

$$(p+1)(p^2-1)/2-(p^2-1)=\frac{1}{2}\left(p+1\right)\left(p-1\right)^2$$

times. The total number of times when the expression is non-zero is

$$p^2\left(p-1\right)-(p^2-1)=\left(p-1\right)\left(p^2-p-1\right)$$

64.9 Grand total

We have p orbits with a default of $(p+1)/2$ descendants in each orbit, and $(p-1)/2$ orbits with a default of p descendants in each orbit. We also have an extra $(p+1)(p-1)/2$ descendants from the situations when $\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$ is a square. So the grand total number of descendants is $\frac{3}{2}p^2-\frac{1}{2}$

65 Appendix E

From the descendants of 6.178.

We consider all $x,y,z\in\mathbb{Z}_p$ with $x^2+yz\neq 0$, and we let k be the number of possibilities for x,y,z with

$$\left(x^2+yz\right)\left(2\left(\omega y+z\right)^2+\omega(x^2+yz)\right)$$

a square. We need to prove that $k=(p+1)(p^2-1)/2$ for $p>3$. (Checked for all p up to 2089.)

If we replace z by ωz then we need to count triples x,y,z such that

$$\left(x^2+\omega yz\right)\left(2\omega^2\left(y+z\right)^2+\omega(x^2+\omega yz)\right)$$

is a square.

First count triples such that $x^2 + \omega yz = 1$ and $y + z = b$. Then y, z are roots of

$$t^2 - bt + \frac{1 - x^2}{\omega}.$$

This quadratic in t has roots if $b^2 - 4\frac{1-x^2}{\omega}$ is a square, or (equivalently) if $4\omega x^2 - (4\omega - b^2\omega^2)$ is a square. Note that $(4\omega - b^2\omega^2)$ cannot be zero. If $(4\omega - b^2\omega^2)$ is a square then $4\omega x^2 - (4\omega - b^2\omega^2)$ cannot be zero, and it is a square for $(p+1)/2$ values of x. For each of these values of x we have two solutions for y, z with $x^2 + \omega yz = 1$ and $y + z = b$. (If y_1, z_1 is one solution then the other is z_1, y_1.) If $(4\omega - b^2\omega^2) = \omega k^2$ then there are two values of x which give $4\omega x^2 - (4\omega - b^2\omega^2) = 0$, each giving a unique pair y, z (with $y = z$) with $x^2 + \omega yz = 1$ and $y + z = b$; and there are $(p-1)/2$ values of x for which $4\omega x^2 - (4\omega - b^2\omega^2)$ is a non-zero square with each of these values of x giving two pairs y, z with $x^2 + \omega yz = 1$ and $y + z = b$. So the total number of triples x, y, z such that $x^2 + \omega yz = 1$ is $p(p+1)$, and in these triples the sum $y + z$ takes all possible p values $p + 1$ times each. Note that if $x^2 + \omega yz = 1$ and $y + z = b$ then

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right) = \omega(2\omega b^2 + 1).$$

Next count triples x, y, z such that $x^2 + \omega yz = \omega$ and $y + z = b$. In this case y, z are roots of

$$t^2 - bt + \frac{\omega - x^2}{\omega}$$

and so we need $b^2 - 4\frac{\omega - x^2}{\omega}$ to be a square, or equivalently $4\omega x^2 - (4\omega^2 - b^2\omega^2)$ to be a square. If $b = \pm 2$ this is only possible when $x = 0$ and $y = z = \pm 1$. But if $b \neq \pm 2$ then (as above) we have $p + 1$ triples x, y, z such that $x^2 + \omega yz = \omega$ and $y + z = b$. In this case

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right) = \omega^3(2b^2 + 1).$$

Now $\omega^3(2b^2 + 1)$ is a square if and only if $\omega(2b^2 + 1)$ is a square.

So we need to count the number of times that $\omega(2\omega b^2 + 1)$ is a square, and add it to the number of times that $\omega(2b^2 + 1)$ is a square. If $b = 0$ then neither of these are squares, and as b takes on all non-zero values then between them $\omega(2\omega b^2 + 1)$ and $\omega(2b^2 + 1)$ take all values other than ω twice each. So as b ranges from 0 to $p - 1$, between them $\omega(2\omega b^2 + 1)$ and $\omega(2b^2 + 1)$ take square values a total of $p + 1$ times. Ignoring the problem with $b = \pm 2$ for the moment, as we run over all possible triples x, y, z with $x^2 + \omega yz = 1$ or ω, we see that each b arises $p + 1$ times, and so

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right)$$

is a square a total of $(p+1)^2$ times.

However this calculation has involved an overcount of the number of times $b = \pm 2$ arises with $x^2 + \omega yz = \omega$. But if $x^2 + \omega yz = \omega$ and $y + z = \pm 2$ then

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right) = \omega(8\omega^2 + \omega^2)$$

which is not a square. So as we run over all possible triples x, y, z with $x^2 + \omega yz = 1$ or ω, then

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right)$$

is a square a total of $(p+1)^2$ times. We get all possible triples x, y, z with $x^2 + \omega yz \neq 0$ by taking all triples

$$\lambda x, \lambda y, \lambda z \text{ with } x^2 + \omega yz = 1 \text{ or } \omega,\ 0 < \lambda \leq (p-1)/2.$$

So the total number of times that

$$\left(x^2 + \omega yz\right)\left(2\omega^2 (y+z)^2 + \omega(x^2 + \omega yz)\right)$$

is a square with $x^2 + \omega yz \neq 0$ is $(p-1)(p+1)^2/2$ as conjectured.

66 Appendix F

From Case 24 in the descendants of 5.14

We are considering the automorphism group of an algebra of order p^7

$$L = \langle a, b, c \,|\, cb = baa = 0,\ caa = kbab + bac,\ cac = \omega bab,\ pa,\ pb,\ pc,\ \text{class } 3\rangle,$$

where $p = 2 \bmod 3$ and where k is any element of $\mathbb{Z}_p$ which is not a value of

$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)}.$$

(Incidentally, the group equivalent to this algebra should occur somewhere in Wilkinson's list of the groups of exponent p and order p^7. Possibly the automorphism group is more transparent in Wilkinson's formulation.)

Actually, we are only interested in the automorphisms induced on L/L_2 by automophisms of L. So we can represent them by elements of $GL(3, p)$. We have $p-1$ automorphisms

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and $2(p-1)$ automorphisms

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} -4\alpha & k\alpha\beta + 3\alpha & 3k\omega^{-1}\alpha + \alpha\beta \\ 0 & 2\alpha & 2\alpha\beta \\ 0 & 2\omega\alpha\beta & 2\alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $\omega\beta^2 = -3$. (I have a proof that these $3(p-1)$ automorphisms exist. The evidence is that there are another $3(p-1)$ automorphisms when k is a value of $\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$, and I have checked this experimentally for all primes $p = 2 \bmod 3$ with $p < 1000$. But I do not have a proof that there are only $3(p-1)$ automorphisms when k is not a value of $\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$, though I have checked this for all primes $p = 2 \bmod 3$ less than 1000.)

It is not too hard to show that there are two possible forms for automorphisms (expressed as elements of $GL(3, p)$).

The ørst form is

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & \omega\varepsilon & \delta \end{pmatrix}$$

with the following equations satisøed:

$$\begin{aligned}
\omega\gamma\varepsilon + k\alpha\varepsilon + \beta\delta &= 0,\\
\alpha\varepsilon + \beta\varepsilon + \gamma\delta &= 0,\\
-\omega k\varepsilon^2 + \omega\beta\varepsilon + \omega\gamma\delta - 2\omega\varepsilon\delta + k\alpha\delta - k\delta^2 &= 0,\\
\omega\gamma\varepsilon - \omega\varepsilon^2 - 2k\varepsilon\delta + \alpha\delta + \beta\delta - \delta^2 &= 0.
\end{aligned}$$

It is clear that we can solve for β, γ in terms of δ, ε. Eliminating β, γ from the last two equations we have

$$\begin{aligned}
-\omega k\varepsilon^2 - \omega\alpha\varepsilon - 2\omega\varepsilon\delta + k\alpha\delta - k\delta^2 &= 0,\\
-k\alpha\varepsilon - \omega\varepsilon^2 - 2k\varepsilon\delta + \alpha\delta - \delta^2 &= 0.
\end{aligned}\tag{1}$$

(It is easy to check that the matrices given above satisfy these conditions.)

Equation (1) gives

$$\begin{aligned}
\alpha(\omega\varepsilon - k\delta) &= -\omega k\varepsilon^2 - 2\omega\varepsilon\delta - k\delta^2,\\
\alpha(k\varepsilon - \delta) &= -\omega\varepsilon^2 - 2k\varepsilon\delta - \delta^2
\end{aligned}$$

and these equations give $\varepsilon = 0$ and $\alpha = \delta$ (which gives the ørst matrix above), or $\alpha = -2\delta$. If $\alpha = -2\delta$ then we have

$$-\omega k\varepsilon^2 - 3k\delta^2 = 0$$

and this gives the second matrix above.

The second possible form is

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & -\delta & \varepsilon \\ 0 & -\omega\varepsilon & \delta \end{pmatrix}$$

with

$$\begin{aligned}
\omega\gamma\varepsilon + k\alpha\varepsilon - \beta\delta &= 0,\\
\alpha\varepsilon + \beta\varepsilon - \gamma\delta &= 0,\\
-\omega k\varepsilon^2 - \omega\beta\varepsilon + \omega\gamma\delta - 2\omega\varepsilon\delta + k\alpha\delta - k\delta^2 &= 0,\\
-\omega\gamma\varepsilon + \omega\varepsilon^2 + 2k\varepsilon\delta + \alpha\delta + \beta\delta + \delta^2 &= 0.
\end{aligned}$$

Again, we can solve for β, γ in terms of δ, ε. Eliminating β, γ from the last two equations we have

$$\begin{aligned}
-\omega k\varepsilon^2 + \omega\alpha\varepsilon - 2\omega\varepsilon\delta + k\alpha\delta - k\delta^2 &= 0,\\
k\alpha\varepsilon + \omega\varepsilon^2 + 2k\varepsilon\delta + \alpha\delta + \delta^2 &= 0.
\end{aligned}\tag{2}$$

So we need to show that when $p = 2 \bmod 3$ and k is not a value of $\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$, then equation (2) has no solutions.

Eliminating k from the two equations (2) we obtain

$$\begin{aligned}
0 &= (\omega\varepsilon^2 + \alpha\delta + \delta^2)(-\omega\varepsilon^2 + \alpha\delta - \delta^2) - (\omega\alpha\varepsilon - 2\omega\varepsilon\delta)(\alpha\varepsilon + 2\varepsilon\delta)\\
&= -\left(\delta^2 - \omega\varepsilon^2\right)\left(\delta^2 - \omega\varepsilon^2 - \alpha^2\right)
\end{aligned}$$

So $\alpha^2 = \left(\delta^2 - \omega\varepsilon^2\right)$. The second equation then gives

$$k = -\frac{\alpha\delta + 2\delta^2 - \alpha^2}{\varepsilon\,(\alpha + 2\delta)}$$

and it is easy to check that both equations in (2) are satisøed when $\alpha^2 = \left(\delta^2 - \omega\varepsilon^2\right)$ and $k = -\frac{\alpha\delta+2\delta^2-\alpha^2}{\varepsilon(\alpha+2\delta)}$. Note that we cannot have $\varepsilon = 0$. If $\alpha = -2\delta$ then the second equation gives $\omega\varepsilon^2 - \delta^2 = 0$, which is impossible. We need to show that all this implies that $\sqrt{\omega} + k$ is a cube in the øeld of order p^2. Now

$$\sqrt{\omega} + k = -\frac{-\sqrt{\omega}\varepsilon\alpha - 2\sqrt{\omega}\varepsilon\delta + \alpha\delta + 2\delta^2 - \alpha^2}{\varepsilon\,(\alpha + 2\delta)}$$

and since $p = 2 \bmod 3$ every element in $\mathbb{Z}_p$ is a cube. So it is suŒcient to show that

$$-\sqrt{\omega}\varepsilon\alpha - 2\sqrt{\omega}\varepsilon\delta + \alpha\delta + 2\delta^2 - \alpha^2$$

is a cube. But $\sqrt{\omega}\varepsilon = \sqrt{\delta^2 - \alpha^2}$, so we need to show that

$$-\sqrt{\delta^2 - \alpha^2}\alpha - 2\sqrt{\delta^2 - \alpha^2}\delta + \alpha\delta + 2\delta^2 - \alpha^2$$

is a cube. Substitute $\alpha\delta$ for δ and we have

$$-\sqrt{\delta^2 - 1}\alpha^2 - 2\sqrt{\delta^2 - 1}\alpha^2\delta + \alpha^2\delta + 2\alpha^2\delta^2 - \alpha^2.$$

We can take out the factor $-\alpha^2$ (which is a cube) and we get

$$\sqrt{\delta^2 - 1} + 2\sqrt{\delta^2 - 1}\delta - \delta - 2\delta^2 + 1$$

Set $t = \sqrt{\delta^2 - 1}$ and we have

$$(x + ty)^3 = x^3 + 3x^2ty - 3xy^2 + 3xy^2\delta^2 - ty^3 + ty^3\delta^2$$

So we want to ønd x, y such that

$$\begin{aligned} x^3 - 3xy^2 + 3xy^2\delta^2 + \delta + 2\delta^2 - 1 &= 0, \\ 3x^2y - y^3 + y^3\delta^2 - 2\delta - 1 &= 0. \end{aligned}$$

$$\begin{aligned} 0 &= 3y(x^3 - 3xy^2 + 3xy^2\delta^2 + \delta + 2\delta^2 - 1) - x(3x^2y - y^3 + y^3\delta^2 - 2\delta - 1) \\ &= -8xy^3 + 8xy^3\delta^2 + 3y\delta + 6y\delta^2 - 3y + 2x\delta + x \end{aligned}$$

Solution is : $x = -\frac{3y\delta+6y\delta^2-3y}{-8y^3+8y^3\delta^2+2\delta+1}$. Substitute this into $3x^2y - y^3 + y^3\delta^2 - 2\delta - 1 = 0$ and we get

$$-1 - 12\delta^2 - 6\delta - 8\delta^3 + 48y^3\delta^3 + 6y^3\delta + 48y^3\delta^4 + 96y^6\delta^2 - 48y^6\delta^4 + 192y^9\delta^2 - 96y^6\delta - 192y^9\delta^4$$
$$+192y^6\delta^3 + 64y^9\delta^6 - 96y^6\delta^5 - 48y^6 - 64y^9 + 42y^3 - 36y^3\delta^2$$

Solution is : $y = (\frac{1}{2\delta+2})^{\frac{1}{3}}$. Plug this into the expression for x.
$x = -\frac{1}{2}\left(\sqrt[3]{(\delta+1)}\right)^2\left(\sqrt[3]{2}\right)^2$

$$(x + y\sqrt{\delta^2 - 1})^3 = -2\delta^2 - \delta + 1 + (2\delta + 1)\sqrt{(-1 + \delta^2)}.$$

So there are no solutions to equations (2) when k is not a value of $\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$.

From Case 5 in the descendants of 6.173

We count the pairs $\lambda, \mu \in \mathbb{Z}_p$ such that $\lambda^2 - \mu$ is not a square, $\mu \neq \omega$, $\lambda \neq 0$ if $\mu = -\omega$, and such that $((\omega + \mu)^2 - 4\omega\lambda^2)$ is a square. (Here ω is a primitive element in $\mathbb{Z}_p$.) I conjecture that the number of such pairs is

$$(p+1)(p-1-\gcd(p-1,4))/4.$$

I have checked this for all primes up to 2039.

It may just be relevant that this arises in the context of considering pairs (λ, μ) under the equivalence relation

$(\lambda, \mu) \sim (\lambda', \mu')$ if $(\lambda, \mu) = (\lambda', \mu')$ or if

$$(\lambda', \mu') = \left(\frac{r^2\lambda + r(\omega + \mu) + \omega\lambda}{r^2 + 2r\lambda + \mu}, \frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu}\right)$$

for some $r \in \mathbb{Z}_p$. (When $r = \infty$ we can interpret the expression above as giving (λ, μ).) There are $(p-3)/2$ orbits of size $p+1$ of pairs satisfying the conditions $\lambda^2 - \mu$ is not a square, $\mu \neq \omega$, $\lambda \neq 0$ if $\mu = -\omega$.

It is not hard to show that the number of pairs (λ, ω) with $\lambda^2 - \omega$ not a square and $((\omega + \omega)^2 - 4\omega\lambda^2)$ a square is $(p+1)/2$ if $p = 1 \bmod 4$, and 0 otherwise. Also if $\lambda = 0$ and $\mu = -\omega$ then $\lambda^2 - \mu$ is not a square and $((\omega + \mu)^2 - 4\omega\lambda^2)$ is a square. So we need to show that the number of pairs (λ, μ) with $\lambda^2 - \mu$ not a square and $((\omega + \mu)^2 - 4\omega\lambda^2)$ a square is $(p-1)^2/4$.

There are $p(p+1)/2$ pairs (λ, μ) with $\lambda^2 - \mu$ a square, and we can write them as $(\lambda, \lambda^2 - k^2)$ with $k \sim -k$. Now

$$((\omega + \lambda^2 - k^2)^2 - 4\omega\lambda^2) = \left((\lambda + k)^2 - \omega\right)\left((\lambda - k)^2 - \omega\right).$$

When $k = 0$ this is always a square, and as k ranges from 1 to $(p-1)/2$ and λ ranges from 0 to $p-1$ the (unordered) pair $\{\lambda + k, \lambda - k\}$ takes on all possibilities for two element subsets of $\mathbb{Z}_p$. We get a square if $(\lambda + k)^2 - \omega$ and $(\lambda - k)^2 - \omega$ are either both squares, or both non-squares. Now $a^2 - \omega$ is a square for $(p-1)/2$ values of a, so $\left((\lambda + k)^2 - \omega\right)\left((\lambda - k)^2 - \omega\right)$ is a square a total of

$$(p-1)(p-3)/8 + (p+1)(p-1)/8 + p = \frac{1}{4}(p+1)^2$$

times.

So we need to show that the total number of times that $((\omega + \mu)^2 - 4\omega\lambda^2)$ is a square over all possible p^2 pairs (λ, μ) is $\frac{1}{2}p^2 + \frac{1}{2}$. (Checked for all $3 \leq p \leq 2999$.) Arggghhh!!!! Once we have reduced it this far then it is clear that all we are really asking is how many times $\mu^2 - 4\omega\lambda^2$ is a square. If we exclude $\lambda = \mu = 0$ then we are back to a familiar situation where we have a group of 2×2 matrices $\begin{pmatrix} \mu & 2\lambda \\ 2\omega\lambda & \mu \end{pmatrix}$ of order $p^2 - 1$, and the determinants take each non-zero value $p+1$ times each. So the total number of times we get a square (including $\lambda = \mu = 0$) is

$$(p+1)(p-1)/2 + 1 = \frac{1}{2}p^2 + \frac{1}{2}.$$

From the descendants of 6.114

$$\langle a, b, c \,|\, pa - ba,\ pb - cb,\ pc - xba - ca,\ \text{class } 2\rangle \ (x = 0, 1, \ldots, p-1)$$

We consider matrices

$$\begin{pmatrix} a & b & c \\ d & y & f \\ u & v & w \end{pmatrix}$$

where

$$xc - ay + a + bd = 0$$

$$-af + cd + c = 0$$

$$-bf + b + cy = 0$$

$$xf - dv + d + yu = 0$$

$$-dw + fu + f = 0$$

$$-yw + y + fv = 0$$

$$-xay + xbd + xw - av + bu + u = 0$$

$$-xaf + xcd - aw + cu + w = 0$$

$$-xbf + xcy - bw + cv + v = 0$$

The following are possibilities (in addition to I)

$$A = \begin{pmatrix} x & 0 & -1 \\ -1 & 0 & 0 \\ x & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} x^2 - x & -1 & -x \\ -x & 0 & 1 \\ x^2 - 1 & 0 & -x \end{pmatrix}, C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & x & 1 \\ -1 & -x & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & -x & -1 \\ -1 & x^2 - x & x \\ 0 & 1 - x^2 & -x \end{pmatrix}$$

If $a = 0$ then C, D are the only possibilities. If $d = 0$ then I, C are the only possibilities. If $y = 0$ then A, B are the only possibilities. If the øst row is $(\lambda, 0, 0)$ then we have I, and if the second row is $(0, \lambda, 0)$ we have I.

If $a \neq 0$ and $x \neq -1$ then the øst row must have the form

$$(\alpha(1-\gamma), -\alpha(1+\gamma x)\gamma, \alpha(1-\gamma)\gamma)$$

for some $\gamma \neq 1$. So there are at most $p-1$ matrices with $a \neq 0$ and $x \neq -1$ (including I when $\gamma = 0$). So there are at most $p+1$ matrices. The experimental evidence is that this is always attained for some, but not all $x \neq -1$. If $x = -1$ then when $a \neq 0$ we have the additional possibility for øst row

$$(a, b, a)$$

which gives another p possibilities, giving at most $2p+1$ matrices in all. However the experimental evidence is that the number is $2p$.

Consider the case when a and d are both non-zero. [If this is not the case we have one of I, A, B, C, D.]

If a and d are non-zero, and if $x \neq -1$ then we can take the matrix to be

$$\begin{pmatrix} \alpha(1-\gamma) & -\alpha(1+\gamma x)\gamma & \alpha(1-\gamma)\gamma \\ \beta(1-\delta) & -\beta(1+\delta x)\delta & \beta(1-\delta)\delta \\ u & v & w \end{pmatrix}$$

with $\gamma, \delta \neq 1$. Note that if $\gamma = 0$ we have I and if $\delta = 0$ we have A.

$a = \alpha(1-\gamma)$
$b = -\alpha(1+\gamma x)\gamma$
$c = \alpha(1-\gamma)\gamma$
$d = \beta(1-\delta)$
$y = -\beta(1+\delta x)\delta$
$f = \beta(1-\delta)\delta$

$$xc - ay + a + bd = \alpha\left(\gamma x - x\gamma^2 + \beta\delta + \beta\delta^2 x - \beta\delta^2\gamma x + 1 - \gamma - \gamma\beta - \gamma^2\beta x + \gamma^2\beta\delta x\right)$$

$$-af + cd + c = \alpha(-1+\gamma)\left(-\gamma - \gamma\beta + \gamma\beta\delta - \beta\delta^2 + \beta\delta\right)$$

$$-bf + b + cy = \alpha\gamma\left(\gamma\beta\delta x - \gamma x + \gamma\beta\delta - \beta\delta^2 - 1 - \beta\delta^2 x\right)$$

$$xf - dv + d + yu = 0$$

Solution is : $v = \frac{-\delta x + \delta^2 x - 1 + \delta + \delta u + \delta^2 ux}{-1+\delta}$

$$-dw + fu + f = 0$$

Solution is : $w = (u+1)\,\delta$

$$-yw + y + fv = 0 \text{ already}$$

$$-xay + xbd + xw - av + bu + u \text{ is a terrible mess}$$

$$\begin{aligned} 0 &= -xaf + xcd - aw + cu + w \\ &= -x\alpha\beta\delta + \alpha\beta\delta^2 x - \alpha\beta\delta^2\gamma x + x\alpha\gamma\beta - \alpha\gamma^2\beta x + \alpha\gamma^2\beta\delta x - \alpha\delta u - \alpha\delta \\ &\quad + \alpha\gamma\delta u + \alpha\gamma\delta + \alpha\gamma u - u\gamma^2\alpha + \delta u + \delta \end{aligned}$$

$$-xbf + xcy - bw + cv + v \text{ is a terrible mess}$$

$$xc - ay + a + bd = \alpha\left(\gamma x - x\gamma^2 + \beta\delta + \beta\delta^2 x - \beta\delta^2\gamma x + 1 - \gamma - \gamma\beta - \gamma^2\beta x + \gamma^2\beta\delta x\right)$$

$$-af + cd + c = \alpha(-1+\gamma)\left(-\gamma - \gamma\beta + \gamma\beta\delta - \beta\delta^2 + \beta\delta\right)$$

$\beta\delta^2 = (-\gamma - \gamma\beta + \gamma\beta\delta + \beta\delta)$

$\left(\gamma x - x\gamma^2 + \beta\delta + (-\gamma - \gamma\beta + \gamma\beta\delta + \beta\delta)x - (-\gamma - \gamma\beta + \gamma\beta\delta + \beta\delta)\gamma x + 1 - \gamma - \gamma\beta - \gamma^2\beta x + \gamma^2\beta\delta x\right) = \beta\delta - x\gamma\beta + \beta\delta x + 1 - \gamma - \gamma\beta$, Solution is : $\delta = \frac{x\gamma\beta - 1 + \gamma + \gamma\beta}{\beta(1+x)}$

$$\left(\gamma x - x\gamma^2 + \beta\delta + \beta\delta^2 x - \beta\delta^2\gamma x + 1 - \gamma - \gamma\beta - \gamma^2\beta x + \gamma^2\beta\delta x\right)$$

$$= -x\frac{-1 + 3\gamma - 3\gamma^2 - 2\gamma^2\beta - \gamma^2\beta x + 2\gamma\beta - \beta + x\gamma\beta - x^2\gamma\beta + x^2\gamma^2\beta + \gamma^3 x\beta + \gamma^3\beta + \gamma^3 - \beta x}{\beta\left(1+x\right)^2}$$

$$= -x\left(-1+\gamma\right)\frac{\gamma^2\beta x + \gamma^2\beta + \gamma^2 - 2\gamma - \gamma\beta + x^2\gamma\beta + \beta x + \beta + 1}{\beta\left(1+x\right)^2}$$

$$\left(-\gamma - \gamma\beta + \gamma\beta\delta - \beta\delta^2 + \beta\delta\right)$$

$$= -\frac{\gamma^2\beta x + \gamma^2\beta + \gamma^2 - 2\gamma - \gamma\beta + x^2\gamma\beta + \beta x + \beta + 1}{\beta\left(1+x\right)^2}$$

$$\left(\gamma\beta\delta x - \gamma x + \gamma\beta\delta - \beta\delta^2 - 1 - \beta\delta^2 x\right)$$

$$= -\frac{\gamma^2\beta x + \gamma^2\beta + \gamma^2 - 2\gamma - \gamma\beta + x^2\gamma\beta + \beta x + \beta + 1}{\beta\left(1+x\right)}$$

So ørst three equations are satisøed if $\gamma^2\beta x + \gamma^2\beta + \gamma^2 - 2\gamma - \gamma\beta + x^2\gamma\beta + \beta x + \beta + 1 = 0$
Solution is : $\beta = -\frac{(-1+\gamma)^2}{(1+x)(\gamma x + \gamma^2 - \gamma + 1)}$. And the next three are satisøed.

$$-xay + xbd + xw - av + bu + u \text{ is a dogs breakfast}$$

$$-xaf + xcd - aw + cu + w \text{ almost as bad}$$

$$-xbf + xcy - bw + cv + v \text{ terrible}$$

$a = -\alpha\left(-1+\gamma\right)$
$b = -\alpha\left(1+\gamma x\right)\gamma$
$c = -\alpha\left(-1+\gamma\right)\gamma$
$d = -\left(-1+\gamma\right)\frac{\gamma}{\gamma x + \gamma^2 - \gamma + 1}$
$y = \left(\gamma x - \gamma + 1\right)\frac{1+\gamma x}{\gamma x + \gamma^2 - \gamma + 1}$
$f = \gamma\frac{1+\gamma x}{\gamma x + \gamma^2 - \gamma + 1}$
$u = u$
$v = -\frac{ux^2\gamma^2 + x^2\gamma^2 - ux\gamma^2 + \gamma x + 2ux\gamma - \gamma^2 - \gamma u + \gamma + u}{(-1+\gamma)\gamma}$
$w = -\left(1+\gamma x\right)\frac{u+1}{-1+\gamma}$
$x\alpha\gamma^3 - 3x\alpha\gamma^2 - \alpha ux\gamma^2 - \gamma x - ux\gamma + 2x\alpha\gamma + \alpha ux\gamma - \alpha\gamma^3 u + 2\alpha u\gamma^2 - 2\alpha\gamma u - \alpha\gamma - 1 - u + \alpha u + \alpha = 0$
Solution is : $u = -\frac{x\alpha\gamma^3 - 3x\alpha\gamma^2 - \gamma x + 2x\alpha\gamma - \alpha\gamma - 1 + \alpha}{-x\alpha\gamma^2 - \gamma x + x\alpha\gamma - \alpha\gamma^3 + 2\alpha\gamma^2 - 2\alpha\gamma - 1 + \alpha}$

$$v = -\frac{-\alpha\gamma^4 + x^3\alpha\gamma^4 - x\alpha\gamma^3 + 2\alpha\gamma^3 + 3x^2\alpha\gamma^3 - x^3\alpha\gamma^3 + x^2\gamma^2 - \gamma^2 x - 2\alpha\gamma^2 - 3x^2\alpha\gamma^2 + 4x\alpha\gamma^2 + 2\gamma x - 3x\alpha\gamma - \gamma + 2\alpha\gamma + 1 - \alpha}{\gamma\left(x\alpha\gamma^2 + \gamma x - x\alpha\gamma + \alpha\gamma^3 - 2\alpha\gamma^2 + 2\alpha\gamma + 1 - \alpha\right)}$$

$$= -\left(\gamma x - \gamma + 1\right)\frac{x^2\alpha\gamma^3 - x^2\alpha\gamma^2 + x\alpha\gamma^3 + x\alpha\gamma^2 - 2x\alpha\gamma + \gamma x + \alpha\gamma^3 - \alpha\gamma^2 + \alpha\gamma + 1 - \alpha}{\gamma\left(x\alpha\gamma^2 + \gamma x - x\alpha\gamma + \alpha\gamma^3 - 2\alpha\gamma^2 + 2\alpha\gamma + 1 - \alpha\right)}$$

$$w = -\left(\gamma x + \gamma - x - 1\right)\alpha\left(1+\gamma x\right)\frac{\gamma}{x\alpha\gamma^2 + \gamma x - x\alpha\gamma + \alpha\gamma^3 - 2\alpha\gamma^2 + 2\alpha\gamma + 1 - \alpha}$$

$$-xay+xbd+xw-av+bu+u$$

$$\begin{aligned} = \quad & ((\alpha^2\gamma^5+2x\alpha^2\gamma^4+x\alpha\gamma^4-3\alpha^2\gamma^4-x\alpha\gamma^3+x^2\alpha^2\gamma^3+5\alpha^2\gamma^3+x^2\alpha\gamma^3-4x\alpha^2\gamma^3-\gamma^2x) \\ & +(x\alpha\gamma^2-x^2\alpha^2\gamma^2+\alpha\gamma^2+4x\alpha^2\gamma^2-5\alpha^2\gamma^2-\gamma-\alpha\gamma+3\alpha^2\gamma-2x\alpha^2\gamma+x\alpha\gamma+\alpha-\alpha^2))\end{aligned}$$

$$\div\gamma\left(x\alpha\gamma^2+\gamma x-x\alpha\gamma+\alpha\gamma^3-2\alpha\gamma^2+2\alpha\gamma+1-\alpha\right)$$

$$\left(x\alpha\gamma-1+\alpha-\alpha\gamma+\alpha\gamma^2\right)\left(x\alpha\gamma^2+\gamma^2x-x\alpha\gamma+\alpha\gamma^3-2\alpha\gamma^2+2\alpha\gamma-\alpha+\gamma\right)=0$$

$$-xbf+xcy-bw+cv+v=$$

$$\begin{aligned} & (-1+\alpha+\gamma-2\gamma x-x^2\gamma^2+\gamma^2x+3x^2\alpha\gamma^2-5x\alpha\gamma^2-2x^2\alpha\gamma^3+4x\alpha\gamma^3+x^2\alpha\gamma^4-x\alpha\gamma^4 \\ & -3\alpha\gamma+3x\alpha\gamma+4\alpha\gamma^2-3\alpha\gamma^3+\alpha\gamma^4+x^3\gamma^3\alpha+x^2\alpha^2\gamma^3-x^2\alpha^2\gamma^4+2x\alpha^2\gamma^2-4x\alpha^2\gamma^3 \\ & +4x\alpha^2\gamma^4-2x\alpha^2\gamma^5+\alpha^2\gamma-3\alpha^2\gamma^2+5\alpha^2\gamma^3-5\alpha^2\gamma^4+3\alpha^2\gamma^5-\alpha^2\gamma^6) \\ & \div\gamma\left(x\alpha\gamma^2+\gamma x-x\alpha\gamma+\alpha\gamma^3-2\alpha\gamma^2+2\alpha\gamma+1-\alpha\right)\end{aligned}$$

$$=\left(x\alpha\gamma-1+\alpha-\alpha\gamma+\alpha\gamma^2\right)(x^2\gamma^2-x\alpha\gamma^3+x\alpha\gamma^2-\gamma^2x+2\gamma x-\alpha\gamma^4+2\alpha\gamma^3-2\alpha\gamma^2+\alpha\gamma-\gamma+1)=0$$

$x\alpha\gamma-1+\alpha-\alpha\gamma+\alpha\gamma^2=0$, Solution is : $\alpha=\frac{1}{\gamma x+\gamma^2-\gamma+1}$
If $\left(x\alpha\gamma-1+\alpha-\alpha\gamma+\alpha\gamma^2\right)\neq 0$ we have
$x\alpha\gamma^2+\gamma^2x-x\alpha\gamma+\alpha\gamma^3-2\alpha\gamma^2+2\alpha\gamma-\alpha+\gamma=0$ and
$x^2\gamma^2-x\alpha\gamma^3+x\alpha\gamma^2-\gamma^2x+2\gamma x-\alpha\gamma^4+2\alpha\gamma^3-2\alpha\gamma^2+\alpha\gamma-\gamma+1=0$
Now

$$x^2\gamma^2-x\alpha\gamma^3+x\alpha\gamma^2-\gamma^2x+2\gamma x-\alpha\gamma^4+2\alpha\gamma^3-2\alpha\gamma^2+\alpha\gamma-\gamma+1+\gamma(x\alpha\gamma^2+\gamma^2x-x\alpha\gamma+\alpha\gamma^3-2\alpha\gamma^2+2\alpha\gamma-\alpha+\gamma)$$

$$=x^2\gamma^2-\gamma^2x+2\gamma x-\gamma+1+\gamma^3x+\gamma^2=\left(\gamma x+\gamma^2-\gamma+1\right)(1+\gamma x)$$

Now if $\gamma x+\gamma^2-\gamma+1=0$ we are in a diœerent ballgame anyway. And if $1+\gamma x=0$ then $b=y=f=0$ and we have A, and this is what we get from the solution $\alpha=\frac{1}{\gamma x+\gamma^2-\gamma+1}$.

$a=-\frac{1}{\gamma x+\gamma^2-\gamma+1}(-1+\gamma)$
$b=-\gamma\frac{1+\gamma x}{\gamma x+\gamma^2-\gamma+1}$
$c=-(-1+\gamma)\frac{\gamma}{\gamma x+\gamma^2-\gamma+1}$
$d=\frac{-1}{(\gamma x+\gamma^2-\gamma+1)}(-1+\gamma)\gamma$
$y=\frac{1}{(\gamma x+\gamma^2-\gamma+1)}(\gamma x-\gamma+1)(1+\gamma x)$
$f=\frac{1}{(\gamma x+\gamma^2-\gamma+1)}\gamma(1+\gamma x)$
$u=-(1+x)\frac{\gamma}{\gamma x+\gamma^2-\gamma+1}$
$v=-\left(\gamma x^2+x-\gamma+1\right)\frac{\gamma}{\gamma x+\gamma^2-\gamma+1}$
$w=-(-1+\gamma)\frac{1+\gamma x}{\gamma x+\gamma^2-\gamma+1}$

So provided $\gamma x + \gamma^2 - \gamma + 1 \neq 0$ we get a unique matrix

$$E = \begin{pmatrix} a & b & c \\ d & y & f \\ u & v & w \end{pmatrix} = \frac{1}{(\gamma x + \gamma^2 - \gamma + 1)} \begin{pmatrix} -\gamma + 1 & -\gamma(1+\gamma x) & -(-1+\gamma)\gamma \\ -(-1+\gamma)\gamma & (\gamma x - \gamma + 1)(1+\gamma x) & \gamma(1+\gamma x) \\ -(1+x)\gamma & -\left(\gamma x^2 + x - \gamma + 1\right)\gamma & -(-1+\gamma)(1+\gamma x) \end{pmatrix}$$

Note that if $\gamma = 1$ this is C.
Now consider the case when $(\gamma x + \gamma^2 - \gamma + 1) = 0$.
We make the following deønitions as above:
$a = \alpha(1-\gamma)$
$b = -\alpha(1+\gamma x)\gamma$
$c = \alpha(1-\gamma)\gamma$
$d = \beta(1-\delta)$
$y = -\beta(1+\delta x)\delta$
$f = \beta(1-\delta)\delta$
$v = \frac{-\delta x + \delta^2 x - 1 + \delta + \delta u + \delta^2 ux}{-1+\delta}$
$w = (u+1)\delta$
$\delta = \frac{x\gamma\beta - 1 + \gamma + \gamma\beta}{\beta(1+x)}$
Then

$$-af + cd + c = -\alpha(-1+\gamma)\frac{1 - 2\gamma + \gamma^2 + x^2\gamma\beta + \beta + \beta x + \gamma^2\beta + x\gamma^2\beta - \gamma\beta}{\beta(1+x)^2}$$

So we need

$$1 - 2\gamma + \gamma^2 + x^2\gamma\beta + \beta + \beta x + \gamma^2\beta + x\gamma^2\beta - \gamma\beta = 0.$$

But if $\gamma x + \gamma^2 - \gamma + 1 = 0$ then $\gamma^2 = (-\gamma x + \gamma - 1)$, and

$$\begin{aligned} & 1 - 2\gamma + \gamma^2 + x^2\gamma\beta + \beta + \beta x + \gamma^2\beta + x\gamma^2\beta - \gamma\beta \\ = \; & 1 - 2\gamma + (-\gamma x + \gamma - 1) + x^2\gamma\beta + \beta + \beta x + (-\gamma x + \gamma - 1)\beta + x(-\gamma x + \gamma - 1)\beta - \gamma\beta \\ = \; & -\gamma - \gamma x \end{aligned}$$

which is non-zero since $\gamma \neq 0$ and $x \neq -1$.

Note that $\gamma = 0$ and $\gamma = 1$ are never roots of $\gamma x + \gamma^2 - \gamma + 1$ for $x \neq -1$. The roots are $\gamma = -\frac{1}{2}x + \frac{1}{2} \pm \frac{1}{2}\sqrt{(x+1)(x-3)}$. So if $x = 3$ we have one root, and if $(x+1)(x-3)$ is a non-zero square we have two roots.

So when $x \neq -1$ and $(x+1)(x-3)$ is not a square we have an automorphism group of order $p+1$ consisting of C, D and $p-1$ matrices E with $\gamma \neq 1$. If $x = 3$ we have an automorphism group of size p consisting of C, D and $p-2$ matrices E with $\gamma \neq \pm 1$. And there are $(p-3)/2$ values of x for which $(x+1)(x-3)$ is a non-zero square, and for these we have an automorphism group of size $p-1$ consisting of C, D and $p-3$ matrices E with $\gamma \neq 1$, and γ not a root of $\gamma x + \gamma^2 - \gamma + 1$.

68.1 $x = -1$

Now consider the case when $x = -1$. The matrix E becomes

$$\begin{pmatrix} -\frac{1}{-1+\gamma} & \frac{\gamma}{-1+\gamma} & -\frac{\gamma}{-1+\gamma} \\ -\frac{\gamma}{-1+\gamma} & \frac{2\gamma-1}{-1+\gamma} & -\frac{\gamma}{-1+\gamma} \\ 0 & 0 & 1 \end{pmatrix}$$

and gives $p-1$ automorphisms for $\gamma \neq 1$. If $a \neq 0$ the other possible ørst row takes the form $(\alpha, \alpha\beta, \alpha)$. So let

$a = \alpha$
$b = \alpha\beta$
$c = \alpha$

$$xc - ay + a + bd = 0$$

Solution is : $y = \beta d$. Note that if $y = 0$ we have A, so we can assume that $\beta, d \neq 0$, if necessary.

$$-af + cd + c = 0$$

Solution is : $d = f - 1$. This gives

$$-bf + b + cy = 0$$

$$xf - dv + d + yu = 0$$

Solution is : $u = -\frac{-vf+v-1}{\beta f - \beta}$

$$-dw + fu + f = 0$$

Solution is : $w = f\frac{vf-v+1+\beta f-\beta}{\beta(f-1)^2}$. We now have

$$0 = -yw + y + fv = -\frac{f + \beta f - \beta}{f - 1}$$

and so $\beta = -1$ is impossible. So we assume that $\beta \neq -1$, which gives $f = \frac{\beta}{1+\beta}$. Then

$$0 = -xay + xbd + xw - av + bu + u = \frac{v\beta - 2\beta - \alpha\beta - \alpha\beta^2 + v - 1}{\beta}$$

which gives $v = -\frac{-2\beta-\alpha\beta-\alpha\beta^2-1}{1+\beta}$. Now we need

$$0 = -xaf + xcd - aw + cu + w = \frac{\alpha + \alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2 - \alpha\beta^2 - \beta}{1 + \beta}$$

and $\alpha + \alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2 - \alpha\beta^2 - \beta = (\alpha\beta + 1 + \alpha)(-\beta + \alpha\beta + \alpha)$. We also need

$$0 = -xbf + xcy - bw + cv + v = \frac{3\alpha\beta^2 + 4\alpha\beta + \alpha^2\beta^3 + 2\alpha^2\beta^2 + \alpha^2\beta + \alpha + 2\beta + 1}{1 + \beta}$$

and $3\alpha\beta^2 + 4\alpha\beta + \alpha^2\beta^3 + 2\alpha^2\beta^2 + \alpha^2\beta + \alpha + 2\beta + 1 = \left(2\beta + \alpha\beta + \alpha\beta^2 + 1\right)(\alpha\beta + 1 + \alpha)$.

We show that we need $\alpha\beta + 1 + \alpha = 0$, which gives $\alpha = -\frac{1}{1+\beta}$. So suppose for the moment that $\alpha\beta + 1 + \alpha = 0$, in which case we need $-\beta + \alpha\beta + \alpha = 0$, which gives $\alpha = \frac{\beta}{1+\beta}$, which in turn gives $2\beta + \alpha\beta + \alpha\beta^2 + 1 = (1 + \beta)^2$.

So we have
$a = -\frac{1}{1+\beta}$
$b = -\frac{\beta}{1+\beta}$
$c = -\frac{1}{1+\beta}$
$d = -\frac{1}{1+\beta}$
$y = -\frac{\beta}{1+\beta}$

$f = \frac{\beta}{1+\beta}$
$u = -1$
$v = 1$
$w = 0$
which gives $p-1$ matrices

$$F = \begin{pmatrix} a & b & c \\ d & y & f \\ u & v & w \end{pmatrix} = \begin{pmatrix} -\frac{1}{1+\beta} & -\frac{\beta}{1+\beta} & -\frac{1}{1+\beta} \\ -\frac{1}{1+\beta} & -\frac{\beta}{1+\beta} & \frac{\beta}{1+\beta} \\ -1 & 1 & 0 \end{pmatrix}$$

Note that $\beta = 0$ gives A.

So if $x = -1$ we have $2p$ automorphisms consisting of C, D, $p-1$ matrices E with $\gamma \neq 1$, and $p-1$ matrices F with $\beta \neq -1$.

68.2 Action on bab, bac

automs[7]
$-xbf^2 + xcyf + ay^2 - ayf - bdy - bdf + bf^2 + 2cdy - cyf,$
$-xaf^2 + xcdf - adf + 2ayf - bdf - bf^2 + cd^2 - cdy + cyf,$
automs[8]
$-xbfw + xcyw + ayv - afv - bdv - bfu + bfw + cdv + cyu - cyw,$
$-xafw + xcdw + ayw - afu + afv - bdw - bfw + cdu - cdv + cyw,$

68.2.1 Matrix C

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & x & 1 \\ -1 & -x & 0 \end{pmatrix}$$

$a = 0$, $b = -1$, $c = 0$, $d = 0$, $y = x$, $f = 1$, $u = -1$, $v = -x$, $w = 0$.
automs[7]
$-xbf^2 + xcyf + ay^2 - ayf - bdy - bdf + bf^2 + 2cdy - cyf = x - 1$
$-xaf^2 + xcdf - adf + 2ayf - bdf - bf^2 + cd^2 - cdy + cyf = 1$
automs[8]
$-xbfw + xcyw + ayv - afv - bdv - bfu + bfw + cdv + cyu - cyw = -1$
$-xafw + xcdw + ayw - afu + afv - bdw - bfw + cdu - cdv + cyw = 0$
So

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \to \begin{pmatrix} x-1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} x-1+z \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} x-1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The only ratios that matrix C øxes are $(1, z)$ where $z^2 - z + zx + 1 = 0$. (Note that if this polynomial has roots then the number of matrices E is reduced.)

68.2.2 Matrix D

$$\begin{pmatrix} 0 & -x & -1 \\ -1 & x^2-x & x \\ 0 & 1-x^2 & -x \end{pmatrix}$$

$a=0$, $b=-x$, $c=-1$, $d=-1$, $y=x^2-x$, $f=x$, $u=0$, $v=1-x^2$, $w=-x$.
automs[7]
$-xbf^2+xcyf+ay^2-ayf-bdy-bdf+bf^2+2cdy-cyf=x^2-2x$
$-xaf^2+xcdf-adf+2ayf-bdf-bf^2+cd^2-cdy+cyf=x-1$
automs[8]
$-xbfw+xcyw+ayv-afv-bdv-bfu+bfw+cdv+cyu-cyw=-x+1$
$-xafw+xcdw+ayw-afu+afv-bdw-bfw+cdu-cdv+cyw=-1$

$$\begin{pmatrix} x^2-2x & x-1 \\ 1-x & -1 \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix}=\begin{pmatrix} x^2-2x+zx-z \\ -x+1-z \end{pmatrix}$$

$$\begin{pmatrix} x^2-2x & x-1 \\ 1-x & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} x-1 \\ -1 \end{pmatrix}$$

If $x=1$ then ration $(0,1)$ is øxed. Otherwise we need

$$0=-x+1-z-z(x^2-2x+zx-z)=-(x-1)\left(zx+1-z+z^2\right)$$

So if $x=1$ then all $p+1$ ratios are øxed. Otherwise, $(1,z)$ is øxed if $zx+1-z+z^2=0$. If $x=3$ we get 1 ratio øxed, and if $(1+x)(x-3)$ is a non-zero square we get two ratios øxed. Note that if $x=-1$ then the only ratio øxed is $(1,-1)$.

68.2.3 Matrix E

$$\frac{1}{(\gamma x+\gamma^2-\gamma+1)}\begin{pmatrix} -\gamma+1 & -\gamma(1+\gamma x) & -(-1+\gamma)\gamma \\ -(-1+\gamma)\gamma & (\gamma x-\gamma+1)(1+\gamma x) & \gamma(1+\gamma x) \\ -(1+x)\gamma & -\left(\gamma x^2+x-\gamma+1\right)\gamma & -(-1+\gamma)(1+\gamma x) \end{pmatrix}$$

We can ignore any scale factor so we take
$a=-\gamma+1$, $b=-\gamma(1+\gamma x)$, $c=-(-1+\gamma)\gamma$, $d=-(-1+\gamma)\gamma$, $y=(\gamma x-\gamma+1)(1+\gamma x)$, $f=\gamma(1+\gamma x)$, $u=-(1+x)\gamma$, $v=-\left(\gamma x^2+x-\gamma+1\right)\gamma$, $w=-(-1+\gamma)(1+\gamma x)$.
automs[7]
$-xbf^2+xcyf+ay^2-ayf-bdy-bdf+bf^2+2cdy-cyf$
$=(1+\gamma x)(\gamma x-2\gamma+1)\left(1+\gamma x-\gamma+\gamma^2\right)^2$
$-xaf^2+xcdf-adf+2ayf-bdf-bf^2+cd^2-cdy+cyf$
$=\gamma(\gamma x+2-\gamma)\left(1+\gamma x-\gamma+\gamma^2\right)^2$
automs[8]
$-xbfw+xcyw+ayv-afv-bdv-bfu+bfw+cdv+cyu-cyw$
$=-\gamma(\gamma x+2-\gamma)\left(1+\gamma x-\gamma+\gamma^2\right)^2$
$-xafw+xcdw+ayw-afu+afv-bdw-bfw+cdu-cdv+cyw$
$=-(-1+\gamma)(\gamma+1)\left(1+\gamma x-\gamma+\gamma^2\right)^2$

$$\begin{pmatrix} (1+\gamma x)(\gamma x-2\gamma+1) & \gamma(\gamma x+2-\gamma) \\ -\gamma(\gamma x+2-\gamma) & -(-1+\gamma)(\gamma+1) \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix}=\begin{pmatrix} \gamma^2x^2+2\gamma x-2\gamma^2x-2\gamma+1+\gamma^2zx+2\gamma z-\gamma^2 z \\ -\gamma^2x-2\gamma+\gamma^2-\gamma^2z+z \end{pmatrix}$$

$$\begin{pmatrix} (1+\gamma x)(\gamma x-2\gamma+1) & \gamma(\gamma x+2-\gamma) \\ -\gamma(\gamma x+2-\gamma) & -(-1+\gamma)(\gamma+1) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(\gamma x+2-\gamma) \\ -(-1+\gamma)(\gamma+1) \end{pmatrix}$$

The ratio $(0,1)$ is øxed if $\gamma = 0$ or if $(\gamma x + 2 - \gamma) = 0$, solution is : $\gamma = -\frac{2}{x-1}$. In this last case we need $x \neq 1$, and then

$$(\gamma x + \gamma^2 - \gamma + 1) = -(1+x)\frac{x-3}{(x-1)^2}$$

so we also need $x \neq -1$ and $x \neq 3$. The ratio $(1,z)$ is øxed if

$$-\gamma\left(1 - z + zx + z^2\right)(\gamma x + 2 - \gamma) = 0$$

so it is øxed if $\gamma = 0$ or $\gamma = -\frac{2}{x-1}$ or if z is a root of $1 - z + zx + z^2 = 0$. Note that if $x = -1$ then the only ratio øxed is $(1,1)$, unless $\gamma = 0$ in which case all ratios are øxed.

68.2.4 Matrix F $(x = -1)$

$$\begin{pmatrix} -\frac{1}{1+\beta} & -\frac{\beta}{1+\beta} & -\frac{1}{1+\beta} \\ -\frac{1}{1+\beta} & -\frac{\beta}{1+\beta} & \frac{\beta}{1+\beta} \\ -1 & 1 & 0 \end{pmatrix}$$

$a = -1$, $b = -\beta$, $c = -1$, $d = -1$, $y = -\beta$, $f = \beta$, $u = -1-\beta$, $v = 1+\beta$, $w = 0$ and $x = -1$.

automs[7]
$-xbf^2 + xcyf + ay^2 - ayf - bdy - bdf + bf^2 + 2cdy - cyf$
$= -2\beta^3 - 4\beta^2 - 2\beta = -2\beta(1+\beta)^2$
$-xaf^2 + xcdf - adf + 2ayf - bdf - bf^2 + cd^2 - cdy + cyf$
$= \beta^2 - \beta + \beta^3 - 1 = (\beta - 1)(1+\beta)^2$
automs[8]
$-xbfw + xcyw + ayv - afv - bdv - bfu + bfw + cdv + cyu - cyw$
$= \beta - \beta^2 - \beta^3 + 1 = -(\beta-1)(1+\beta)^2$
$-xafw + xcdw + ayw - afu + afv - bdw - bfw + cdu - cdv + cyw$
$= -4\beta - 2\beta^2 - 2 = -2(1+\beta)^2$

$$\begin{pmatrix} -2\beta & (\beta-1) \\ -(\beta-1) & -2 \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} -2\beta + z\beta - z \\ -\beta + 1 - 2z \end{pmatrix}$$

$$\begin{pmatrix} -2\beta & (\beta-1) \\ -(\beta-1) & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \beta - 1 \\ -2 \end{pmatrix}$$

The ratio $(0,1)$ is øxed if $\beta = 1$, and the ratio $(1,z)$ is øxed if

$$0 = -\beta + 1 - 2z - z(-2\beta + z\beta - z) = -(z-1)^2(\beta - 1),$$

in other words if $z = 1$ or $\beta = 1$.

68.3 Orbits

68.3.1 $x = -1$

If $x = -1$ then the automorphism group has order $2p$, the identity transformation øxes all $p+1$ ratios, and everything øxes $(1,1)$, and F with $\beta = 1$ øxes everything. So if we exclude the ratio $(1,1)$, which is in an orbit on its own, a Burnside's Lemma count gives the number of other orbits as 1. This agrees with experiment!

68.3.2 $x = 3$

The automorphism group has size p, consisting of C, D, and E with $\gamma \neq \pm 1$. The ratio $(1, -1)$ is øxed by everthing, and E with $\gamma = 0$ øxes everything. so we have two orbits: $(1, -1)$ and the rest.

68.3.3 $\gamma x + \gamma^2 - \gamma + 1$ has no roots

The automorphism group has size $p+1$ and consists of C, D and E with $\gamma \neq -1$. Matrix C øxes nothing, matrix D øxes all $p+1$ ratios if $x = 1$, but otherwise øxes nothing, matrix E with $\gamma = 0$ øxes everything, and if $x \neq 1$ then E with $\gamma = -\frac{2}{x-1}$ øxes everything. So a Burnside's Lemma count gives 2 orbits.

68.3.4 $\gamma x + \gamma^2 - \gamma + 1$ has two roots

The automorphism group has size $p - 1$ and consists of C, D and E with $\gamma \neq -1$ and $\gamma x + \gamma^2 - \gamma + 1 \neq 0$. The ratio $(1, z)$ is øxed by everything if $1 - z + zx + z^2 = 0$. So there are two ratios øxed by everything. Matrix C øxes nothing else and nor does D unless $x = 1$ when if øxes everthing. Matrix E with $\gamma = 0$ øxes everything, as does $\gamma = -\frac{2}{x-1}$ if $x \neq 1$. So a Burnside's Lemma count gives 4 orbits.

69 Appendix I

From Case 6 in the immediate descendants of algebra 5.1.

We want to compute the number of orbits of 3×2 matrices under transformations of the form

$$\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \longmapsto (\alpha\eta - \gamma\varepsilon)^{-2} \begin{pmatrix} (\alpha\eta + \gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} \eta & -\gamma \\ -\varepsilon & \alpha \end{pmatrix}.$$

For a given matrix $\begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}$ we want to compute the number of matrices $\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix}$ which are øxed by $\begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}$. The condition for $\begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix}$ to be øxed is

$$\begin{pmatrix} (\alpha\eta + \gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} = (\alpha\eta - \gamma\varepsilon) \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} u\alpha\eta + u\gamma\varepsilon + 2\gamma\eta w - 2\alpha\varepsilon y & v\alpha\eta + v\gamma\varepsilon + 2\gamma\eta x - 2\alpha\varepsilon z \\ \varepsilon\eta u + \eta^2 w - \varepsilon^2 y & \varepsilon\eta v + \eta^2 x - \varepsilon^2 z \\ -\alpha\gamma u - \gamma^2 w + \alpha^2 y & -\alpha\gamma v - \gamma^2 x + \alpha^2 z \end{pmatrix} = (\alpha\eta - \gamma\varepsilon) \begin{pmatrix} u\alpha + v\varepsilon & u\gamma + v\eta \\ w\alpha + x\varepsilon & w\gamma + x\eta \\ y\alpha + z\varepsilon & y\gamma + z\eta \end{pmatrix},$$

which in its turn is equivalent to

$$\begin{pmatrix} \alpha\eta + \gamma\varepsilon - \alpha(\alpha\eta - \gamma\varepsilon) & -\varepsilon(\alpha\eta - \gamma\varepsilon) & 2\gamma\eta & 0 & -2\alpha\varepsilon & 0 \\ -\gamma(\alpha\eta - \gamma\varepsilon) & \alpha\eta + \gamma\varepsilon - \eta(\alpha\eta - \gamma\varepsilon) & 0 & 2\gamma\eta & 0 & -2\alpha\varepsilon \\ \varepsilon\eta & 0 & \eta^2 - \alpha(\alpha\eta - \gamma\varepsilon) & -\varepsilon(\alpha\eta - \gamma\varepsilon) & -\varepsilon^2 & 0 \\ 0 & \varepsilon\eta & -\gamma(\alpha\eta - \gamma\varepsilon) & \eta^2 - \eta(\alpha\eta - \gamma\varepsilon) & 0 & -\varepsilon^2 \\ -\alpha\gamma & 0 & -\gamma^2 & 0 & \alpha^2 - \alpha(\alpha\eta - \gamma\varepsilon) & -\varepsilon(\alpha\eta - \gamma\varepsilon) \\ 0 & -\alpha\gamma & 0 & -\gamma^2 & -\gamma(\alpha\eta - \gamma\varepsilon) & \alpha^2 - \eta(\alpha\eta - \gamma\varepsilon) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So we want to compute the nullity of the 6×6 matrix

$$\left(\begin{array}{llllll}
\alpha\eta+\gamma\varepsilon-\alpha(\alpha\eta-\gamma\varepsilon) & -\varepsilon(\alpha\eta-\gamma\varepsilon) & 2\gamma\eta & 0 & -2\alpha\varepsilon & 0 \\
-\gamma(\alpha\eta-\gamma\varepsilon) & \alpha\eta+\gamma\varepsilon-\eta(\alpha\eta-\gamma\varepsilon) & 0 & 2\gamma\eta & 0 & -2\alpha\varepsilon \\
\varepsilon\eta & 0 & \eta^2-\alpha(\alpha\eta-\gamma\varepsilon) & -\varepsilon(\alpha\eta-\gamma\varepsilon) & -\varepsilon^2 & 0 \\
0 & \varepsilon\eta & -\gamma(\alpha\eta-\gamma\varepsilon) & \eta^2-\eta(\alpha\eta-\gamma\varepsilon) & 0 & -\varepsilon^2 \\
-\alpha\gamma & 0 & -\gamma^2 & 0 & \alpha^2-\alpha(\alpha\eta-\gamma\varepsilon) & -\varepsilon(\alpha\eta-\gamma\varepsilon) \\
0 & -\alpha\gamma & 0 & -\gamma^2 & -\gamma(\alpha\eta-\gamma\varepsilon) & \alpha^2-\eta(\alpha\eta-\gamma\varepsilon)
\end{array}\right)$$

This nullity should only depend on the conjugacy class of $\begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}$. So we consider the possible canonical forms for $\begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix}$.

70 Case 1

First we let $\begin{pmatrix} \alpha & \gamma \\ \varepsilon & \eta \end{pmatrix} = \begin{pmatrix} 0 & \gamma \\ 1 & \eta \end{pmatrix}$ with $\gamma \neq 0$, and $\eta^2 + 4\gamma$ not a square.

If $\alpha = 0$, $\varepsilon = 1$, then the matrix becomes

$$\begin{pmatrix}
\gamma & \gamma & 2\eta\gamma & 0 & 0 & 0 \\
\gamma^2 & \gamma+\eta\gamma & 0 & 2\eta\gamma & 0 & 0 \\
\eta & 0 & \eta^2 & \gamma & -1 & 0 \\
0 & \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\
0 & 0 & -\gamma^2 & 0 & 0 & \gamma \\
0 & 0 & 0 & -\gamma^2 & \gamma^2 & \eta\gamma
\end{pmatrix}$$

We show that the only situation that this has non-zero nullity is when $\gamma = \eta = -1$ and $p = 2 \bmod 3$. In this case the matrix has nullity 2.

Since $\gamma \neq 0$, we can take out the factor γ from rows 1,2,5,6:

$$\begin{pmatrix}
1 & 1 & 2\eta & 0 & 0 & 0 \\
\gamma & 1+\eta & 0 & 2\eta & 0 & 0 \\
\eta & 0 & \eta^2 & \gamma & -1 & 0 \\
0 & \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\
0 & 0 & -\gamma & 0 & 0 & 1 \\
0 & 0 & 0 & -\gamma & \gamma & \eta
\end{pmatrix}$$

Use column 1 to purge row 1:

$$\begin{pmatrix}
1 & 1 & 2\eta & 0 & 0 & 0 \\
\gamma & 1+\eta & 0 & 2\eta & 0 & 0 \\
\eta & 0 & \eta^2 & \gamma & -1 & 0 \\
0 & \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\
0 & 0 & -\gamma & 0 & 0 & 1 \\
0 & 0 & 0 & -\gamma & \gamma & \eta
\end{pmatrix}
\begin{pmatrix}
1 & -1 & -2\eta & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\gamma & 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 & 0 \\
\eta & -\eta & -\eta^2 & \gamma & -1 & 0 \\
0 & \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\
0 & 0 & -\gamma & 0 & 0 & 1 \\
0 & 0 & 0 & -\gamma & \gamma & \eta
\end{pmatrix}$$

Remove row 1 and column 1:

$$\begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 & 0 \\ -\eta & -\eta^2 & \gamma & -1 & 0 \\ \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\ 0 & -\gamma & 0 & 0 & 1 \\ 0 & 0 & -\gamma & \gamma & \eta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\eta & 1 \end{pmatrix} \begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 & 0 \\ -\eta & -\eta^2 & \gamma & -1 & 0 \\ \eta & \gamma^2 & \eta^2+\eta\gamma & 0 & -1 \\ 0 & -\gamma & 0 & 0 & 1 \\ 0 & 0 & -\gamma & \gamma & \eta \end{pmatrix}$$

$$= \begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 & 0 \\ -\eta & -\eta^2 & \gamma & -1 & 0 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma & 0 & 0 \\ 0 & -\gamma & 0 & 0 & 1 \\ 0 & \eta\gamma & -\gamma & \gamma & 0 \end{pmatrix}$$

Remove row 4 and column 5:

$$\begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 \\ -\eta & -\eta^2 & \gamma & -1 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma & 0 \\ 0 & \eta\gamma & -\gamma & \gamma \end{pmatrix}$$

Take factor γ out of last row:

$$\begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 \\ -\eta & -\eta^2 & \gamma & -1 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma & 0 \\ 0 & \eta & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 \\ -\eta & -\eta^2 & \gamma & -1 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma & 0 \\ 0 & \eta & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta & 0 \\ -\eta & -\eta^2+\eta & -1+\gamma & 0 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma & 0 \\ 0 & \eta & -1 & 1 \end{pmatrix}$$

Take out last row and column:

$$\begin{pmatrix} 1+\eta-\gamma & -2\eta\gamma & 2\eta \\ -\eta & -\eta^2+\eta & -1+\gamma \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma \end{pmatrix}$$

This matrix has determinant $\left(\gamma^2-3\eta\gamma-\gamma-\eta^3\right)(-1+\gamma+\eta)^2$

If $\eta = 1-\gamma$ then we have

$$\begin{pmatrix} 2-2\gamma & 2(-1+\gamma)\gamma & 2-2\gamma \\ -1+\gamma & \gamma-\gamma^2 & -1+\gamma \\ 1-\gamma & \gamma^2-\gamma & 1-\gamma \end{pmatrix}$$

If $\gamma = 1$ and $\eta = 0$ then we have nullity 3. But if $\gamma \neq 1$ then we can take a factor $1-\gamma$ out of all three rows, leaving:

$$\begin{pmatrix} 1 & -\gamma & 1 \\ -1 & \gamma & -1 \\ 1 & -\gamma & 1 \end{pmatrix}$$

which has nullity 2. In any case, if $\eta = 1-\gamma$ then $\eta^2 + 4\gamma = (1+\gamma)^2$, so this situation does not arise.

Now consider the case when $\gamma - 1 + \eta \neq 0$, but $\gamma^2 - 3\eta\gamma - \gamma - \eta^3 = 0$. so $\gamma^2 - \gamma = 3\eta\gamma + \eta^3$

$$\begin{pmatrix} 1+\eta-\gamma & -2\gamma\eta & 2\eta \\ -\eta & -\eta^2+\eta & \gamma-1 \\ \eta & \gamma^2-\gamma & \eta^2+\eta\gamma \end{pmatrix}$$

We substitute $3\eta\gamma + \eta^3$ for $\gamma^2 - \gamma$ in the $(3,2)$-entry:

$$\begin{pmatrix} 1+\eta-\gamma & -2\gamma\eta & 2\eta \\ -\eta & -\eta^2+\eta & \gamma-1 \\ \eta & 3\eta\gamma+\eta^3 & \eta^2+\eta\gamma \end{pmatrix}$$

which has determinant $3\eta(\gamma-1+\eta)\left(\gamma^2 - 3\eta\gamma - \gamma - \eta^3\right)$.

Take out the factor η from the third row:

$$\begin{pmatrix} 1+\eta-\gamma & -2\gamma\eta & 2\eta \\ -\eta & -\eta^2+\eta & \gamma-1 \\ 1 & 3\gamma+\eta^2 & \eta+\gamma \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1-\eta+\gamma \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\eta-\gamma & -2\gamma\eta & 2\eta \\ -\eta & -\eta^2+\eta & \gamma-1 \\ 1 & 3\gamma+\eta^2 & \eta+\gamma \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5\eta\gamma - 3\gamma - \eta^2 - \eta^3 + 3\gamma^2 + \eta^2\gamma & \eta - \gamma - \eta^2 + \gamma^2 \\ 0 & -\eta^2 + \eta + 3\eta\gamma + \eta^3 & \gamma - 1 + \eta^2 + \eta\gamma \\ 1 & 3\gamma + \eta^2 & \eta + \gamma \end{pmatrix}$$

$$\begin{pmatrix} -5\eta\gamma - 3\gamma - \eta^2 - \eta^3 + 3\gamma^2 + \eta^2\gamma & \eta - \gamma - \eta^2 + \gamma^2 \\ -\eta^2 + \eta + 3\eta\gamma + \eta^3 & \gamma - 1 + \eta^2 + \eta\gamma \end{pmatrix}$$

which has determinant $3(\gamma-1+\eta)\left(\gamma^2 - 3\eta\gamma - \gamma - \eta^3\right)$.

We know that this has determinant zero if $\gamma^2 - 3\eta\gamma - \gamma - \eta^3 = 0$. But can it have nullity 2? Suppose it does. Then looking at the $(1,2)$-entry we see that $\gamma^2 - \gamma = \eta^2 - \eta$. So $(\gamma-\eta)(\gamma+\eta) = (\gamma-\eta)$. But $\gamma + \eta \neq 1$, and so $\gamma = \eta$. Then we have

$$\begin{pmatrix} -3\eta^2 - 3\eta & 0 \\ 2\eta^2 + \eta + \eta^3 & \eta - 1 + 2\eta^2 \end{pmatrix}.$$

So we do get nullity 2 if $\gamma = \eta = -1$, but only nullity one otherwise.

We need to compute how many pairs γ, η there are with $\gamma^2 - 3\eta\gamma - \gamma - \eta^3 = 0$, $\gamma \neq 0$, $\eta^2 + 4\gamma$ not a square. We can assume that $\eta \neq 0$, as the case $\gamma = 1$, $\eta = 0$ is dealt with above. (In any case $\eta^2 + 4\gamma$ is then a square.) For there to be a solution we must have $(3\eta + 1)^2 + 4\eta^3$ equal to a square. Now

$$(3\eta + 1)^2 + 4\eta^3 = (4\eta + 1)(1 + \eta)^2.$$

So there is a solution to the equation if $\eta = -1$ or if $4\eta + 1 = k^2$ for some k. Now in the latter case $\eta = (k^2 - 1)/4$, and so

$$\gamma^2 - 3\eta\gamma - \gamma - \eta^3 = \gamma^2 - \frac{3}{4}\gamma k^2 - \frac{1}{4}\gamma - \frac{1}{64}k^6 + \frac{3}{64}k^4 - \frac{3}{64}k^2 + \frac{1}{64}.$$

This polynomial has roots:

$$\begin{pmatrix} \frac{3}{8}k^2 + \frac{1}{8} + \frac{3}{8}k + \frac{1}{8}k^3 \\ \frac{3}{8}k^2 + \frac{1}{8} - \frac{3}{8}k - \frac{1}{8}k^3 \end{pmatrix}.$$

But if $\gamma = \frac{3}{8}k^2 + \frac{1}{8} + \frac{3}{8}k + \frac{1}{8}k^3$ then

$$\begin{aligned} \eta^2 + 4\gamma &= \frac{1}{16}k^4 + \frac{11}{8}k^2 + \frac{9}{16} + \frac{3}{2}k + \frac{1}{2}k^3 \\ &= \frac{1}{16}(k+3)^2(k+1)^2. \end{aligned}$$

So in this case $\eta^2 + 4\gamma$ is a square.

If $\eta = -1$ then $\gamma^2 - 3\eta\gamma - \gamma - \eta^3 = \gamma^2 + 3\gamma - \gamma + 1 = (\gamma + 1)^2$. So we have the possibility $\eta = \gamma = -1$ dealt with above. In this case $\eta^2 + 4\gamma = -3$, and for this not to be a square we need $p = 2 \bmod 3$.

What about $(\gamma - 1 + \eta) = (\gamma^2 - 3\eta\gamma - \gamma - \eta^3) = 0$?

$\eta = 1 - \gamma$

$\gamma^2 - 3\eta\gamma - \gamma - \eta^3 =$

$\gamma^2 - \gamma - 1 + \gamma^3 = (-1 + \gamma)(\gamma + 1)^2$

The two case that arise are $\gamma = 1$, $\eta = 0$ and $\gamma = -1$, $\eta = 2$, and in both these cases $\eta^2 + 4\gamma$ is a square.

71 Case 2

Now consider the case when $\gamma = \varepsilon = 0$. Then we have

$$\begin{pmatrix} \alpha\eta - \alpha^2\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\eta - \eta^2\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 - \alpha^2\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 - \eta^2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha^2 - \alpha^2\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2 - \eta^2\alpha \end{pmatrix}$$

Since α and η are non-zero, this has the same rank as

$$\begin{pmatrix} 1-\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta - \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha - \eta^2 \end{pmatrix}$$

If $\alpha = \eta = 1$ then this has nullity 6.
If $\alpha = 1$, $\eta = -1$ it has nullity 3.
If $\alpha = 1$, $\eta \neq \pm 1$ it has nullity 2.
If $\alpha = -1$, $\eta = 1$ it has nullity 3.
If $\alpha \neq \pm 1$, $\eta = 1$ it has nullity 2.

If $\alpha \neq 1$, $\eta \neq 1$ then it has nullity 0 unless $\eta = \alpha^2$ or $\alpha = \eta^2$, in which case it has nullity 1 unless α and η are inverse cube roots of unity in $\mathbb{Z}_p$. (Note that this can only happen when $p = 1 \bmod 3$.)

72 Case 3

Finally we consider the case when $\gamma = 1$, $\varepsilon = 0$, $\eta = \alpha$. Then we have

$$\begin{pmatrix} \alpha^2-\alpha^3 & 0 & 2\alpha & 0 & 0 & 0 \\ -\alpha^2 & \alpha^2-\alpha^3 & 0 & 2\alpha & 0 & 0 \\ 0 & 0 & \alpha^2-\alpha^3 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^2 & \alpha^2-\alpha^3 & 0 & 0 \\ -\alpha & 0 & -1 & 0 & \alpha^2-\alpha^3 & 0 \\ 0 & -\alpha & 0 & -1 & -\alpha^2 & \alpha^2-\alpha^3 \end{pmatrix}$$

We show that the nullity is 2 if $\alpha = 1$ and zero otherwise.

Take out factors α from columns:

$$\begin{pmatrix} \alpha-\alpha^2 & 0 & 2\alpha & 0 & 0 & 0 \\ -\alpha & \alpha-\alpha^2 & 0 & 2\alpha & 0 & 0 \\ 0 & 0 & \alpha^2-\alpha^3 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^2 & \alpha^2-\alpha^3 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1-\alpha & 0 \\ 0 & -1 & 0 & -1 & -1 & 1-\alpha \end{pmatrix}$$

Now take out factors α from rows

$$\begin{pmatrix} 1-\alpha & 0 & 2 & 0 & 0 & 0 \\ -1 & 1-\alpha & 0 & 2 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 \\ 0 & 0 & -1 & 1-\alpha & 0 & 0 \\ -1 & 0 & -1 & 0 & 1-\alpha & 0 \\ 0 & -1 & 0 & -1 & -1 & 1-\alpha \end{pmatrix}$$

Consider the situation when $\alpha = 1$.

$$\begin{pmatrix} 1-\alpha & 0 & 2 & 0 & 0 & 0 \\ -1 & 1-\alpha & 0 & 2 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 \\ 0 & 0 & -1 & 1-\alpha & 0 & 0 \\ -1 & 0 & -1 & 0 & 1-\alpha & 0 \\ 0 & -1 & 0 & -1 & -1 & 1-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Take out penultimate column and last row.

$$\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Take out ørst row and third column

$$\begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The nullity is clearly 2.
Now consider the case when $\alpha \neq 1$.

$$\begin{pmatrix} 1-\alpha & 0 & 2 & 0 & 0 & 0 \\ -1 & 1-\alpha & 0 & 2 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 \\ 0 & 0 & -1 & 1-\alpha & 0 & 0 \\ -1 & 0 & -1 & 0 & 1-\alpha & 0 \\ 0 & -1 & 0 & -1 & -1 & 1-\alpha \end{pmatrix}$$

Take out last two columns and last two rows:

$$\begin{pmatrix} 1-\alpha & 0 & 2 & 0 \\ -1 & 1-\alpha & 0 & 2 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & -1 & 1-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1-\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1-\alpha & 0 & 2 & 0 \\ -1 & 1-\alpha & 0 & 2 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & -1 & 1-\alpha \end{pmatrix}$$

$$\begin{pmatrix} 0 & (-1+\alpha)^2 & 2 & 2-2\alpha \\ -1 & 1-\alpha & 0 & 2 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & -1 & 1-\alpha \end{pmatrix}$$

Take out ørst column, second row:

$$\begin{pmatrix} (-1+\alpha)^2 & 2 & 2-2\alpha \\ 0 & 1-\alpha & 0 \\ 0 & -1 & 1-\alpha \end{pmatrix}$$

Take out ørst row and ørst column:

$$\begin{pmatrix} 1-\alpha & 0 \\ -1 & 1-\alpha \end{pmatrix}$$

So the nullity is zero.

73 Orbits when $p = 1 \bmod 3$

Scalar matrices: $p^6 + (p-2)$

Diagonal, not scalar: $(\alpha, \eta) = (1, -1)$, $(1, k)$ $(k \neq 0, 1, -1)$, (α, α^2) $(\alpha \neq 0, 1, -1$, but if $p = 1 \bmod 3$ and $\lambda^3 = 1$, $\lambda \neq 1$ then $(\lambda, \lambda^2) \sim (\lambda^2, \lambda)$ is double counted), and the rest.

$(p^3 + (p-3)p^2 + (p-5)p + p^2 + ((p-2)(p-3)/2 - (p-4)))p(p+1)$

$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$: $(p^2 + (p-2))(p^2 - 1)$

$\begin{pmatrix} 0 & \gamma \\ 1 & \eta \end{pmatrix}$ with $\gamma \neq 0$, and $\eta^2 + 4\gamma$ not a square: $(p(p-1)/2)p(p-1)$

$$p^6+(p-2)+(p^3+(p-3)p^2+(p-5)p+p^2+((p-2)(p-3)/2-(p-4)))p(p+1)+(p^2+(p-2))(p^2-1)+(p(p-1)/2)p(p-1)$$

$$= p^6 + 7p + 2p^5 + 3p^4 - 9p^3 - 4p^2.$$

$$\frac{p^6 + 7p + 2p^5 + 3p^4 - 9p^3 - 4p^2}{(p^2-1)(p^2-p)} = p^2 + 3p + 7$$

So if $p = 1 \bmod 3$ we have $p^2 + 3p + 7$ orbits. There are 6 orbits in which the matrix A does not have rank 2. So there are $p^2 + 3p + 1$ orbits in which it has rank 2.

74 Orbits when $p = 2 \bmod 3$

Scalar matrices: $p^6 + (p-2)$

Diagonal, not scalar: $(\alpha, \eta) = (1, -1)$, $(1, k)$ $(k \neq 0, 1, -1)$, (α, α^2) $(\alpha \neq 0, 1, -1$, but if $p = 1 \bmod 3$ and $\lambda^3 = 1$, $\lambda \neq 1$ then $(\lambda, \lambda^2) \sim (\lambda^2, \lambda)$ is double counted), and the rest.

$(p^3 + (p-3)p^2 + (p-3)p + ((p-2)(p-3)/2 - (p-3)))p(p+1)$

$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$: $(p^2 + (p-2))(p^2 - 1)$

$\begin{pmatrix} 0 & \gamma \\ 1 & \eta \end{pmatrix}$ with $\gamma \neq 0$, and $\eta^2 + 4\gamma$ not a square. Here we have to treat $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ separately:

$(p(p-1)/2 - 1 + p^2)p(p-1)$

$$p^6+(p-2)+(p^3+(p-3)p^2+(p-3)p+((p-2)(p-3)/2-(p-3)))p(p+1)+(p^2+(p-2))(p^2-1)+(p(p-1)/2-1+p^2)p(p-1)$$

$$= p^6 + 7p + 2p^5 + 3p^4 - 9p^3 - 4p^2$$

$$\frac{p^6 + 7p + 2p^5 + 3p^4 - 9p^3 - 4p^2}{(p^2-1)(p^2-p)} = p^2 + 3p + 7$$

So if $p = 2 \bmod 3$ we have $p^2 + 3p + 7$ orbits. There are 6 orbits in which the matrix A does not have rank 2. So there are $p^2 + 3p + 1$ orbits in which it has rank 2.

75 Orbits when $p = 3$

Scalar matrices: $p^6 + 1$

Diagonal, not scalar: $(\alpha, \eta) = (1, -1)$: $p^3 p(p+1)$

$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$: $(p^2 + 1)(p^2 - 1)$

$\begin{pmatrix} 0 & \gamma \\ 1 & \eta \end{pmatrix}$ with $\gamma \neq 0$, and $\eta^2 + 4\gamma$ not a square. This does not include $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ since -3 is a square.

$(p(p-1)/2)p(p-1)$

$$p^6 + 1 + p^3 p(p+1) + (p^2+1)(p^2-1) + (p(p-1)/2)p(p-1)$$

$$= p^6 + p^5 + \frac{5}{2}p^4 + \frac{1}{2}p^2 - p^3$$

$$\frac{p^6 + p^5 + \frac{5}{2}p^4 + \frac{1}{2}p^2 - p^3}{(p^2-1)(p^2-p)} = 24$$

So if $p = 3$ we have 24 orbits. There are 6 orbits in which the matrix A does not have rank 2. So there are 18 orbits in which it has rank 2.

76 Can't get $pa = pd = 0$

We consider the general situation in attempting to get $pa = pd = 0$. We consider a change of generators of the form

$$\begin{aligned}
a' &= \alpha a + (\alpha\eta - \gamma\varepsilon)^{-1}(\alpha\beta\varepsilon + \gamma\delta\eta - \alpha^2\zeta - \gamma^2\theta)b + \beta c + \gamma d + \delta e, \\
b' &= \lambda((\alpha\eta + \gamma\varepsilon)b + 2\gamma\eta c - 2\alpha\varepsilon e), \\
c' &= \lambda(\varepsilon\eta b + \eta^2 c - \varepsilon^2 e), \\
d' &= \varepsilon a + (\alpha\eta - \gamma\varepsilon)^{-1}(-\alpha\varepsilon\zeta - \gamma\eta\theta + \beta\varepsilon^2 + \delta\eta^2)b + \zeta c + \eta d + \theta e, \\
e' &= \lambda(-\alpha\gamma b - \gamma^2 c + \alpha^2 e),
\end{aligned}$$

with $\alpha = \eta = 1$, $\gamma = \varepsilon = 0$. We have

$$\begin{aligned}
a' &= a + -\zeta b + \beta c + \delta e, \\
b' &= \lambda b, \\
c' &= \lambda c, \\
d' &= \delta b + \zeta c + d + \theta e, \\
e' &= \lambda e.
\end{aligned}$$

We assume that pb, pc, pe span $Sp\langle ba, ca\rangle$, and so we can assume that $pa = 0$. We want to show that if we cannot also take $pd = 0$ then (with a general change of generators as above) we can take $pb = ca$ and take $pc = 0$ or $pe = 0$. It is easy to see that if $pb = 0$ then we can take $pa = pd = 0$. It is also possible to take $pa = pd = 0$ if pc, pe are both non-zero but linearly dependant. Similarly it is possible to take $pa = pd = 0$ if pb, pc are both non-zero but linearly dependant, or if pb, pe are both non-zero but linearly dependant.

So we assume that it is not possible to ønd $\zeta, \beta, \delta, \theta$ so that $pa' = pd' = 0$, and we suppose that none of pb, pc, pe are zero, and that no two of them are linearly dependant. So we have a dependance relation

$$pb = \rho pc + \sigma pe$$

for some unique non-zero ρ, σ. So, for any given ζ, $pa' = 0$ if and only if $\beta = \zeta\rho$ and $\delta = \zeta\sigma$. So we can take

$$d' = \zeta\sigma b + \zeta c + d + \theta e$$

for any ζ, θ. Clearly we can choose ζ, θ so that $pd' = 0$ unless $\sigma pb + pc$ and pe are linearly dependant. So

$$\sigma pb + pc + \tau pe = 0$$

for some unique non-zero τ. It follows that $\rho = -\sigma^{-1}$ and so

$$\rho pb - \rho^2 pc + pe = 0.$$

If we take $\alpha = -1$, $\gamma = \rho$ above, then we have $pe' = 0$. So we can assume that $pe = 0$.

We still need to show that we can take $pb = ca$. If we consider a change of generators of the form above, then we see that $pe' = 0$ if and only if $\gamma = 0$. And we also see that if $\gamma = 0$ then pb' is a scalar multiple of pb. Furthermore if $\gamma = 0$ then

$$\begin{aligned} b'a' &= \lambda\alpha^2\eta ba, \\ c'a' &= \lambda\varepsilon\alpha\eta ba + \lambda\alpha\eta^2 ca, \end{aligned}$$

so we can assume that pb is either a scalar multiple of ba or a scalar multiple of ca. And if pb is a scalar multiple of ba and $pe = 0$ then a further change of generators of the form above with $\alpha = \eta = 0$ gives pb' a scalar multiple of $c'a'$ and $pc' = 0$. So we can assume that pb is a scalar multiple of ca and that either $pc = 0$ or $pe = 0$. Finally, by scaling we can take $pb = ca$.

Let

$$\begin{pmatrix} pb \\ pc \\ pe \end{pmatrix} = \begin{pmatrix} u & v \\ w & x \\ y & z \end{pmatrix} \begin{pmatrix} ba \\ ca \end{pmatrix}.$$

Here we assume that pb, pc, pe span a space of dimension 2 with basis ba, ca. We show that we can always get $pa = pd = 0$ unless $\Delta = 0$ where $\Delta = (wz - xy)^2 - (ux - vw)(uz - vy)$.

First we note that we can always take $pa = pd = 0$ if $pb = 0$ and that in this case $\Delta \neq 0$. Next note that $\Delta = 0$ if $pc = 0$ or $pe = 0$. Futhermore it is easy to see that if $pc = 0$ or $pe = 0$ then it is impossible to guarantee that $pa = pd = 0$. And, as we showed above, if it is impossible to guarantee that $pa = pd = 0$ then it is possible to take $pe = 0$. Also, as we showed above, if it is not possible to guarantee that $pa = pd = 0$ then

$$\rho pb - \rho^2 pc + pe = 0$$

for some $\rho \neq 0$. But then $(ux - vw) = \rho^{-1}(wz - xy)$ and $(uz - vy) = \rho(wz - xy)$ so that $\Delta = 0$.

On the other hand, if none of pb, pc, pe are zero, but two of them are linearly dependant, then it is easy to see that we can always take $pa = pd = 0$. Furthermore, in this case $\Delta \neq 0$. And if no two of pb, pc, pe are linearly dependant then they satisfy a unique relation

$$pb = \rho pc + \sigma pe.$$

And if $\rho\sigma \neq -1$ then $\Delta \neq 0$ and we can guarantee that $pa = pd = 0$.

76.1 One of pc, pe are zero

We consider a general change of generators of the form

$$\begin{aligned} a' &= \alpha a + (\alpha\eta - \gamma\varepsilon)^{-1}(\alpha\beta\varepsilon + \gamma\delta\eta - \alpha^2\zeta - \gamma^2\theta)b + \beta c + \gamma d + \delta e, \\ b' &= \lambda((\alpha\eta + \gamma\varepsilon)b + 2\gamma\eta c - 2\alpha\varepsilon e), \\ c' &= \lambda(\varepsilon\eta b + \eta^2 c - \varepsilon^2 e), \\ d' &= \varepsilon a + (\alpha\eta - \gamma\varepsilon)^{-1}(-\alpha\varepsilon\zeta - \gamma\eta\theta + \beta\varepsilon^2 + \delta\eta^2)b + \zeta c + \eta d + \theta e, \\ e' &= \lambda(-\alpha\gamma b - \gamma^2 c + \alpha^2 e). \end{aligned}$$

It is clear that if $pb = ca$, $pb' = c'a'$, then you cannot have $pc = 0$, $pe' = 0$, or $pe = 0$, $pc' = 0$.

So let $pb = ca$, $pc = 0$ and consider a change of generators such that $pb' = c'a'$, $pc' = 0$. If $pc = 0$ and pb, pe are linearly independant, then to ensure that $pc' = 0$ we need $\varepsilon = 0$. Then

$$(\alpha\eta - \gamma\varepsilon)^{-2} \begin{pmatrix} (\alpha\eta + \gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ y & z \end{pmatrix} \begin{pmatrix} \eta & -\gamma \\ -\varepsilon & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\eta} \\ 0 & 0 \\ \frac{1}{\eta}y & -\frac{y\gamma+\gamma-\alpha z}{\eta^2} \end{pmatrix}$$

and so we need $\eta = 1$. Then

$$\begin{pmatrix} 0 & \frac{1}{\eta} \\ 0 & 0 \\ \frac{1}{\eta}y & -\frac{y\gamma+\gamma-\alpha z}{\eta^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ y & -y\gamma - \gamma + \alpha z \end{pmatrix}.$$

So if $y \neq -1$ we can take $z = 0$, and if $y = -1$ then we can take $z = 0$ or 1.

Next consider the situation when $pb = ca$ and $pe = 0$. If $pe = 0$ and pb, pc are linearly independant, then to ensure that $pe' = 0$ we require $\gamma = 0$. Then

$$(\alpha\eta - \gamma\varepsilon)^{-2} \begin{pmatrix} (\alpha\eta + \gamma\varepsilon) & 2\gamma\eta & -2\alpha\varepsilon \\ \varepsilon\eta & \eta^2 & -\varepsilon^2 \\ -\alpha\gamma & -\gamma^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ w & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta & -\gamma \\ -\varepsilon & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\varepsilon}{\eta\alpha} & \frac{1}{\eta} \\ \frac{w\eta^2-\varepsilon^2-\varepsilon x\eta}{\alpha^2\eta} & \frac{\varepsilon+x\eta}{\alpha\eta} \\ 0 & 0 \end{pmatrix}.$$

So we need $\varepsilon = 0$ and $\eta = 1$. Then we have

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{\alpha^2}w & \frac{x}{\alpha} \\ 0 & 0 \end{pmatrix}.$$

So if $x = 0$ we can take $w = 1$ or the smallest non-square, and if $x \neq 0$ we can take $x = 1$ and let w have any non-zero value. (So there are $p + 1$ orbits of this form.)

So there are $2p + 1$ orbits with $\Delta = 0$. We know that there are $(p^2 + 2p - 1)/2$ orbits with Δ a non-zero square when $p \neq 1 \bmod 3$, and $(p^2 + 2p + 1)/2$ orbits with Δ a non-zero square when $p = 1 \bmod 3$. And we know that the total number of orbits (for all possible Δ) is 18 when $p = 3$ and $p^2 + 3p + 1$ otherwise. So the number of orbits of each type is given by the following table.

	$\Delta \neq 0$ a square	$\Delta = 0$	Δ not a square
$p = 3$	7	7	4
$p = 1 \bmod 3$	$(p^2 + 2p + 1)/2$	$2p + 1$	$(p^2 - 1)/2$
$p = 2 \bmod 3$	$(p^2 + 2p - 1)/2$	$2p + 1$	$(p^2 + 1)/2$